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NOTE ON THE CONVERSES OF INEQUALITIES OF HARDY AND LITTLEWOOD

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Dedicated to Árpád Varecza on his 60th birthday

Abstract. The inequalities of Hardy–Littlewood type play very important role in many theorems concerning convergence or summability of orthogonal series. In the applications many times their converses are also very useful. Both the original inequalities and their converses have been generalized in several directions by many authors, among others Leindler (see [5], [6], [7], [8]), Mulholland (see [10]), Chen Yung-Ming (see [1]), etc.

The aim of this paper is to give further generalization of the inequalities of converse type replacing the power functions by more general ones.

1. Introduction; preliminaries

Suppose throughout that $\{a_n\}$ is a sequence of nonnegative numbers, that $\{\lambda_n\}$ is a sequence of positive numbers and the following notations will be used:

$$A_{1,n} = \sum_{k=1}^{n} a_k; \quad \Lambda_{n,m} = \sum_{k=n}^{m} \lambda_k \quad (1 \le n \le m \le \infty).$$

G.H. Hardy [2] proved the following inequality for $p > 1$:

$$\sum_{n=1}^{\infty} n^{-p} A_{1,n}^p \le K \sum_{n=1}^{\infty} a_n^p.^1 \tag{1.1}$$

This result was generalized and extended for $0 < p \le 1$ by Hardy and Littlewood [3] as follows:

$$\sum_{n=1}^{\infty} n^{-c} A_{1,n}^p \le K \sum_{n=1}^{\infty} n^{-c} (n a_n)^p \quad \text{with} \quad p \ge 1,\ c > 1, \tag{1.2}$$

$$\sum_{n=1}^{\infty} n^{-c} A_{1,n}^p \ge K \sum_{n=1}^{\infty} n^{-c} (n a_n)^p \quad \text{with} \quad 0 < p \le 1,\ c > 1. \tag{1.3}$$

Inequalities (1.2) and (1.3) were generalized by H.P. Mulholland [10] and Chen Yung-Ming [1] replacing the function x^p by more general function $\Phi(x)$.

We shall denote by $\Delta(p, q)$ $(p \ge q > 0)$ the set of all nonnegative functions $\Phi(x)$ defined on $[0, \infty)$ such that $\Phi(0) = 0$ and $\Phi(x)/x^p$ is nonincreasing and $\Phi(x)/x^q$ is nondecreasing. (This notation was introduced by M. Mateljevič and M. Pavlovič

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[1] K and K_i denote positive absolute constants, not necessarily the same at each occurrence.

in [9]). Further on the case $q = 0$ also will be allowed, so the results indicated by $q = 0$ are valid for a wider class of functions containing for example the function $\log(1 + x)$.

Now we formulate the results of Mulholland and Chen Yung-Ming:

$$\sum_{n=1}^{\infty} n^{-c}\ \Phi(A_{1,n}) \le K \sum_{n=1}^{\infty} n^{-c}\ \Phi(na_n), \tag{1.4}$$

for $\Phi(x) \in \Delta(p,q)$ $(p \ge q \ge 1)$, $c > 1$, and

$$\sum_{n=1}^{\infty} n^{-c}\ \Phi(A_{1,n}) \ge K \sum_{n=1}^{\infty} n^{-c}\ \Phi(na_n), \tag{1.5}$$

if $\Phi(x) \in \Delta(p,q)(1 \ge p \ge q \ge 0)$, $c > 1$.

Inequalities (1.2) and (1.3) were generalized by L. Leindler [5], who replaced n^{-c} by an arbitrary positive sequence $\{\lambda_n\}$. Namely he showed the following inequalities:

$$\sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \le K \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{n,\infty}^p\ a_n^p, \quad \text{if} \quad p \ge 1, \ \text{and} \tag{1.6}$$

$$\sum_{n=1}^{\infty}\ \lambda_n^{1-p}\ \Lambda_{n,\infty}^p\ a_n^p \le K \sum_{n=1}^{\infty} \lambda_n\ A_{1,n}^p, \quad \text{if} \quad 0 < p \le 1. \tag{1.7}$$

Later we in [11] generalized (1.6) and (1.7) using the functions $\Phi(x) \in \Delta(p,q)$. More precisely we proved the following inequalities:

$$\sum_{n=1}^{\infty} \lambda_n\ \Phi(A_{1,n}) \le K \sum_{n=1}^{\infty} \lambda_n \Phi\left(\frac{a_n}{\lambda_n}\Lambda_{n,\infty}\right), \tag{1.8}$$

for $\Phi(x) \in \Delta(p,q)$ $(p \ge q \ge 1)$, and

$$\sum_{n=1}^{\infty} \lambda_n \Phi\left(\frac{a_n}{\lambda_n}\Lambda_{n,\infty}\right) \le K \sum_{n=1}^{\infty} \lambda_n \Phi(A_{1,n}), \tag{1.9}$$

for $\Phi(x) \in \Delta(p,q)(1 \ge p \ge q \ge 0)$.

It is easy to see that the converses of these inequalities, in general, do not hold. But in the particular case, if $n^{-\gamma}a_n \downarrow$ $(\gamma > 0)$ Konyushkov [4] proved that (1.2) holds for $0 < p \le 1$, too, what is the converse of (1.3).

Later L. Leindler [6] generalized Konyuskhov's result proving inequalities of type of (1.6) and (1.7) by using the so called quasi monotone sequences.

A nonnegative sequence $\{a_n\}$ is said to be quasi increasing (decreasing) if there exists a constant M such that

$$Ma_{n+j} \ge a_n \quad (a_{n+j} \le Ma_n)$$

holds for any natural number n and $j \le n$. More precisely this definition means that $\{a_n\}$ is quasi monotone by section, namely e.g. a quasi increasing sequence can be decreasing, see $a_n = n^{-2}$.

Using the above notion Leindler's result reads as follows: If $\{a_n\}$ is quasi decreasing sequence, then

$$\sum_{n=1}^{\infty} a_n^p n^{p-1} \Lambda_{n,\infty} \le K \sum_{n=1}^{\infty} \lambda_n\ A_{1,n}^p, \quad \text{for} \quad p \ge 1, \tag{1.10}$$

and

$$\sum_{n=1}^{\infty} \lambda_n\ A_{1,n}^p \le K \sum_{n=1}^{\infty} a_n^p n^{p-1} \Lambda_{n,\infty}, \quad \text{for} \quad 0 < p \le 1. \tag{1.11}$$

It can be shown that (1.10) and (1.11) are not the converses of (1.6) and (1.7) but in the special case $\lambda_n = n^{-c}$ they give the converses of (1.2) and (1.3). Later L. Leindler [7], [8] showed that under some restriction on the sequence $\{\lambda_n\}$ the exact converses of (1.6) and (1.7) can be obtained.

Finally we remark that P.F. Renaud [13] proved certain converse of (1.1). Namely he showed that if $\{a_n\}$ is a monotone nonincreasing sequence of nonnegative numbers and $p > 1$, then

$$\sum_{n=1}^{\infty} n^{-p} A_{1,n}^p \geq \zeta(p) \sum_{n=1}^{\infty} a_n^p, \quad \text{where} \quad \zeta \text{ is the} \tag{1.12}$$

Riemann Zeta function.

It can be seen that (1.10) implies (1.12) disregarding the constant $\zeta(p)$.

The aim of the present paper is to give a generalization of (1.10) and (1.11) replacing the function x^p by $\Phi(x) \in \Delta(p,q)$. Our theorem will give the converses of (1.4) and (1.5) in the special case $\lambda_n = n^{-c}$.

Now we formulate our result.

2. Theorem

Theorem. *If the nonnegative sequence $\{a_n\}$ is quasi decreasing and*

$$\Phi(x) \in \Delta(p,q) \ (p \geq q \geq 1),$$

then

$$\sum_{n=1}^{\infty} \Phi(na_n) n^{-1} \Lambda_{n,\infty} \leq K \sum_{n=1}^{\infty} \lambda_n \ \Phi(A_{1,n}) \tag{2.1}$$

and if $\Phi(x) \in \Delta(p,q)$ $(1 \geq p \geq q \geq 0)$, then

$$\sum_{n=1}^{\infty} \lambda_n \ \Phi(A_{1,n}) \leq K \sum_{n=1}^{\infty} \Phi(na_n) n^{-1} \ \Lambda_{n,\infty}. \tag{2.2}$$

3. Lemmas

Lemma 1 ([9], [12])**.** *Let $\Phi \in \Delta(p,q)$ $(p \geq q \geq 0)$ and $t_n \geq 0$ $(n = 1, 2, \ldots)$. Then*

$$\Theta^p \Phi(t) \leq \Phi(\Theta t) \leq \Theta^q \Phi(t) \quad for \quad 0 \leq \Theta \leq 1, \ t \geq 0, \tag{3.1}$$

$$\Phi\left(\sum_{n=1}^{\infty} t_n\right) \leq \sum_{n=1}^{\infty} \Phi(t_n), \quad for \quad 0 \leq q \leq p \leq 1. \tag{3.2}$$

Lemma 2 ([12])**.** *Let $\Phi \in \Delta(p,q)$ $(p \geq q \geq 1)$ and $\overline{\Phi}$ denote the inverse of Φ. Then for $x \geq 0$, $y \geq 0$ and $\alpha \geq 1$ the following inequalities hold:*

$$\overline{\Phi}(x+y) \leq \overline{\Phi}(x) + \overline{\Phi}(y), \tag{3.3}$$

$$\overline{\Phi}(\alpha x) \leq \alpha \overline{\Phi}(x). \tag{3.4}$$

Furthermore for any $\Phi \in \Delta(p,q)$ $(p \geq q \geq 0)$ and for $\alpha \geq 1$

$$\Phi(\alpha x) \leq \alpha^p \Phi(x) \tag{3.5}$$

holds.

4. Proof

Proof of (2.1). Since the Abel-transformation gives that

$$\sum_{n=1}^{\infty} \Phi(na_n)n^{-1} \sum_{k=n}^{\infty} \lambda_k = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{n} \Phi(ka_k)k^{-1},$$

it is enough to prove that

$$\sum_{k=1}^{n} \Phi(ka_k)k^{-1} \le K\Phi\left(\sum_{k=1}^{n} a_k\right). \tag{4.1}$$

Let $2^k \le n < 2^{k+1}$. Using (3.3) we get

$$\overline{\Phi}\left(\sum_{k=1}^{n} \Phi(ka_k)k^{-1}\right) \le \sum_{m=0}^{k-1} \overline{\Phi}\left(\sum_{\ell=2^m}^{2^{m+1}} \Phi(\ell a_\ell)\ell^{-1}\right) + \overline{\Phi}\left(\sum_{\ell=2^k}^{n} \Phi(\ell a_\ell)\ell^{-1}\right) = I \tag{4.2}$$

Applying (3.4) and taking into account that $\{a_n\}$ is quasi decreasing, we obtain that

$$I \le K_1 \sum_{m=0}^{k} 2^m a_{2^m} \le K_2 \sum_{\ell=1}^{n} a_\ell. \tag{4.3}$$

From (4.2) and (4.3) one can get (4.1), which proves (2.1).

Proof of (2.2). First we estimate $\Phi\left(\sum_{k=1}^{n} a_k\right)$. If $2^k \le n < 2^{k+1}$ then by using (3.2) we have

$$\Phi\left(\sum_{k=1}^{n} a_k\right) \le \sum_{m=0}^{k-1} \Phi\left(\sum_{k=2^m}^{2^{m+1}} a_k\right) + \Phi\left(\sum_{\ell=2^k}^{n} a_\ell\right) = I. \tag{4.4}$$

Taking into account that $\{a_n\}$ is quasi decreasing and applying (3.5) we get

$$\begin{aligned} I &\le K_1 \sum_{m=0}^{k} \Phi(2^m a_{2^m}) \le K_2 \sum_{m=1}^{k} \sum_{\ell=2^{m-1}}^{2^m} \Phi(\ell a_\ell)\ell^{-1} \le \\ &\le K_3 \sum_{\ell=1}^{n} \Phi(\ell a_\ell)\ell^{-1}. \end{aligned} \tag{4.5}$$

Using (4.4) and (4.5) a simple Abel-transformation gives that

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n \Phi\left(\sum_{k=1}^{n} a_k\right) &\le K \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{n} \Phi(ka_k)k^{-1} = \\ &= K \sum_{n=1}^{\infty} \Phi(na_n)n^{-1} \sum_{k=n}^{\infty} \lambda_k, \end{aligned}$$

which proves (2.2).

The proof of Theorem is completed.

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