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CONVERGENCE IN NORM OF THE FEJÉR MEANS ON $\mathbf{R}^+$ WITH RESPECT TO WALSH-KACZMARZ SYSTEM

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Abstract. In this paper we prove that the Fejér integrals of an integrable function f defined on $\mathbb{R}^+$ converges in norm to the function itself.

Let denote by $\mathbf{Z}_2$ the discrete cyclic group of order 2, that is $\mathbf{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. Haar measure on $\mathbf{Z}_2$ is given in the way that the measure of a singleton is $\frac{1}{2}$. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbf{Z}_2$. The elements of G are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$. The group operation on G is the coordinate-wise addition, the measure and the topology are the product measure and the topology. G is a compact Abelian group, called Walsh group.

For $k \in \mathbf{N}$ and $x \in G$ denote r_k the k-th Rademacher function:

$$r_k(x) := (-1)^{x_k}.$$

Let $n \in \mathbf{N}$, n can be written in the form $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$, that is, n is expressed in the number system based 2. Denote by $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$. We define the Walsh-Paley system as the set of Walsh-Paley functions:

$$\omega_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{N}).$$

The Walsh-Paley system can be given in the Kaczmarz enumeration as follows:

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-k-1}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-k-1}},$$

for $x \in G,\ n \in \mathbf{P}$, $\kappa_0(x) := 1 (x \in G)$. Define the norm of $x \in G$ by

$$\|x\|_G := \sum_{n \in \mathbf{N}} x_n 2^{-n-1}.$$

The Fine's map $x \mapsto \|x\|_G$ takes G onto $[0, 1)$, the Haar measure to Lebesgue measure on $[0, 1)$, and is $1 - 1$ off a countable subset of $[0, 1)$.

The Walsh system in the Kaczmarz enumeration was studied by a lot of authors. A.A. Šneider [10] showed that for the Dirichlet kernel of Walsh-Kaczmarz system the inequality $\limsup_{n\to\infty} \frac{D_n(x)}{\log n} \geq C > 0$ holds a.e. L.A. Balašov [1] constructed

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an examples for divergent Fourier series. On the other hand F. Schipp [6] and Wo-Sang Young [12] proved that the Walsh-Kaczmarz system is a convergence system, V.A. Skvorcov [9] proved for continuous functions f, that the Fejér means of f converges uniformly to the function f itself. G. Gát [3] proved for integrable functions f, that the Fejér means converges to the function f a.e. and the maximal operator of Fejér means is of type (q,q) for all $1 < q \leq \infty$ and weak type $(1,1)$. G. Gát and K. Nagy [4] proved that the maximal operator of Fejér means is of type (H, L).

F. Schipp, W.R. Wade, P. Simon and J. Pál [7] showed the following. Let $f \in L^1(\mathbf{R}^+)$, the Fejér means of f with respect to generalised Walsh-Paley system converges in L^1-norm to the function f. In this paper we prove the same result for the generalised Walsh-Kaczmarz system.

Let $\mathbf{F}$ denote the set of doubly infinite sequences

$$\{x_n : n \in \mathbf{Z}\}$$

where $x_n = 0$ or 1 $(n \in \mathbf{Z})$ and $\lim_{n\to-\infty} x_n = 0$. The group operation on $\mathbf{F}$ let be the coordinate-wise addition by modulo 2. $(\mathbf{F}, +)$ is an Abelian group. The norm of $x = (x_n : n \in \mathbf{Z}) \in \mathbf{F}$ is

$$\|x\| := \sum_{n\in\mathbf{Z}} x_n 2^{-n-1}.$$

There is a unique Haar measure μ on $\mathbf{F}$ for which the measure of the unit ball is 1. The Fine's map $x \to \|x\|$ takes $\mathbf{F}$ onto $\mathbf{R}^+ := [0,\infty)$ and is 1-1 off a countable subset of $\mathbf{F}$. (The set $\mathbf{Q}^+ := \{p2^q : p \in \mathbf{N}, q \in \mathbf{Z}\}$ represents the diadic rationals in $\mathbf{R}^+$ and every point in $\mathbf{Q}^+$ has two preimages in $\mathbf{F}$.) At any rate, the Fine's map allows us to identify $\mathbf{F}$ with $\mathbf{R}^+$. Define the integer part of an $x \in \mathbf{F}$ by

$$[x] := (\dots, x_{-2}, x_{-1},, 0, 0, \dots)$$

and the fractional part of x by

$$\{x\} := (\dots, 0, 0,, x_0, x_1, \dots).$$

Thus $\|[x]\|$ is an integer in $\mathbf{R}^+$ and $0 \leq \|\{x\}\| \leq 1$ for each $x \in \mathbf{F}$, we shall also denote these corresponding numbers in $\mathbf{R}^+$ by $[x]$ and $\{x\}$, if no confusion arises. For $x \neq 0$ let

$$|x| := \max\{n \in \mathbf{N} : x_{-n} \neq 0\},$$

that is $2^{|x|-1} \leq [x] < 2^{|x|}$. Denote $\{\psi_y : y \in \mathbf{F}\}$ the set of the characters of the additive group $(\mathbf{F}, +)$. The characters can be generated in the following way. Set

$$\psi_y(x) := (-1)^{\sum_{i+j=-1} x_i y_j}$$

for each $x, y \in \mathbf{F}$. Fix an $n \in \mathbf{N}$ and set

$$\chi_n(x) := \kappa_n(x - [x]) \text{ for all } x \in \mathbf{F}.$$

Define the generalized Walsh-Kaczmarz functions by the equation

$$\chi_y(x) := \chi_{[y]}(x)\chi_{[x]}(y) \quad (x, y \in \mathbf{F}).$$

Let the transformation $\tau_y \colon \mathbf{F} \to \mathbf{F}$ be defined by

$$\tau_y(x) := (\dots, x_{-|x|-1}, x_{-1}, x_{-2}, \dots, x_{-|x|},, x_{|y|-1}, x_{|y|-2}, \dots, x_1, x_0, x_{|y|}, \dots)$$

where $(x, y \in \mathbf{F})$ (and two commas show the "centre" of the sequence).

We have

$$\begin{aligned} \chi_y(x) &= \chi_{[y]}(x)\chi_{[x]}(y) = \psi_{[y]}(\tau_y(x))\psi_{[x]}(\tau_x(y)) \\ &= \psi_{[y]}(\tau_y(x))\psi_{[\tau_y(x)]}(y) = \psi_y(\tau_y(x)) \end{aligned}$$

for all $x, y \in \mathbf{F}$.

It is sometimes convenient to work on $\mathbf{R}^+$ instead of $\mathbf{F}$, the Fine's map takes Haar measure μ to Lebesgue measure on $\mathbf{R}^+$, the characters of $\mathbf{F}$ to generalized Walsh functions on $\mathbf{R}^+$. We can leave all notation the same. It is easy to see that

$$\psi_k(x) = \omega_k(x) \text{ and } \chi_k(x) = \kappa_k(x)$$

for $k \in \mathbf{N}$ and $x \in [0,1)$, where ω_k denotes the k-th Walsh-Paley function and κ_k denotes the k-th Walsh-Kaczmarz function. The functions $\{\psi_j : j \in \mathbf{N}\}$ and $\{\chi_j : j \in \mathbf{N}\}$ form a complete orthonormal system in each interval of the form $[k, k+1)$, $k = 0, 1, \ldots$. For $x \in [k, k+1)$ we have $\psi_j(x) = \omega_j(x - [x])$ and $\chi_j(x) = \kappa_j(x - [x])$ $j \in \mathbf{N}$. That is ψ_j (χ_j) is a periodic extension of ω_j (κ_j) from $[0,1)$ to $\mathbf{R}^+$.

Define the generalized Dirichlet kernel, the generalized Fejér kernel by

$$D_t^\alpha(y) := \int_0^t \alpha_y(x)dx, \quad \mathcal{K}_u^\alpha(y) := \frac{1}{u}\int_0^u D_t^\alpha(y)dt$$

for $t, y, u \in \mathbf{R}^+$ and $\alpha = \psi$ or χ.

For $t = n \in \mathbf{N}$, the generalized Dirichlet kernel is a zero extension of the Walsh-Paley (Walsh-Kaczmarz)-Dirichlet kernel outside $[0,1)$, namely

$$D_n^\psi(y) = \begin{cases} \sum_{k=0}^{n-1} \omega_k(y) & 0 \le y < 1 \\ 0 & y \ge 1, \end{cases} \qquad D_n^\chi(y) = \begin{cases} \sum_{k=0}^{n-1} \kappa_k(y) & 0 \le y < 1 \\ 0 & y \ge 1. \end{cases}$$

(The generalized Fejér kernel is not simply a zero extension of the Walsh-Fejér kernel.) For each $f \in L^1(\mathbf{R}^+)$ we introduce the Walsh-Fourier transform, the Walsh-Dirichlet integrals and the Walsh-Fejér integrals by

$$\hat{f}^\alpha(y) := \int_0^\infty f(x)\alpha_y(x)dx, \quad (S_t^\alpha f)(x) := \int_0^t \hat{f}^\alpha(y)\alpha_y(x)dy, \quad (y, x, t > 0)$$

$$(\sigma_u^\alpha f)(x) := \frac{1}{u}\int_0^u (S_t^\alpha f)(x)dt \quad (u, x > 0),$$

where $\alpha = \psi$ or χ. By definitions and Fubini's theorem we have

$$S_t^\alpha f = f * D_t^\alpha, \quad \sigma_u^\alpha f = f * \mathcal{K}_u^\alpha.$$

Introduce some useful notation, set

$$\mathbf{J}_0(t) := t - [t], \quad \mathbf{J}_l(t) := \int_0^t \chi_l(x)dx \quad (l \in \mathbf{P}, t \in \mathbf{R}^+)$$

and

$$\mathbf{J}_l^{(2)}(t) := \int_0^t \mathbf{J}_l(x)dx \quad (l \in \mathbf{N}, t \in [0,1)),$$

extend the $\mathbf{J}_l^{(2)}$ to $\mathbf{R}^+$ by periodicity of period 1.

Lemma 1. *For each $u, y > 0$ holds the following equation*

$$u\mathcal{K}_u^\chi(y) = \sum_{k=0}^{[u]-1}\left(D_k^\chi(y) + \chi_k(y)\mathbf{J}_{[y]}^{(2)}(1)\right) + (u - [u])D_{[u]}^\chi(y) + \chi_{[u]}(y)\mathbf{J}_{[y]}^{(2)}(u).$$

Proof. Consequently, by definitions

$$\begin{aligned} D_t^\chi(x) &= \sum_{k=0}^{[t]-1}\int_k^{k+1}\chi_y(x)dy + \int_{[t]}^t \chi_y(x)dy \\ &= \sum_{k=0}^{[t]-1}\chi_k(x)\int_0^1 \chi_{[x]}(y)dy + \chi_{[t]}(x)\int_0^{t-[t]}\chi_{[x]}(y)dy \\ &= D_{[t]}^\chi(x) + \chi_{[t]}(x)\mathbf{J}_{[x]}(t) \end{aligned}$$

for $t, x \in \mathbf{R}^+$ and

$$u\mathcal{K}_u^\chi(y) := \sum_{l=0}^{[u]-1} \int_l^{l+1} D_t^\chi(y)dt + \int_{[u]}^u D_t^\chi(y)dt$$

for $u, y \in \mathbf{R}^+$. These imply that

$$\int_{[u]}^u D_t^\chi(y)dt = (u-[u])D_{[u]}^\chi(y) + \chi_{[u]}(y)\mathbf{J}_{[y]}^{(2)}(u)$$

and

$$\begin{aligned}\int_l^{l+1} D_t^\chi(y)dt &= \int_l^{l+1} D_{[t]}^\chi(y) + \chi_{[t]}(y)\mathbf{J}_{[y]}(t)dt \\ &= \int_l^{l+1} \sum_{k=0}^{[t]-1} \chi_k(y) \int_0^1 \chi_{[y]}(s)dsdt + \chi_l(y)\int_l^{l+1} \mathbf{J}_{[y]}(t)dt \\ &= D_l^\chi(y) + \chi_l(y)\mathbf{J}_{[y]}^{(2)}(1).\end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2 (V.A. Skvorcov)**.** *There exists a constant $C>0$ such that*

$$\|K_n^\kappa\|_1 \le C \quad (n \in \mathbf{N}).$$

Lemma 3. *There exists a constant $C>0$ such that*

$$\|\mathcal{K}_u^\chi\|_1 \le C, \quad (u \ge 1).$$

Proof. It follows from Lemma 1 that

$$|\mathcal{K}_u^\chi(y)| \le |K_{[u]}^\kappa(y)| + |\mathbf{J}_{[y]}^{(2)}(1)| + |\frac{1}{u}D_{[u]}^\chi(y)| + |\frac{1}{u}\mathbf{J}_{[y]}^{(2)}(u)| \quad (u \ge 1, y \in \mathbf{R}^+).$$

By Skvorcov's lemma, there exists a $C>0$ such that

$$\|K_{[u]}^\kappa\| \le C \quad (u \ge 1).$$

By the fact, that for integer values of t, the generalized Dirichlet kernel D_t^χ is a zero extension of the Walsh-Kaczmarz-Dirichlet kernel outside $[0,1)$, we have that

$$\|\frac{1}{u}D_{[u]}^\chi\| \le 1.$$

Let $2^n \le l < 2^{n+1}$, $t \in \mathbf{R}^+$ and denote $\bar{t} := \{t\}$, by definitions

$$\begin{aligned}\mathbf{J}_l(t) &= \int_{[t]}^t \chi_l(x)dx = \int_0^{\{t\}} \chi_l(x)dx = \int_{\frac{\bar{t}_0}{2}+\cdots+\frac{\bar{t}_{n-1}}{2^n}}^{\bar{t}} \chi_l(x)dx \\ &= (-1)^{\sum_{k=0}^{n-1} l_k \bar{t}_{n-1-k}} \int_{\frac{\bar{t}_0}{2}+\cdots+\frac{\bar{t}_{n-1}}{2^n}}^{\bar{t}} (-1)^{x_n}dx \\ &= \chi_l(\frac{\bar{t}_0}{2}+\cdots+\frac{\bar{t}_{n-1}}{2^n})\left[\frac{\bar{t}_n}{2^{n+1}} + (-1)^{\bar{t}_n}\left(\frac{\bar{t}_{n+1}}{2^{n+2}} + \frac{\bar{t}_{n+2}}{2^{n+3}} + \dots\right)\right].\end{aligned}$$

Consequently,

$$|\mathbf{J}_l(t)| \le 2^{-n}$$

for $2^n \le l < 2^{n+1}$, $n \in \mathbf{N}$ and $t \in \mathbf{R}^+$.

Investigate the second integral $\mathbf{J}_l^{(2)}(1)$. Let $l = 2^n$

$$|\mathbf{J}_l^{(2)}(1)| \le \int_0^1 |\mathbf{J}_l(t)|dt \le 2^{-n},$$

and let $l = 2^n + 2^m + k$ $(0 \le m < n,\ 0 \le k < 2^m)$

$$\mathbf{J}_l^{(2)}(1) =$$

$$\begin{aligned}
&= \int_0^1 (-1)^{\sum_{j=0}^{m-1} l_j t_{n-1-j} + t_{n-1-m}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n}\left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots\right)\right] dt \\
&= \int_{\{x\in[0,1):x_{n-m-1}=0\}} (-1)^{\sum_{j=0}^{m-1} l_j t_{n-1-j}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n}\left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots\right)\right] dt \\
&- \int_{\{x\in[0,1):x_{n-m-1}=1\}} (-1)^{\sum_{j=0}^{m-1} l_j t_{n-1-j}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n}\left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots\right)\right] dt \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\mathbf{J}_{[.]}^{(2)}(1)\|_1 &= \int_0^\infty |\mathbf{J}_{[y]}^{(2)}(1)|dy = \int_0^1 |\mathbf{J}_0^{(2)}(1)|dy + \sum_{i=1}^\infty \int_i^{i+1} |\mathbf{J}_i^{(2)}(1)|dy \\
&\le \frac{1}{2} + \sum_{n=1}^\infty 2^{-n} = \frac{3}{2}.
\end{aligned}$$

Last, we have to see the L^1-norm of $\frac{1}{u}\mathbf{J}_{[.]}^{(2)}(u)$ for $u \ge 1$.

$$\begin{aligned}
\frac{1}{u}\int_0^\infty |\mathbf{J}_{[y]}^{(2)}(u)|dy &= \frac{1}{u}\int_0^1 |\mathbf{J}_0^{(2)}(u)|dy + \frac{1}{u}\int_1^4 |\mathbf{J}_0^{(2)}(u)|dy \\
&+ \frac{1}{u}\int_4^\infty |\mathbf{J}_{[y]}^{(2)}(u)|dy =: I_1(u) + I_2(u) + I_3(u).
\end{aligned}$$

Using the periodicity of $\mathbf{J}_0^{(2)}$, it is easy to estimate the first integral

$$\mathbf{J}_0^{(2)}(u) = \int_0^{u-[u]} t - [t]dt = \frac{(u-[u])^2}{2}, \text{ and } I_1(u) = \left|\frac{(u-[u])^2}{2u}\right| \le \frac{1}{2}$$

for $u \ge 1$.

Similarly, the second integral $I_2(u) \le c$.

Now we will estimate the third integral $I_3(u)$. Let $l := 2^n$, by the periodicity of $\mathbf{J}_l^{(2)}$ and by the inequality on $\mathbf{J}_l$ we conclude that

$$|\mathbf{J}_l^{(2)}(u)| = \int_0^{u-[u]} 2^{-n}dt \le 2^{-n}.$$

Let $l := 2^n + 2^m + k$, where $m := \min\{i : l_i \ne 0\}$ and $2^m < k < 2^n - 1$. Set

$$z^0 := (\dots, 0, 0, z_0, z_1, \dots, z_{n-m-2}, 0, z_{n-m}, z_{n-m+1}, \dots)$$

and

$$z^1 := (\dots, 0, 0, z_0, z_1, \dots, z_{n-m-2}, 1, z_{n-m}, z_{n-m+1}, \dots)$$

for any $z \in [0,1)$.

$$\begin{aligned}
&\int_{I_{n-m}(z^0)} (-1)^{\sum_{j=m+1}^{n-1} l_j t_{n-1-j} + t_{n-1-m}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n}\left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots\right)\right] dt \\
&+ \int_{I_{n-m}(z^1)} (-1)^{\sum_{j=m+1}^{n-1} l_j t_{n-1-j} + t_{n-1-m}} \left[\frac{t_n}{2^{n+1}} + (-1)^{t_n}\left(\frac{t_{n+1}}{2^{n+2}} + \frac{t_{n+2}}{2^{n+3}} + \dots\right)\right] dt \\
&= 0.
\end{aligned}$$

That is

$$\int_{I_{n-m}(z^0)\cup I_{n-m}(z^1)} \mathbf{J}_l(t)dt = 0$$

for any $z \in [0,1)$. Therefore

$$J_l^{(2)}(u) = \int_0^{\{u\}} \mathbf{J}_l(t)dt = \int_{\frac{\overline{u}_0}{2}+\dots+\frac{\overline{u}_{n-m-1}}{2^{n-m}}}^{\overline{u}} \mathbf{J}_l(t)dt$$

and

$$|J_l^{(2)}(u)| \leq 2^{-n}2^{-(n-m)},$$

where $\overline{u} := \{u\}$.

$$\int_4^\infty |J_{[y]}^{(2)}(u)|dy = \sum_{n=2}^\infty \int_{2^n}^{2^{n+1}} |J_{[y]}^{(2)}(u)|dy$$

$$= \sum_{n=2}^\infty [\int_{2^n}^{2^n+1} |J_{[y]}^{(2)}(u)|dy + \sum_{\{k:k_0=0\}} \int_{2^n+1+k}^{2^n+k+2} |J_{[y]}^{(2)}(u)|dy$$

$$+ \sum_{\{k:k_0=k_1=0\}} \int_{2^n+2^1+k}^{2^n+2^1+k+1} |J_{[y]}^{(2)}(u)|dy + \dots$$

$$+ \sum_{\{k:k_l=0,l\leq n-1\}} \int_{2^n+2^{n-1}+k}^{2^n+2^{n-1}+k+1} |J_{[y]}^{(2)}(u)|dy]$$

$$\leq \sum_{n=2}^\infty \left[\frac{1}{2^n} + \frac{1}{2^n}\frac{1}{2^n}2^{n-1} + \frac{1}{2^n}\frac{1}{2^{n-1}}2^{n-2} + \dots + \frac{1}{2^n}\frac{1}{2^1}2^0\right]$$

$$= c + \frac{1}{2}\sum_{n=2}^\infty \frac{n}{2^n} \leq c.$$

The proof is complete. □

By standard argument the following theorem holds.

Theorem 1. *For every $f \in L^1(\mathbf{R}^+)$ holds*

$$\lim_{u\to\infty} \|\sigma_u^\chi f - f\|_1 = 0.$$

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