

## ON $\pi$ -IMAGES OF METRIC SPACES

YING GE

ABSTRACT. In this paper, we prove that sequence-covering,  $\pi$ -images of metric spaces and spaces with a  $\sigma$ -strong network consisting of  $fcs$ -covers are equivalent. We also investigate  $\pi$ -images of separable metric spaces.

### 1. INTRODUCTION

A study of images of metric spaces is an important question in general topology ([2, 7, 9, 10, 16]). In recent years,  $\pi$ -images of metric spaces cause attention once again ([4, 13, 18, 19]). It is known that a space is a strong-sequence-covering (resp. sequentially-quotient),  $\pi$ -image of a metric space if and only if it has a  $\sigma$ -strong network consisting of  $cs$ -covers (resp.  $cs^*$ -covers) (see [13], for example). Note that strong-sequence-covering mapping  $\implies$  sequence-covering mapping  $\implies$  (if the domain is metric) sequentially-quotient mapping and that  $cs$ -cover  $\implies fcs$ -cover  $\implies cs^*$ -cover. It is natural to raised the following question.

**Question 1.1.** Can sequence-covering,  $\pi$ -images of metric spaces be characterized as spaces with a  $\sigma$ -strong network consisting of  $fcs$ -covers?

On the other hand, whether sequentially-quotient,  $\pi$ -images of metric spaces and sequence-covering,  $\pi$ -images of metric spaces are equivalent? This question is still open (see [13, Question 3.1.14] or [19, Question 4.4(2)], for example). This leads us to consider the following question.

**Question 1.2.** Are sequentially-quotient,  $\pi$ -images of separable metric spaces and sequence-covering,  $\pi$ -images of separable metric spaces equivalent?

In this paper, we give a positive answer for Question 1.1. We also investigate  $\pi$ -images of separable metric spaces, and answer Question 1.2 affirmatively.

---

2000 *Mathematics Subject Classification.* 54E35, 54E40.

*Key words and phrases.* Metric space,  $\pi$ -mapping, sequence-covering mapping,  $\sigma$ -strong network,  $fcs$ -cover,  $cs^*$ -cover.

This project was supported by NSFC(No.10571151).

Throughout this paper, all spaces are assumed to be Hausdorff, and all mappings are continuous and onto.  $\mathbb{N}$  denotes the set of all natural numbers,  $\{x_n\}$  denotes a sequence, where the  $n$ -th term is  $x_n$ . Let  $X$  be a space and let  $A$  be a subset of  $X$ . We say that a sequence  $\{x_n\}$  converging to  $x$  in  $X$  is eventually in  $A$  if  $\{x_n : n > k\} \cup \{x\} \subset A$  for some  $k \in \mathbb{N}$ . Let  $\mathcal{P}$  be a family of subsets of  $X$  and let  $x \in X$ .  $\bigcup \mathcal{P}$ ,  $st(x, \mathcal{P})$  and  $(\mathcal{P})_x$  denote the union  $\bigcup\{P : P \in \mathcal{P}\}$ , the union  $\bigcup\{P \in \mathcal{P} : x \in P\}$  and the subfamily  $\{P \in \mathcal{P} : x \in P\}$  of  $\mathcal{P}$  respectively. For a sequence  $\{\mathcal{P}_n : n \in \mathbb{N}\}$  of covers of a space  $X$ , we abbreviate  $\{\mathcal{P}_n : n \in \mathbb{N}\}$  to  $\{\mathcal{P}_n\}$ . A point  $b = (\beta_n)_{n \in \mathbb{N}}$  of a Tychonoff-product space is abbreviated to  $(\beta_n)$ , where  $\beta_n$  is the  $n$ -th coordinate of  $b$ . If  $f: X \rightarrow Y$  is a mapping, then  $f(\mathcal{P})$  denotes  $\{f(P) : P \in \mathcal{P}\}$ .

## 2. $\pi$ -IMAGES OF METRIC SPACES

**Definition 2.1.** Let  $f: X \rightarrow Y$  be a mapping.

(1)  $f$  is called a strong-sequence-covering mapping ([11]) if for every convergent sequence  $S$  in  $Y$ , there exists a convergent sequence  $L$  in  $X$  such that  $f(L) = S$ .

(2)  $f$  is called a sequence-covering mapping ([6]) if for every sequence  $S$  converging to  $y$  in  $Y$ , there exists a compact subset  $K$  of  $X$  such that  $f(K) = S \cup \{y\}$ .

(3)  $f$  is called a sequentially-quotient mapping ([1]) if for every convergent sequence  $S$  in  $Y$ , there exists a convergent sequence  $L$  in  $X$  such that  $f(L)$  is a subsequence of  $S$ .

(4)  $f$  is called a compact-covering mapping ([15]) if for every compact subset  $C$  of  $Y$ , there exists a compact subset  $K$  of  $X$  such that  $f(K) = C$ .

(5)  $f$  is called a  $\pi$ -mapping ([16]), if for every  $y \in Y$  and for every neighborhood  $U$  of  $y$  in  $Y$ ,  $d(f^{-1}(y), X - f^{-1}(U)) > 0$ , where  $X$  is a metric space with a metric  $d$ .

**Definition 2.2.** Let  $\mathcal{P}$  be a cover of a space  $X$ .

(1)  $\mathcal{P}$  is called an  $fcs$ -cover of  $X$  ([5]) if for every sequence  $S$  converging to  $x$  in  $X$ , there exists a finite subfamily  $\mathcal{P}'$  of  $(\mathcal{P})_x$  such that  $S$  is eventually in  $\bigcup \mathcal{P}'$ .

(2)  $\mathcal{P}$  is called a  $cs^*$ -cover ([13]) if for every convergent sequence  $S$  in  $X$ , there exist  $P \in \mathcal{P}$  and a subsequence  $S'$  of  $S$  such that  $S'$  is eventually in  $P$ .

**Definition 2.3.** (1) Let  $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$  be a cover of a space  $X$ , where  $\mathcal{P}_x \subset (\mathcal{P})_x$ .  $\mathcal{P}$  is called a network of  $X$  ([15]), if for every  $x \in U$  with  $U$  open in  $X$ , there exists  $P \in \mathcal{P}_x$  such that  $x \in P \subset U$ , where  $\mathcal{P}_x$  is called a network at  $x$  in  $X$ .

(2) Let  $\{\mathcal{P}_n\}$  be a sequence of covers of a space  $X$  and every  $\mathcal{P}_{n+1}$  is an refinement of  $\mathcal{P}_n$ .  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$  is called a  $\sigma$ -strong network ([8]), if  $\{st(x, \mathcal{P}_n)\}$  is a network at  $x$  in  $X$  for every  $x \in X$ .

(3) A  $\sigma$ -strong network  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$  is called a  $\sigma$ -strong network consisting of (countable) *fcs*-covers (resp. *cs\**-covers) if  $\mathcal{P}_n$  is a (countable) *fcs*-cover (resp. *cs\**-cover) for every  $n \in \mathbb{N}$ .

(4) A  $\sigma$ -strong network  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$  is called a  $\sigma$ -point-countable strong network if  $\mathcal{P}_n$  is point-countable for every  $n \in \mathbb{N}$ .

**Theorem 2.4.** *For a space  $X$ , the following are equivalent.*

- (1)  $X$  is a sequence-covering,  $\pi$ -image of a metric space.
- (2)  $X$  has a  $\sigma$ -strong network consisting of *fcs*-covers.

*Proof.* (1) $\implies$ (2): Let  $M$  be a metric space with a metric  $d$ , and let  $f: M \rightarrow X$  be a sequence-covering,  $\pi$ -mapping. We write  $B(a, \varepsilon) = \{b \in M : d(a, b) < \varepsilon\}$  for every  $a \in M$ , where  $\varepsilon > 0$ . For every  $n \in \mathbb{N}$ , put  $\mathcal{B}_n = \{B(a, 1/n) : a \in M\}$ , and put  $\mathcal{P}_n = f(\mathcal{B}_n)$ , then  $\mathcal{P}_n$  is a cover of  $X$ .

Claim 1.  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$  is a  $\sigma$ -strong network of  $X$ .

It is clear that  $\mathcal{P}_{n+1}$  is a refinement of  $\mathcal{P}_n$  for every  $n \in \mathbb{N}$ . We only need to prove that  $\{st(x, \mathcal{P}_n)\}$  is a network at  $x$  in  $X$  for every  $x \in X$ . Let  $x \in U$  with  $U$  open in  $X$ . Since  $f$  is a  $\pi$ -mapping, there exists  $n \in \mathbb{N}$  such that  $d(f^{-1}(x), M - f^{-1}(U)) > 1/n$ . Pick  $m \in \mathbb{N}$  such that  $m > 2n$ . It suffices to prove that  $st(x, \mathcal{P}_m) \subset U$ . Let  $a \in M$  and let  $x \in f(B(a, 1/m)) \in \mathcal{P}_m$ . We claim that  $B(a, 1/m) \subset f^{-1}(U)$ . In fact, if  $B(a, 1/m) \not\subset f^{-1}(U)$ , then pick  $b \in B(a, 1/m) - f^{-1}(U)$ . Note that  $f^{-1}(x) \cap B(a, 1/m) \neq \emptyset$ , pick  $c \in f^{-1}(x) \cap B(a, 1/m) \neq \emptyset$ , then  $d(f^{-1}(x), M - f^{-1}(U)) \leq d(c, b) \leq d(c, a) + d(a, b) < 2/m < 1/n$ . This is a contradiction. So  $B(a, 1/m) \subset f^{-1}(U)$ , thus  $f(B(a, 1/m)) \subset ff^{-1}(U) = U$ . This proves that  $st(x, \mathcal{P}_m) \subset U$ .

Claim 2.  $\mathcal{P}_n$  is an *fcs*-cover of  $X$  for every  $n \in \mathbb{N}$ .

Let  $n \in \mathbb{N}$ . Suppose  $S$  is a sequence converging to  $x$  in  $X$ . Since  $f$  is sequence-covering, there exists a compact subset  $K$  in  $M$  such that  $f(K) = S \cup \{x\}$ . Note that  $f^{-1}(x) \cap K$  is compact in  $M$ . There exists a finite subset  $M'$  of  $M$  such that  $f^{-1}(x) \cap K \subset \bigcup_{a \in M'} B(a, 1/n)$ . We can assume that  $f^{-1}(x) \cap B(a, 1/n) \neq \emptyset$  for every  $a \in M'$ . Put  $\mathcal{B} = \{B(a, 1/n) : a \in M'\}$  and  $B = \bigcup \mathcal{B}$ , then  $K - B$  is compact in  $M$ . Put  $\mathcal{P}' = \{f(B(a, 1/n)) : a \in M'\}$ . Then  $\mathcal{P}'$  is a finite subfamily of  $(\mathcal{P}_n)_x$ . We prove that  $S$  is eventually in  $\bigcup \mathcal{P}'$  as follows. If not, there exists a subsequence  $\{x_k\}$  of  $S$  converging to  $x$  such that  $x_k \notin \bigcup \mathcal{P}'$  for every  $k \in \mathbb{N}$ . Thus there exists  $a_k \in K - B$  such that  $f(a_k) = x_k$  for every  $k \in \mathbb{N}$ . Since  $K - B$  is compact in  $M$ , there exists a subsequence  $\{a_{k_i}\}$  of  $\{a_k\}$  such that the sequence  $\{a_{k_i}\}$  converges to a point  $a \in K - B$ . Thus  $f(a) \neq x$ . This contradicts the continuity of  $f$ . So  $S$  is eventually in  $\bigcup \mathcal{P}'$ . This proves that  $\mathcal{P}_n$  is an *fcs*-cover of  $X$ .

By the above,  $X$  has a  $\sigma$ -strong network  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$  consisting of *fcs*-covers.

(2) $\implies$ (1): Let  $X$  have a  $\sigma$ -strong network  $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$  consisting of *fcs*-covers. For every  $n \in \mathbb{N}$ , put  $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ , and  $\Lambda_n$  is endowed

with discrete topology. Put

$$M = \{a = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ is a network at some } x_a \text{ in } X\}.$$

Then  $M$ , which is a subspace of the product space  $\prod_{n \in \mathbb{N}} \Lambda_n$ , is a metric space with metric  $d$  described as follows.

Let  $a = (\alpha_n), b = (\beta_n) \in M$ . If  $a = b$ , then  $d(a, b) = 0$ . If  $a \neq b$ , then  $d(a, b) = 1/\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}$ .

Define  $f: M \rightarrow X$  by choosing  $f(a) = x_a$  for every  $a = (\alpha_n) \in M$ , where  $\{P_{\alpha_n}\}$  is a network at  $x_a$  in  $X$ . It is not difficult to check that  $f$  is continuous and onto.

Claim 1.  $f$  is a  $\pi$ -mapping.

Let  $x \in U$  with  $U$  open in  $X$ . Since  $\{\mathcal{P}_n\}$  is a  $\sigma$ -strong network of  $X$ , there exists  $n \in \mathbb{N}$  such that  $st(x, \mathcal{P}_n) \subset U$ . Then  $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$ . In fact, if  $a = (\alpha_n) \in M$  such that  $d(f^{-1}(x), a) < 1/2n$ , then there is  $b = (\beta_n) \in f^{-1}(x)$  such that  $d(a, b) < 1/n$ , so  $\alpha_k = \beta_k$  if  $k \leq n$ . Notice that  $x \in P_{\beta_n} \in \mathcal{P}_n$ ,  $P_{\alpha_n} = P_{\beta_n}$ , so  $f(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_n) \subset U$ , hence  $a \in f^{-1}(U)$ . Thus  $d(f^{-1}(x), a) \geq 1/2n$  if  $a \in M - f^{-1}(U)$ , so  $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$ . This proves that  $f$  is a  $\pi$ -mapping.

Claim 2.  $f$  is a sequence-covering mapping.

Let  $S = \{x_n\}$  be a sequence converging to  $x$  in  $X$ . For every  $n \in \mathbb{N}$ , since  $\mathcal{P}_n$  is an  $fcs$ -cover, there exists a finite subfamily  $\mathcal{F}_n$  of  $(\mathcal{P}_n)_x$  such that  $S$  is eventually in  $\bigcup \mathcal{F}_n$ . Note that  $S - \bigcup \mathcal{F}_n$  is finite. There exists a finite subfamily  $\mathcal{G}_n$  of  $\mathcal{P}_n$  such that  $S - \bigcup \mathcal{F}_n \subset \bigcup \mathcal{G}_n$ . Put  $\mathcal{F}_n \cup \mathcal{G}_n = \{P_{\alpha_n} : \alpha_n \in \Gamma_n\}$ , where  $\Gamma_n$  is a finite subset of  $\Lambda_n$ . For every  $\alpha_n \in \Gamma_n$ , if  $P_{\alpha_n} \in \mathcal{F}_n$ , put  $S_{\alpha_n} = (S \cup \{x\}) \cap P_{\alpha_n}$ , otherwise, put  $S_{\alpha_n} = (S - \bigcup \mathcal{F}_n) \cap P_{\alpha_n}$ . It is easy to see that  $S \cup \{x\} = \bigcup_{\alpha_n \in \Gamma_n} S_{\alpha_n}$  and  $\{S_{\alpha_n} : \alpha_n \in \Gamma_n\}$  is a family of compact subsets of  $X$ . Put  $K = \{(\alpha_n) \in \prod_{n \in \mathbb{N}} \Gamma_n : \bigcap_{n \in \mathbb{N}} S_{\alpha_n} \neq \emptyset\}$ . Then

(i)  $K \subset M$  and  $f(K) \subset S \cup \{x\}$ : Let  $a = (\alpha_n) \in K$ , then  $\bigcap_{n \in \mathbb{N}} S_{\alpha_n} \neq \emptyset$ . Pick  $y \in \bigcap_{n \in \mathbb{N}} S_{\alpha_n}$ , then  $y \in \bigcap_{n \in \mathbb{N}} P_{\alpha_n}$ . Note that  $\{P_{\alpha_n} : n \in \mathbb{N}\}$  is a network at  $y$  in  $X$  if and only if  $y \in \bigcap_{n \in \mathbb{N}} P_{\alpha_n}$ . So  $a \in M$  and  $f(a) = y \in S \cup \{x\}$ . This proves That  $K \subset M$  and  $f(K) \subset S \cup \{x\}$ .

(ii)  $S \cup \{x\} \subset f(K)$ : Let  $y \in S \cup \{x\}$ . For every  $n \in \mathbb{N}$ , pick  $\alpha_n \in \Gamma_n$  such that  $y \in S_{\alpha_n}$ . Put  $a = (\alpha_n)$ , then  $a \in K$  and  $f(a) = y$ . This proves That  $S \cup \{x\} \subset f(K)$ .

(iii)  $K$  is a compact subset of  $M$ : Since  $K \subset M$  and  $\prod_{n \in \mathbb{N}} \Gamma_n$  is a compact subset of  $\prod_{n \in \mathbb{N}} \Lambda_n$ . We only need to prove that  $K$  is a closed subset of  $\prod_{n \in \mathbb{N}} \Gamma_n$ . It is clear that  $K \subset \prod_{n \in \mathbb{N}} \Gamma_n$ . Let  $a = (\alpha_n) \in \prod_{n \in \mathbb{N}} \Gamma_n - K$ . Then  $\bigcap_{n \in \mathbb{N}} S_{\alpha_n} = \emptyset$ . There exists  $n_0 \in \mathbb{N}$  such that  $\bigcap_{n \leq n_0} S_{\alpha_n} = \emptyset$ . Put  $W = \{(\beta_n) \in \prod_{n \in \mathbb{N}} \Gamma_n : \beta_n = \alpha_n \text{ for } n \leq n_0\}$ . Then  $W$  is open in  $\prod_{n \in \mathbb{N}} \Gamma_n$  and  $a \in W$ . It is easy to see that  $W \cap K = \emptyset$ . So  $K$  is a closed subset of  $\prod_{n \in \mathbb{N}} \Gamma_n$ .

By the above (i), (ii) and (iii),  $f$  is a sequence-covering mapping.

By the above,  $X$  is a sequence-covering,  $\pi$ -image of a metric space.  $\square$

**Lemma 2.5.** *Let  $\mathcal{P}$  be a point-countable cover of a space  $X$ . Then  $\mathcal{P}$  is an fcs-cover if and only if  $\mathcal{P}$  is a  $cs^*$ -cover.*

*Proof.* Necessity holds by Definition 2.2. We only need to prove sufficiency.

Let  $\mathcal{P}$  be a point-countable  $cs^*$ -cover of  $X$ . Let  $S = \{x_n\}$  be a sequence converging to  $x$  in  $X$ . Since  $\mathcal{P}$  is point-countable, put  $(\mathcal{P})_x = \{P_n : n \in \mathbb{N}\}$ . Then  $S$  is eventually in  $\bigcup_{n \leq k} P_n$  for some  $k \in \mathbb{N}$ . If not, then for any  $k \in \mathbb{N}$ ,  $S$  is not eventually in  $\bigcup_{n \leq k} P_n$ . So, for every  $k \in \mathbb{N}$ , there exists  $x_{n_k} \in S - \bigcup_{n \leq k} P_n$ . We may assume  $n_1 < n_2 < \dots < n_{k-1} < n_k < n_{k+1} < \dots$ . Put  $S' = \{x_{n_k}\}$ , then  $S'$  is a sequence converging to  $x$  in  $X$ . Since  $\mathcal{P}$  is a  $cs^*$ -cover, there exists  $m \in \mathbb{N}$  and a subsequence  $S''$  of  $S'$  such that  $S''$  is eventually in  $P_m$ . This contradicts the construction of  $S'$ .  $\square$

Recall a mapping  $f: X \rightarrow Y$  is an  $s$ -mapping, if  $f^{-1}(y)$  is a separable subset of  $X$  for every  $y \in Y$ . Combining [13, Theorem 3.3.12] and [19, Lemma 2.2(2)], we have the following corollary.

**Corollary 2.6.** *Let  $X$  be a space. Then the following are equivalent.*

- (1)  $X$  is a sequence-covering,  $s$  and  $\pi$ -image of a metric space.
- (2)  $X$  is a sequentially-quotient,  $s$  and  $\pi$ -image of a metric space.
- (3)  $X$  has a  $\sigma$ -point-countable strong network consisting of fcs-covers.
- (4)  $X$  has a  $\sigma$ -point-countable strong network consisting of  $cs^*$ -covers.

*Proof.* (1)  $\implies$  (2): it is clear.

(2)  $\implies$  (4): It holds by [13, Theorem 3.3.12].

(4)  $\implies$  (1): It holds by [19, Lemma 2.2(2)].

(3)  $\iff$  (4): It holds by Lemma 2.5.  $\square$

### 3. $\pi$ -IMAGES OF SEPARABLE METRIC SPACES

Now we discuss sequence-covering (resp. sequentially-quotient),  $\pi$ -images of separable metric spaces.

**Definition 3.1.** Let  $X$  be a space, and let  $x \in X$ . A subset  $P$  of  $X$  is called a sequential neighborhood of  $x$  ([3]) if every sequence  $\{x_n\}$  converging to  $x$  in  $X$  is eventually in  $P$ .

**Definition 3.2.** Let  $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$  be a cover of a space  $X$ .  $\mathcal{P}$  is called an  $sn$ -network of  $X$  ([14]), if  $\mathcal{P}_x$  satisfies the following (a),(b) and (c) for every  $x \in X$ , where  $\mathcal{P}_x$  is called an  $sn$ -network at  $x$  in  $X$ .

- (a)  $\mathcal{P}_x$  is a network at  $x$  in  $X$ ;
- (b) if  $P_1, P_2 \in \mathcal{P}_x$ , then  $P \subset P_1 \cap P_2$  for some  $P \in \mathcal{P}_x$ ;
- (c) every element of  $\mathcal{P}_x$  is a sequential neighborhood of  $x$ .

*Remark 3.3.* In [12], a sequential neighborhood of  $x$  and an  $sn$ -network is called a sequence barrier at  $x$  and a universal  $cs$ -network respectively.

**Theorem 3.4.** *For a space  $X$ , the following are equivalent.*

- (1)  $X$  is a sequence-covering,  $\pi$ -image of a separable metric space;
- (2)  $X$  is a sequentially-quotient,  $\pi$ -image of a separable metric space;
- (3)  $X$  has a  $\sigma$ -strong network consisting of countable fcs-covers;
- (4)  $X$  has a  $\sigma$ -strong network consisting of countable  $cs^*$ -covers.

*Proof.* The proofs of (1) $\iff$ (3) and (2) $\iff$ (4) are as the proof of Theorem 2.4. (3) $\iff$ (4) from Lemma 2.5.  $\square$

Ge proved that for a regular space  $X$ , conditions in Theorem 3.4 are equivalent to that  $X$  has a countable  $sn$ -network ([4]). The following example shows that "regular" can not be omitted here.

*Example 3.5.* A space with a countable  $sn$ -network is not a sequentially-quotient,  $\pi$ -image of a metric space.

*Proof.* Let  $R$  be the set of all real numbers, and let  $\tau$  be the Euclidean topology on  $R$ . Put  $X = R$  with the topology  $\tau^* = \{\{x\} \cup (D \cap U) : x \in U \in \tau\}$ , where  $D$  is the set of all irrational numbers. That is,  $X$  is the pointed irrational extension of  $R$ . Then  $X$  is Hausdorff, non-regular, and has a countable base ([17, Example 69]), so  $X$  has a countable  $sn$ -network. Lin showed that  $X$  is not a symmetric space ([13, Example 3.13(5)]), so  $X$  is not a quotient,  $\pi$ -image of a metric space ([18]). Note that every sequentially-quotient mapping onto a first countable space is quotient ([1]). Thus  $X$  is not a sequentially-quotient,  $\pi$ -image of a metric space.  $\square$

However, by the proofs of [14, Theorem 4.6 (3) $\implies$ (2)] and [4, Theorem 2.7(3) $\implies$ (1)], we have the following results without requiring the regularity of the spaces involved.

**Proposition 3.6.** *For a space  $X$ , the following are true.*

- (1) If  $X$  is a sequentially-quotient,  $\pi$ -image of a separable metric space, then  $X$  has a countable  $sn$ -network.
- (2) If  $X$  has a countable closed  $sn$ -network, then  $X$  is a compact-covering, compact image of a separable metric space.

The author would like to thank the referees for their valuable amendments and suggestions.

## REFERENCES

- [1] J. R. Boone and F. Siwiec. Sequentially quotient mappings. *Czech. Math. J.*, 26:174–182, 1976.
- [2] D. K. Burke. Cauchy sequences in semimetric spaces. *Proc. Am. Math. Soc.*, 33:161–164, 1972.
- [3] S. Franklin. Spaces in which sequences suffice. *Fundam. Math.*, 57:107–115, 1965.
- [4] Y. Ge. Spaces with countable  $sn$ -networks. *Comment. Math. Univ. Carolinae*, 45(1):169–176, 2004.
- [5] Y. Ge and G. J. On  $\pi$ -images of separable metric spaces. *Mathematical Sciences*, 10:65–71, 2004.

- [6] G. Gruenhagen, E. Michael, and Y. Tanaka. Spaces determined by point-countable covers. *Pac. J. Math.*, 113:303–332, 1984.
- [7] R. Heath. On open mappings and certain spaces satisfying the first countability axiom. *Fundam. Math.*, 57:91–96, 1965.
- [8] Y. Ikeda, C. Liu, and Y. Tanaka. Quotient compact images of metric spaces, and related matters. *Topology Appl.*, 122(1-2):237–252, 2002.
- [9] Y. Kofner. On a new class of spaces and some problems of symmetrizable theory. *Dokl. Akad. Nauk SSSR*, 187:270–273, 1969.
- [10] K. B. Lee. On certain  $g$ -first countable spaces. *Pac. J. Math.*, 65:113–118, 1976.
- [11] Z. Li. A note on  $\aleph$ -spaces and  $g$ -metrizable spaces. *Czech. Math. J.*, 55(3):803–808, 2005.
- [12] S. Lin. A note on the Arens' space and sequential fan. *Topology Appl.*, 81(3):185–196, 1997.
- [13] S. Lin. *Point-countable covers and sequence-covering mappings*. Beijing: Science Press., 2002.
- [14] S. Lin and P. Yan. Sequence-covering maps of metric spaces. *Topology Appl.*, 109(3):301–314, 2001.
- [15] E. Michael.  $\aleph_0$ -spaces. *J. Math. Mech.*, 15:983–1002, 1966.
- [16] V. Ponomarev. Axioms of countability and continuous mapping. *Bull. Pol. Acad. Math.*, 8:127–133, 1960.
- [17] L. A. Steen and J. j. Seebach. *Counterexamples in topology. 2nd ed.* New York - Heidelberg - Berlin: Springer-Verlag. XI, 244 p. , 1978.
- [18] Y. Tanaka. Symmetric spaces,  $g$ -developable spaces and  $g$ -metrizable spaces. *Math. Jap.*, 36(1):71–84, 1991.
- [19] Y. Tanaka and Y. Ge. Around quotient compact images of metric spaces, and symmetric spaces. *Houston J. Math.*, 32(1):99–117, 2006.

*Received September 28, 2004.*

DEPARTMENT OF MATHEMATICS,  
SUZHOU UNIVERSITY,  
SUZHOU 215006,  
P.R.CHINA  
*E-mail address:* geying@pub.sz.jsinfo.net