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THE UNIT GROUP OF FS_3

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Abstract. In this paper we give a complete characterization of the unit group $\mathscr{U}(FS_3)$ of the group algebra FS_3 of the symmetric group S_3 of degree 3 over a finite field F. Moreover, over the prime field $\mathbb{Z}_2$ and $\mathbb{Z}_3$, presentation of the unit groups of group algebras $\mathbb{Z}_2S_3$ and $\mathbb{Z}_3S_3$ in terms of generators and relators have also been obtained.

1. Introduction

Let FG denote the group algebra of a group G over a field F. For a normal subgroup H of G, the natural homomorphism $g \mapsto gH : G \longrightarrow G/H$ can be extended to an F-algebra homomorphism from FG onto $F[G/H]$ defined by $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g gH$. Kernel of this homomorphism, denoted by $\omega(H)$, is an ideal of FG generated by $\{h-1 \mid h \in H\}$. Thus, $FG/\omega(H) \cong F[G/H]$. The augmentation ideal, $\omega(FG)$, of the group algebra FG is defined by

$$\omega(FG) = \left\{ \sum_{g\in G} a_g g \,\middle|\, a_g \in F, \sum_{g\in G} a_g = 0 \right\}.$$

Clearly, $\omega(G) = \omega(FG)$. In general, $\omega(H) = \omega(FH)FG = FG\omega(FH)$. Also $FG/\omega(G) \cong F$ implies that the Jacobson radical $J(FG) \subseteq \omega(FG)$. It is known that, the natural homomorphism $x \mapsto x + J(FG) : FG \longrightarrow FG/J(FG)$ induces an epimorphism: $\mathscr{U}(FG) \longrightarrow \mathscr{U}(FG/J(FG))$ with kernel $1 + J(FG)$ so that $\mathscr{U}(FG)/(1 + J(FG)) \cong \mathscr{U}(FG/J(FG))$.

This is also known that for any prime p and for any positive integer n, there is a monic irreducible polynomial of degree n over $\mathbb{Z}_p$ [7].

Here we shall use the presentation of S_3 as

$$S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau\sigma = \sigma^2\tau \rangle.$$

Thus, the elements of S_3 are $\{1, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$. The alternating group A_3 of degree 3 is given by $A_3 = \{1, \sigma, \sigma^2\}$. The distinct conjugacy classes of S_3

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are $\mathscr{C}_0 = \{1\}, \mathscr{C}_1 = \{\sigma, \sigma^2\}$ and $\mathscr{C}_2 = \{\tau, \sigma\tau, \sigma^2\tau\}$. Hence, $\{\widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}\}$ form a basis of center $\mathcal{Z}(FS_3)$ of FS_3 (cf. Lemma 4.1.1 of [5]), where $\widehat{\mathscr{C}_i}$ denotes the class sum.

We shall use V_1 for the unit subgroup $1 + J(FS_3)$.

The unit group of integral group ring $\mathbb{Z}S_3$ has been studied by Hughes and Pearson [2] and by Allen and Hobby [1]. The unit group has been discussed in terms of the bicyclic units by Jespers and Parmenter [3]. Sharma et al. [6] studied chains of subgroups of the unit group $\mathscr{U}(\mathbb{Z}S_3)$. However, so far it seems the structure of the unit group $\mathscr{U}(FS_3)$, for $\operatorname{char} F = p > 0$ is not known.

This paper gives a complete characterization of the unit group $\mathscr{U}(FS_3)$ over a finite field F. Also we give the presentation of the unit groups of group algebras $\mathbb{Z}_2S_3$ and $\mathbb{Z}_3S_3$ over the prime field $\mathbb{Z}_2$ and $\mathbb{Z}_3$ in terms of generators and relators.

2. The Unit Group of FS_3

In this Section, the following theorems gives a complete structure of the unit group $\mathscr{U}(FS_3)$ over an arbitrary finite field F.

Let $\operatorname{char} F = p$ and $|F| = p^n$.

Theorem 2.1. *If $p = 2$, then $\mathscr{U}(FS_3)/V_1 \cong GL(2, F) \times F^*$ and V_1 is central elementary abelian 2-group of order 2^n, where $GL(2, F)$ denotes the general linear group of degree 2 over F.*

Theorem 2.2. *If $p = 3$ and $\mathcal{Z}(V_1)$ is the center of V_1, then $\mathcal{Z}(V_1)$ and $V_1/\mathcal{Z}(V_1)$ both are elementary abelian 3-groups.*

Theorem 2.3. *If $p > 3$, then*

$$\mathscr{U}(FS_3) \cong GL(2, F) \times F^* \times F^*.$$

Proof of the Theorem 2.1. We define a matrix representation of S_3,

$$\rho : S_3 \longrightarrow \mathbb{M}(2, F) \oplus F$$

by the assignment

$$\sigma \mapsto \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 1 \right)$$

and

$$\tau \mapsto \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

Thus, ρ can be extended to an F-algebra homomorphism

$$\rho^* : FS_3 \longrightarrow \mathbb{M}(2, F) \oplus F.$$

Let $x = \alpha_0 + \alpha_1\sigma + \alpha_2\sigma^2 + \alpha_3\tau + \alpha_4\sigma\tau + \alpha_5\sigma^2\tau \in \operatorname{Ker}\rho^*$, where α_i's $\in F$. Therefore, $\rho^*(x) = 0$ gives the following system of equations:

$$\begin{aligned}\alpha_0 + \alpha_2 + \alpha_3 + \alpha_5 &= 0\\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0\\ \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 &= 0\\ \alpha_0 + \alpha_1 + \alpha_3 + \alpha_5 &= 0\\ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 &= 0\end{aligned}$$

By solving this system of equations we get all α_i's are same. Thus,

$$\operatorname{Ker}\rho^* = \{\alpha(1 + \sigma + \sigma^2 + \tau + \sigma\tau + \sigma^2\tau) \mid \alpha \in F\}.$$

If $\widehat{S_3}$ is the sum of all elements in S_3, then $\widehat{S_3}^2 = 0$, because F is a field of characteristic 2. It follows that $\operatorname{Ker}\rho^*$ is a nilpotent ideal of FS_3. Hence, $\operatorname{Ker}\rho^* \subseteq J(FS_3)$. Since, ρ^* is onto, we have $\rho^*(J(FS_3)) \subseteq J(\mathbb{M}(2, F) \oplus F) = 0$ and hence $J(FS_3) \subseteq \operatorname{Ker}\rho^*$. Hence, $J(FS_3) = \operatorname{Ker}\rho^* = F\widehat{S_3}$ and so $FS_3/J(FS_3) \cong \mathbb{M}(2, F)\oplus F$. It follows that $\mathscr{U}(FS_3)/V_1 \cong \mathscr{U}(FS_3/J(FS_3)) \cong GL(2, F) \times F^*$.

Further, assume $f(X)$ is a monic irreducible polynomial of degree n over the field $\mathbb{Z}_2$. Then $\mathbb{Z}_2[X]/\langle f(X)\rangle \cong F$. Assume ξ is the residue class of X *mod* $\langle f(X)\rangle$. So the structure of V_1 is

$$V_1 = \prod_{i=0}^{n-1}\langle 1 + \xi^i x \mid x = \widehat{S_3}\rangle,$$

a central subgroup of order 2^n. □

Proof of the Theorem 2.2. Since A_3 is a normal subgroup of S_3 and $[S_3 : A_3] = 2$, which is invertible in F, we have $J(FS_3) = J(FA_3)FS_3$ (cf. Lemma 7.2.7 of [5]). Further, since $\operatorname{char} F = 3$ and A_3 is a 3-group, we get $J(FA_3) = \omega(FA_3)$ (cf. Lemma 8.1.17 of [5]). Consequently,

$$J(FS_3) = \omega(FA_3)FS_3 = \omega(A_3).$$

Hence,

$$FS_3/J(FS_3) = FS_3/\omega(A_3) \cong F[S_3/A_3] \cong FC_2 \cong F \oplus F.$$

Thus,

$$\mathscr{U}(FS_3)/V_1 \cong \mathscr{U}(FS_3/J(FS_3)) \cong F^* \times F^*.$$

Now, $V_1 = 1 + J(FS_3) = 1 + \omega(A_3) = 1 + \omega(FA_3)FS_3$ and $\omega(FA_3)^3 = 0$, then $\omega(A_3)^3 = 0$. Thus, every non identity element of V_1 is of order 3. For $\alpha \in F$ and $x = 1 + \sigma + \sigma^2$, let $u_\alpha = 1 + \alpha x$ and $v_\alpha = 1 + \alpha x\tau$. Both u_α and v_α are central elements of FS_3 as well as elements of V_1. Take $U = \{u_\alpha \mid \alpha \in F\}$ and $V = \{v_\alpha \mid \alpha \in F\}$. Since, $u_\alpha u_\beta = u_{\alpha+\beta}$, and $v_\alpha v_\beta = v_{\alpha+\beta}$, it follows that both U and V are central subgroups of V_1. Further, since all the elements in U and V are distinct we have $|U| = |V| = 3^n$. If possible, let $u \in U \cap V$, i.e. $u = u_\alpha = v_\beta$ for some $\alpha, \beta \in F$. Thus, we have $\alpha(1+\sigma+\sigma^2) = \beta(1+\sigma+\sigma^2)\tau$, which implies that $\alpha = \beta = 0$ and so $U \cap V = \{1\}$. Then $U \times V \subseteq \mathcal{Z}(V_1)$, which gives us that $|\mathcal{Z}(V_1)| \geq 3^{2n}$.

Assume $w_\alpha = 1 + \alpha(\sigma - 1)$ and $t_\alpha = 1 + \alpha(\sigma - 1)\tau$ are two noncommuting elements in $V_1 \setminus Z(V_1)$, where

$$w_\alpha^2 = 1 + 2\alpha(\sigma - 1) + \alpha^2(1 + \sigma + \sigma^2) = w_{2\alpha} u_{\alpha^2},$$
$$t_\alpha^2 = 1 + 2\alpha(\sigma - 1)\tau + 2\alpha^2(1 + \sigma + \sigma^2) = t_{2\alpha} u_{2\alpha^2}.$$

It can be verified that $w_\alpha \mathcal{Z}(V_1) w_\beta \mathcal{Z}(V_1) = w_{\alpha+\beta}\mathcal{Z}(V_1)$. Therefore, we get that $\{w_\alpha \mathcal{Z}(V_1) \mid \alpha \in F\}$ is a subgroup of $V_1/\mathcal{Z}(V_1)$. If possible, let $w_\alpha \mathcal{Z}(V_1) = w_\beta \mathcal{Z}(V_1)$. Then $w_\alpha w_\beta^2 \in \mathcal{Z}(V_1)$, i.e. $w_\alpha w_{2\beta} \in \mathcal{Z}(V_1)$. But, $w_\alpha w_{2\beta} = w_{\alpha+2\beta}$ $(mod\ \mathcal{Z}(V_1))$. Hence, $w_\alpha w_\beta^2 \in \mathcal{Z}(V_1)$ implies $\alpha = \beta$. This shows that all the elements in $\{w_\alpha \mid \alpha \in F\}$ $(mod\ \mathcal{Z}(V_1))$ are distinct. Thus, the number of elements in $\{w_\alpha \mathcal{Z}(V_1) \mid \alpha \in F\}$ are 3^n. Also, since $t_\alpha t_\beta = t_{\alpha+\beta} u_{2\alpha\beta}$, by using the similar argument we get $\{t_\alpha \mathcal{Z}(V_1) \mid \alpha \in F\}$ is a subgroup of $V_1/\mathcal{Z}(V_1)$ with order 3^n. Note that $w_\alpha \mathcal{Z}(V_1)$ and $t_\beta \mathcal{Z}(V_1)$ commute with each other.

Since, $\omega(FA_3)$ is F-linear combination of $(\sigma - 1)$ and $(\sigma^2 - 1)$, we have $\omega(A_3)$ is F-linear combination of $(\sigma - 1), (\sigma^2 - 1), (\sigma - 1)\tau$ and $(\sigma^2 - 1)\tau$ so that any element $1 + x$ in V_1, for $x \in \omega(A_3)$, can be written as

$$1 + x = 1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau + \alpha_3(\sigma^2 - 1)\tau,$$

where α_i's $\in F$. Now,

$$\begin{aligned} 1 + \alpha_1(\sigma^2 - 1) &= 1 + 2\alpha_1(\sigma - 1) + \alpha_1(1 + \sigma + \sigma^2) \\ &= (1 + 2\alpha_1(\sigma - 1))(1 + \alpha_1(1 + \sigma + \sigma^2)) \\ &= w_{2\alpha_1} u_{\alpha_1}, \end{aligned}$$

and so,

$$\begin{aligned} &(1 + \alpha_0(\sigma - 1))(1 + \alpha_1(\sigma^2 - 1)) \\ &\qquad = 1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + 2\alpha_0\alpha_1(1 + \sigma + \sigma^2) \\ &\qquad = (1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1)) u_{2\alpha_0\alpha_1}. \end{aligned}$$

Thus, $(1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1)) = w_{\alpha_0} w_{2\alpha_1} u_{\alpha_1} u_{\alpha_0\alpha_1}$. Further,

$$\begin{aligned} &(1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1))(1 + \alpha_2(\sigma - 1)\tau) \\ &\qquad = (1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau) \times \\ &\qquad \times (1 + \alpha_0\alpha_2(1 + \sigma + \sigma^2)\tau)(1 + 2\alpha_1\alpha_2(1 + \sigma + \sigma^2)\tau) \\ &\qquad = (1 + \alpha_0(\sigma - 1) + \alpha_1(\sigma^2 - 1) + \alpha_2(\sigma - 1)\tau) v_{\alpha_0\alpha_2} v_{2\alpha_1\alpha_2}. \end{aligned}$$

Thus, $(1+\alpha_0(\sigma-1)+\alpha_1(\sigma^2-1)+\alpha_2(\sigma-1)\tau) = w_{\alpha_0} w_{2\alpha_1} u_{\alpha_1} u_{\alpha_0\alpha_1} t_{\alpha_2} v_{2\alpha_0\alpha_2} v_{\alpha_1\alpha_2}$. In similar way one can show that any element of V_1 can be expressed as a linear combination of w_α $(mod\ \mathcal{Z}(V_1)), t_\alpha$ $(mod\ \mathcal{Z}(V_1))$, for $\alpha \in F$.

If possible, let $w_\alpha \mathcal{Z}(V_1) = t_\beta \mathcal{Z}(V_1)$ for some $\alpha, \beta \in F$. Then $w_\alpha t_\beta^2 \in \mathcal{Z}(V_1)$, i.e. $w_\alpha t_{2\beta} \in \mathcal{Z}(V_1)$. But,

$$\begin{aligned} w_\alpha t_{2\beta} &= (1+\alpha(\sigma-1))(1+2\beta(\sigma-1)\tau) \\ &= (1+\alpha(\sigma-1)+2\beta(\sigma-1)\tau)\ (mod\ \mathcal{Z}(V_1)) \end{aligned}$$

Then $w_\alpha t_{2\beta} \in \mathcal{Z}(V_1)$ when $\alpha = \beta = 0$. Thus,

$$\{w_\alpha \mathcal{Z}(V_1) \mid \alpha \in F\} \cap \{t_\alpha \mathcal{Z}(V_1) \mid \alpha \in F\} = \mathcal{Z}(V_1).$$

Hence, the order of $V_1/\mathcal{Z}(V_1)$ is 3^{2n}, so that the order of $\mathcal{Z}(V_1)$ is 3^{2n}.

Let $f(X)$ be a monic irreducible polynomial of degree n in $\mathbb{Z}_3[X]$. Therefore, $\mathbb{Z}_3[X]/\langle f(X)\rangle \cong F$. Further, since order of each u_α, v_α is 3, $\mathcal{Z}(V_1)$ is an elementary abelian 3-group and the structure of $\mathcal{Z}(V_1)$ is given as

$$\mathcal{Z}(V_1) = \prod_{i=0}^{n-1} \langle 1+\alpha^i x\rangle \times \prod_{j=0}^{n-1} \langle 1+\alpha^j x\tau\rangle,$$

where α is residue class of X modulo $\langle f(X)\rangle$.

The presentation of $V_1/\mathcal{Z}(V_1)$ is given by

$$V_1/\mathcal{Z}(V_1) = \prod_{i=0}^{n-1} \langle (1+\alpha^i(\sigma-1)\mathcal{Z}(V_1)\rangle \times \prod_{j=0}^{n-1} \langle (1+\alpha^j(\sigma-1)\tau)\mathcal{Z}(V_1)\rangle.$$

□

Proof of the Theorem 2.3. Since $p \nmid |S_3|$, by Maschke's theorem FS_3 is a semi-simple Artinian algebra over F. Then by Wedderburn structure theorem we get

$$FS_3 \cong \bigoplus_{i=1}^{r} \mathbb{M}(n_i, D_i),$$

where D_i's are finite dimensional division algebras over F. Since F is a finite field, D_i's are finite division algebras, and hence they are fields. In this case denote D_i by F_i. Thus,

$$FS_3 \cong \bigoplus_{i=1}^{r} \mathbb{M}(n_i, F_i),$$

where F_i's are finite field extension of F.

Since, $\dim_F(FS_3) = 3$, FS_3 is noncommutative, and not simple, the possible structures of the group algebra FS_3 are given by

$$\begin{aligned} FS_3 &\cong \mathbb{M}(2,F) \oplus F \oplus F \text{ or} \\ FS_3 &\cong \mathbb{M}(2,F) \oplus F_2, \end{aligned}$$

where F_2 is a quadratic extension of F. No other case is possible. Since, if $\mathbb{M}(2, F_2)$ occurs in the right hand side in the place of $\mathbb{M}(2,F)$, but then $\dim_F(\mathbb{M}(2,F_2)) = 8$, a contradiction. Therefore, only $\mathbb{M}(2,F)$ will occur in the right hand side. Since $\dim_F(FS_3) = 6$, we get $\mathbb{M}(2,F)$ to be a direct summand of FS_3 of codimension 2. So only two cases as mentioned above may arise.

We will prove that second case is not possible. If possible, let second case holds. In this case $\mathscr{U}(FS_3) \cong GL(2,F) \times F_2^*$. In F_2^*, there is an element of order $p^{2n}-1$, i.e. there is an element in the center of $\mathscr{U}(FS_3)$ of order $p^{2n}-1$. Now, $\mathcal{Z}(FS_3)$ is F-linear combination of $\widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_1}$ and $\widehat{\mathscr{C}_2}$, so any element $x \in \mathcal{Z}(FS_3)$ can be written as $x = \alpha_0\widehat{\mathscr{C}_0} + \alpha_1\widehat{\mathscr{C}_1} + \alpha_2\widehat{\mathscr{C}_2}$, where $\alpha_i \in F$. Since, $p > 3$, we get either $3|(p^n-1)$ or $3|(p^n+1)$. In both the cases it can be verified that $(\widehat{\mathscr{C}_1})^{p^n} = \widehat{\mathscr{C}_1}$ and $(\widehat{\mathscr{C}_2})^{p^n} = \widehat{\mathscr{C}_2}$. This gives $x^{p^n} = (\alpha_0 + \alpha_1\widehat{\mathscr{C}_1} + \alpha_2\widehat{\mathscr{C}_2})^{p^n} = \alpha_0 + \alpha_1\widehat{\mathscr{C}_1} + \alpha_2\widehat{\mathscr{C}_2} = x$. Hence, $x^{p^n} = x$ for all $x \in \mathcal{Z}(FS_3)$. But then $\mathscr{U}(\mathcal{Z}(FS_3)))$ is a group of exponent (p^n-1), a contradiction. Hence, second case does not arise. Thus,

$$FS_3 \cong \mathbb{M}(2,F) \oplus F \oplus F.$$

Hence,

$$\mathscr{U}(FS_3) \cong GL(2,F) \times F^* \times F^*.$$

□

3. Unit Groups of $\mathbb{Z}_2S_3$ and $\mathbb{Z}_3S_3$

In this section we give presentation of the unit group $\mathscr{U}(\mathbb{Z}_pS_3)$ for the prime field $\mathbb{Z}_p$, when $p = 2, 3$.

Theorem 3.1. *The unit group $\mathscr{U}(\mathbb{Z}_2S_3)$ is isomorphic to D_{12}, the dihedral group of order 12. In particular, if $S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau\sigma = \sigma^2\tau\rangle$ then $\mathscr{U}(\mathbb{Z}_2S_3) = \langle \omega, \tau \mid \omega^6 = \tau^2 = 1, \tau\omega = \omega^5\tau\rangle$, where $\omega = 1 + \sigma^2 + \tau + \sigma\tau + \sigma^2\tau$.*

Proof. Any element of even length in $\mathbb{Z}_2S_3$ cannot be a unit, since any such element belongs to the augmentation ideal $\omega(\mathbb{Z}_2S_3)$. Elements of length 1 are trivial units in $\mathbb{Z}_2S_3$. Let $x = g_1 + g_2 + g_3 \in \mathbb{Z}_2S_3$, be an element of length 3. Then $x = g_1(1 + g_1^{-1}g_2 + g_1^{-1}g_3)$ is a unit if and only if $1 + g_1^{-1}g_2 + g_1^{-1}g_3$ is a unit. Hence, we can assume that any element of length 3 is of the form $x = 1 + g_1 + g_2$ for some non-identity elements $g_1, g_2 \in S_3$. The following two cases arise:

Case 1. Elements g_1 and g_2 commute with each other. First, note that, $x^2 = (1 + g_1 + g_2)^2 = 1 + g_1^2 + g_2^2$. Since σ and σ^2 are the only elements of S_3 which commute each other, we get $x = 1 + g_1 + g_2 = 1 + \sigma + \sigma^2$. Since, x is an idempotent, it can not be a unit.

Case 2. If g_1 and g_2 do not commute with each other, then also x can not be a unit in $\mathbb{Z}_2S_3$. For that, take $g_1, g_2 \in \{\tau, \sigma\tau, \sigma^2\tau\}$, then $x^2 = 1 + g_1g_2 + g_2g_1 = 1 + \sigma + \sigma^2$, an idempotent; hence x^2 and therefore x cannot be a unit. Next, assume $g_1 \in \{\tau, \sigma\tau, \sigma^2\tau\}$ and $g_2 \in \{\sigma, \sigma^2\}$, then $x^2 = g_2^2 + g_1g_2 + g_2g_1 = g_2^2(1 + g_2g_1g_2 + g_2^2g_1)$. If x is a unit then $y = 1 + g_2g_1g_2 + g_2^2g_1$ is also a unit. But, this is not possible, because $g_2g_1g_2$ and $g_2^2g_1 \in \{\tau, \sigma\tau, \sigma^2\tau\}$. Hence, no element of length 3 is a unit.

This leaves only one case to explore, namely the elements of length 5. All elements of length 5 are units. These are given by

$$
\begin{aligned}
u_1 = u_1^{-1} &= 1+\sigma+\sigma^2+\sigma\tau+\sigma^2\tau;\\
u_2 = u_2^{-1} &= 1+\sigma+\sigma^2+\tau+\sigma\tau;\\
u_3 = u_3^{-1} &= 1+\sigma+\sigma^2+\tau+\sigma^2\tau;\\
v = v^{-1} &= \sigma+\sigma^2+\tau+\sigma\tau+\sigma^2\tau \text{ and}\\
w &= 1+\sigma^2+\tau+\sigma\tau+\sigma^2\tau, \text{ with}\\
w^{-1} &= 1+\sigma+\tau+\sigma\tau+\sigma^2\tau;.
\end{aligned}
$$

Hence, the unit group $\mathscr{U}(\mathbb{Z}_2S_3)$ of $\mathbb{Z}_2S_3$ is

$$\mathscr{U}(\mathbb{Z}_2S_3) = \{u_1, u_2, u_3, v, w, w^{-1}, 1, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}.$$

Further, $w^2 = \sigma^2$, $w^3 = \sigma^2 w = v$, $w^4 = \sigma^4 = \sigma$, $w^5 = w\sigma = w^{-1}$, $w^6 = 1$ and $w\tau = u_3, w^3\tau = u_1$ and $w^5\tau = u_2$. We get

$$\mathscr{U}(\mathbb{Z}_2S_3) = \langle w, \tau \mid w^6 = \tau^2 = 1, w\tau = \tau w^5\rangle,$$

which is a dihedral group of order 12. This completes the proof of this theorem. □

Next, we will discuss about the unit group $\mathscr{U}(\mathbb{Z}_3S_3)$ over the prime field $\mathbb{Z}_3$. For the field $\mathbb{Z}_3$, structure of the unit group $\mathscr{U}(\mathbb{Z}_3S_3)$ is given as follows:

Theorem 3.2. *Let $V_1 = 1 + J(\mathbb{Z}_3S_3)$ and let $\mathcal{Z}(V_1)$ denotes the center of V_1. Then*

(i) *both the groups $\mathcal{Z}(V_1)$ and $V_1/\mathcal{Z}(V_1)$ are isomorphic to $C_3 \times C_3$.*
(ii) *the unit group $\mathscr{U}(\mathbb{Z}_3S_3)/V_1$ is isomorphic to $C_2 \times C_2$. In particular, order of $\mathscr{U}(\mathbb{Z}_3S_3)$ is 324.*

The above theorem is direct consequence of the Theorem 2.2.

Now, we give more precise presentations of the unit group $\mathscr{U}(\mathbb{Z}_3S_3)$. In fact, we present all units in their canonical forms.

In Example 8, Kulshammer and Sharma [4] showed that

$$\omega(A_3) = \mathbb{Z}_3u + \mathbb{Z}_3v + \mathbb{Z}_3uv + \mathbb{Z}_3vu$$

for some $u, v \in \mathbb{Z}_3S_3$. Let $u = (\sigma-\sigma^2)(1-\tau)$ and $v = (\sigma-\sigma^2)(1+\tau)$. Thus, $uv = 2(1+\sigma+\sigma^2) + 2(1+\sigma+\sigma^2)\tau$ and $vu = 2(1+\sigma+\sigma^2) + (1+\sigma+\sigma^2)\tau$ and so $\mathbb{Z}_3u + \mathbb{Z}_3v + \mathbb{Z}_3uv + \mathbb{Z}_3vu \subseteq \omega(A_3)$.

Further, $\{(1-\sigma), (1-\sigma^2), (1-\sigma)\tau, (1-\sigma^2)\tau\}$ form a basis of $\omega(A_3)$. One can see that

$$
\begin{aligned}
1-\sigma &= uv+vu-u-v,\\
1-\sigma^2 &= uv+vu+u+v,\\
(1-\sigma)\tau &= uv-vu-v+u,\\
(1-\sigma^2)\tau &= uv-vu+v-u.
\end{aligned}
$$

Thus, any element of $\omega(A_3)$ can be expressed as $\mathbb{Z}_3$-linear combination of u, v, uv and vu. Hence $\omega(A_3) = \mathbb{Z}_3 u + \mathbb{Z}_3 v + \mathbb{Z}_3 uv + \mathbb{Z}_3 vu$.

Since $J(\mathbb{Z}_3 S_3) = \omega(A_3)$, we have

$$V_1 = 1 + J(\mathbb{Z}_3 S_3) = \{1 + \alpha_1 u + \alpha_2 v + \alpha_3 uv + \alpha_4 vu \mid 0 \leq \alpha_i \leq 2\}$$

for $i = 1, 2, 3, 4$. Let

$$x = uv + vu, \ y = uv - vu, \ \omega_1 = 1 + v, \ \omega_2 = 1 + u.$$

Assume $H_1 = \langle 1 + x, 1 + y \rangle$. Now, $1 + x, 1 + y \in \mathcal{Z}(\mathbb{Z}_3 S_3)$ and $u^2 = 0, \ v^2 = 0$ and $uvu = 0$, implies $x^2 = y^2 = 0$. Thus,

$$H_1 = \langle 1 + x \mid (1 + x)^3 = 1 \rangle \times \langle 1 + y \mid (1 + y)^3 = 1 \rangle \subseteq \mathcal{Z}(\mathbb{Z}_3 S_3).$$

Hence, $H_1 \subseteq \mathcal{Z}(V_1)$. For the converse, observe that $uv, vu \in \mathcal{Z}(\mathbb{Z}_3 S_3)$. Therefore, if $z = 1 + \alpha_1 u + \alpha_2 v + \alpha_3 uv + \alpha_4 vu \in \mathcal{Z}(V_1)$, then $\alpha_1 u + \alpha_2 v$ commutes with every element of V_1. In particular, $\alpha_1 u + \alpha_2 v$ commutes with $1 + v$ but, then it commutes with v also. This implies that $\alpha_1 u$ commutes with v. This gives that $\alpha_1 y = \alpha_1(uv - vu) = \alpha_1(uv) - \alpha_1(vu) = (\alpha_1 u)v - v(\alpha_1 u) = (\alpha_1 u)v - (\alpha_1 u)v = 0$. But, then $\alpha_1(1 + y) = \alpha_1$. Since, $(1 + y)$ is a unit, we get $\alpha_1 = 0$. Similarly, we get $\alpha_2 = 0$. Hence, $z = 1 + \alpha_3 uv + \alpha_4 vu$, i.e. $\mathcal{Z}(V_1) = 1 + \mathbb{Z}_3 uv + \mathbb{Z}_3 vu$. Since, $H_1 \subseteq \mathcal{Z}(V_1)$ and $|H_1| = |\mathcal{Z}(V_1)| = 9$ we get

$$\begin{aligned}
\mathcal{Z}(V_1) &= 1 + \mathbb{Z}_3 uv + \mathbb{Z}_3 vu \\
&= \langle 1 + x \mid (1 + x)^3 = 1 \rangle \times \langle 1 + y \mid (1 + y)^3 = 1 \rangle \\
&= \langle 2 + \sigma + \sigma^2 \ \mid \ (2 + \sigma + \sigma^2)^3 = 1 \rangle \times \\
&\times \langle (1 + (1 + \sigma + \sigma^2)\tau \mid (1 + (1 + \sigma + \sigma^2)\tau)^3 = 1 \rangle.
\end{aligned}$$

We have so far got that

$$H_1 = \langle 1 + x \mid (1 + x)^3 = 1 \rangle \times \langle 1 + y \mid (1 + y)^3 = 1 \rangle = \mathcal{Z}(V_1).$$

Next, $\omega_1, \omega_2 \notin \mathcal{Z}(V_1)$ as $\omega_1 \omega_2 \neq \omega_2 \omega_1$. Otherwise,

$$(1 + v)(1 + u) = (1 + u)(1 + v) \Rightarrow uv - vu = y = x\tau = 0.$$

But, then $x = 1 + \sigma + \sigma^2 = 0$, a contradiction. Further, since $v^2 = 0$, $\omega_1^3 = (1 + v)^3 = 1$. Similarly, we get $\omega_2^3 = 1$. Also,

$$(\omega_1, \omega_2) = \omega_1^{-1} \omega_2^{-1} \omega_1 \omega_2 = \omega_1^2 \omega_2^2 \omega_1 \omega_2.$$

Observe that $\omega_1^2 = (1 + v)^2 = 1 + 2v + v^2 = 1 + 2v = 1 - v$. Similarly, $\omega_2^2 = 1 - u$.

So $\omega_1^2 \omega_2^2 = (1 - v)(1 - u) = 1 - u - v + vu$ and

$$\omega_1 \omega_2 = (1 + v)(1 + u) = 1 + u + v + vu$$

and therefore,

$$\begin{aligned}\omega_1^2\omega_2^2\omega_1\omega_2 &= (1-u-v+vu)(1+u+v+vu)\\ &= (1-u-v)(1+u+v)+vu+vu \text{ since } vu\in\mathcal{Z}(\mathbb{Z}_3S_3), u^2=v^2=0\\ &= 1-(u+v)^2+2vu\\ &= 1-(uv+vu)-vu\\ &= 1-2vu-uv\\ &= 1+vu-uv\\ &= 1-y=(1+y)^2.\end{aligned}$$

The equation $(\omega_1,\omega_2)=(1+y)^2\in\mathcal{Z}(V_1)$ implies that $\omega_1\mathcal{Z}(V_1)$ and $\omega_2\mathcal{Z}(V_1)$ commute with each other. Also $(\omega_1\mathcal{Z}(V_1))^3=(\omega_2\mathcal{Z}(V_1))^3=\mathcal{Z}(V_1)$ as

$$\omega_1^3=\omega_2^3=1.$$

Since, $|V_1/\mathcal{Z}(V_1)|=9$, we get $V_1/\mathcal{Z}(V_1)=\langle\omega_1\mathcal{Z}(V_1)\rangle\times\langle\omega_2\mathcal{Z}(V_1)\rangle$. This discussion summarizes the following:

Lemma 3.3. *Let V_1 be $1+J(\mathbb{Z}_3S_3)$ and $\mathcal{Z}(V_1)$ be its center. Then*

(i) $\mathcal{Z}(V_1)=\langle 1+x\rangle\times\langle 1+y\rangle$, *where* $x=1+\sigma+\sigma^2$ *and*

$$y=(1+\sigma+\sigma^2)\tau; (1+x)^3=(1+y)^3=1.$$

(ii) $\mathcal{Z}(V_1)=\{1+\alpha uv+\beta vu \mid \alpha,\beta\in\mathbb{Z}_3\}$, *where* $u=(\sigma-\sigma^2)(1-\tau)$, $v=(\sigma-\sigma^2)(1+\tau)$

(iii) $V_1/\mathcal{Z}(V_1)=\langle\omega_1\mathcal{Z}(V_1)\rangle\times\langle\omega_2\mathcal{Z}(V_1)\rangle$, *where* $\omega_1=1+v,\ \omega_2=1+u$.

This gives

Theorem 3.4. *If* $x=1+\sigma+\sigma^2, y=(1+\sigma+\sigma^2)\tau, u=(\sigma-\sigma^2)(1-\tau)$ *and* $v=(\sigma-\sigma^2)(1+\tau)$, *then*

(i) $V_1=\{1+\alpha_1u+\alpha_2v+\alpha_3uv+\alpha_4vu \mid \alpha_i\in\mathbb{Z}_3 \text{ for } i=1,2,3,4\}$

(ii)

$$\begin{aligned}V_1=\langle 1+x, 1+y, 1+v, 1+u \mid\ & (1+x)^3=(1+y)^3=(1+v)^3=(1+u)^3=1,\\ & (1+u)(1+v)=(1+y)(1+v)(1+u)\\ & \text{and } 1+x,\ 1+y \text{ commute with every generator}\rangle;\end{aligned}$$

(iii) $V_1=\{(1+x)^i(1+y)^j(1+v)^k(1+u)^l \mid 0\le i,j,k,l\le 2\}$;

(iv) $V_1=[H]K$, *the semidirect product of H by K, where*

$$H=\langle 1+x\rangle\times\langle 1+y\rangle\times\langle 1+v\rangle$$

and $K=\langle 1+u\rangle$ *or* $\langle\sigma\rangle$;

(v) $V_1=W\times\langle 1+x\rangle$ *where*

$$W=\langle 1+u,1+v\rangle=[\langle 1+y\rangle\times\langle 1+v\rangle]\langle 1+u\rangle.$$

Proof. Proof of Part (i) directly follows from our earlier discussion. First, we prove part (iv). Observe that $H = \langle 1+x, 1+y, 1+v\rangle = \langle 1+x\rangle \times \langle 1+y\rangle \times \langle 1+v\rangle$ is an abelian subgroup of the form $C_3 \times C_3 \times C_3$ of V_1, because $\langle 1+x\rangle \times \langle 1+y\rangle = H_1 = \mathcal{Z}(V_1)$. It is known that for a finite group G of order $|G|$, if p is the smallest prime such that p divides $|G|$, then a subgroup of index p is normal in G. Hence, $H \trianglelefteq V_1$. Already we have checked that $(1+v)(1+u) \neq (1+u)(1+v)$. Hence $(1+u) \notin H$. Thus, $V_1 = HK$ and $H \cap K = \{1\}$, where $K = \langle 1+u\rangle$. Therefore, $V_1 = [H]\langle 1+u\rangle$, the semi direct product of H and $\langle 1+u\rangle$. Further, observe that

$$\begin{aligned}
&(1+u)(1+v)(1+y)\\
&\qquad = (1+u+v+uv)(1+uv-vu)\\
&\qquad = 1+(u+v+uv)+(uv-vu), \text{ since } u^2 = v^2 = 0 \text{ and } uv \in \mathcal{Z}(\mathbb{Z}_3S_3)\\
&\qquad = 1+(\sigma-\sigma^2)(1-\tau)+(\sigma-\sigma^2)(1+\tau)+2(1+\sigma+\sigma^2)\\
&\qquad = 1+2(\sigma-\sigma^2)+2(1+\sigma+\sigma^2)\\
&\qquad = 1+2(1+2\sigma)\\
&\qquad = \sigma.
\end{aligned}$$

The equation $\sigma = (1+u)(1+v)(1+y)$ gives that $\sigma \in \langle (1+y), (1+v), (1+u)\rangle$.

Also $\sigma(1+v) \neq (1+v)\sigma \Rightarrow \sigma \notin H$. Hence, $\sigma \in [H]\langle 1+u\rangle$. This proves that $[H]\langle\sigma\rangle \subseteq [H]\langle 1+u\rangle$.

For the converse, observe that $(1+u)(1+v) = (1+y)(1+v)(1+u)$.

$$\begin{aligned}
(1+y)(1+v) &= (1+uv-vu)(1+v)\\
&= 1+v+(uv-vu)+(uv-vu)v\\
&= 1+v+uv-vu.
\end{aligned}$$

Hence,

$$\begin{aligned}
(1+y)(1+v)(1+u) &= (1+v+uv-vu)(1+u)\\
&= 1+v+uv-vu+u+vu+(uv-vu)u\\
&= 1+u+v+uv\\
&= (1+u)(1+v).
\end{aligned}$$

Thus,

$$\begin{aligned}
(1+y)(1+v)^2\sigma &= (1+y)(1+v)^2(1+u)(1+v)(1+y)\\
&= (1+y)^2(1+v)^2\{(1+u)(1+v)\}\\
&= (1+y)^2(1+v)^2\{(1+y)(1+v)(1+u)\}\\
&= (1+y)^3(1+v)^3(1+u)\\
&= (1+u).
\end{aligned}$$

The equation $(1+u) = (1+y)(1+v)^2\sigma$ gives that $1+u \in [H]\langle\sigma\rangle$. But, then $[H]\langle 1+u\rangle \subseteq [H]\langle\sigma\rangle$. Hence, $V_1 = [H]\langle 1+u\rangle = [H]\langle\sigma\rangle$. This proves part (iv).

Now, for part (ii), observe that each of $(1+x),(1+y),(1+v),(1+u)$ is a unit of order 3. Also $(1+u)(1+v)=(1+y)(1+v)(1+u)$ and that $(1+x),(1+y)$ commute with each generator. This proves part (ii) as

$$V_1=[H]\langle 1+u\rangle=\langle 1+x,1+y,1+u,1+v\rangle.$$

The canonical form of part (iii) now, follows from part (ii). For the proof of the part (v), observe that $W=\langle 1+u,1+v\rangle$ is a nonabelian normal subgroup of V_1 of order 27. The following relations can be verified:

$$(1+u)^3=(1+v)^3=1 \text{ and } 1+y=((1+u),(1+v))\in\mathcal{Z}(V_1).$$

Hence,

$$W=\langle 1+u,1+v\rangle=\langle 1+u,1+v,1+y\rangle$$

satisfies the following relations:

$$\begin{aligned}(1+u)^3=(1+v)^3&=(1+y)^3=1,\\(1+u)(1+v)&=(1+v)(1+u)(1+y),\\(1+u)(1+y)&=(1+y)(1+u),\\(1+v)(1+y)&=(1+y)(1+v).\end{aligned}$$

It can be easily seen that $W=[\langle 1+v,1+y\rangle]\langle 1+u\rangle$, the semidirect product of $\langle 1+v,1+y\rangle$ by $\langle 1+u\rangle$. Further, $1+x\notin W$ otherwise $1+x\in\mathcal{Z}(W)=\langle 1+y\rangle$, a contradiction. Hence, $V_1=W\times\langle 1+x\rangle$. The proof of the theorem is now complete. □

Further, V_1 is a 3-group, τ and -1 are units in $\mathbb{Z}_3S_3$ of order 2, we get $\tau,-1\notin V_1$. Also, V_1 is a normal subgroup of $\mathscr{U}(\mathbb{Z}_3S_3)$ of index 4 with $\mathscr{U}(\mathbb{Z}_3S_3)/V_1\cong C_2\times C_2$. Hence we can explicitly write all the units as follows:

Theorem 3.5. *The unit group*

$$\begin{aligned}\mathscr{U}(\mathbb{Z}_3S_3)&=[V_1](\langle -1\rangle\times\langle\tau\rangle)=(\pm V_1)\cup(\pm V_1\tau)\\&=\{\pm(1+\alpha_1u+\alpha_2v+\alpha_3uv+\alpha_4vu),\\&\pm(1+\alpha_1'u+\alpha_2'v+\alpha_3'uv+\alpha_4'vu)\tau\mid\alpha_i,\alpha_i'\in\mathbb{Z}_3\}.\end{aligned}$$

We can write a presentation of the unit group as follows:

Theorem 3.6.

$$\mathscr{U}(\mathbb{Z}_3S_3)=\{(1+x)^i(1+y)^j\omega_1^k\omega_2^l(-1)^m\tau^n\mid 0\le i,j,k,l\le 2;0\le m,n\le 1\}.$$

The canonical form obtained here uses 6 generators. Let $u_1=2+u+v+uv+vu$, $u_2=1+u+v+uv+vu$, $u_3=\tau+u+v+uv+vu$, and $u_4=1+u$. They can be re-written as $u_1=-\sigma^2,u_2=-(1+\sigma^2),u_3=1-\sigma^2+\tau,u_4=1+(\sigma-\sigma^2)(1-\tau)$. The following relations can be verified:

$$\begin{aligned}\omega_1=u_1^4u_2^2u_3^4u_4^2,\ \omega_2=u_4,\ 1+x=u_1^2u_2^2,\\(1+y)=u_1^2u_2^2u_3^4,-1=u_1^3,\ \tau=u_2^2u_3u_4.\end{aligned}$$

For example

$$
\begin{aligned}
u_1^4 &= (-\sigma^2)^4 = \sigma^8 = \sigma^2 \\
u_2^2 &= \{-(1+\sigma^2)\}^2 = (1+\sigma^2)^2 = 1+\sigma+2\sigma^2 = 1+\sigma-\sigma^2 \\
u_3^2 &= (1-\sigma^2+\tau)^2 = (1-\sigma^2)^2+\tau^2+(1-\sigma^2)\tau+\tau(1-\sigma^2) \\
&= (1+\sigma^4-2\sigma^2)+\tau^2+(1-\sigma^2)\tau+(1-\sigma)\tau \\
&= (1+\sigma+\sigma^2)+1+(2-\sigma-\sigma^2)\tau \\
&= (2+\sigma+\sigma^2)-(1+\sigma+\sigma^2)\tau \\
&= 1+(1+\sigma+\sigma^2)-(1+\sigma+\sigma^2)\tau \\
&= 1+x-y. \\
u_3^4 &= (1+x-y)^2 = 1+x^2+y^2+2x-2y-2xy \\
&= 1-x+y = 1-x+x\tau \\
&\quad \text{since } x,y\in\mathcal{Z}(\mathcal{Z}_3S_3),\quad x^2=0, y^2=0, \text{ and } y=x\tau \\
&= 1-x(1-\tau) = 1-(1+\sigma+\sigma^2)(1-\tau),
\end{aligned}
$$

Since, $(1-\tau)(\sigma-\sigma^2) = (\sigma-\sigma^2)-(\sigma^2-\sigma)\tau = (\sigma-\sigma^2)(1+\tau)$, we get

$$
\begin{aligned}
u_4^2 &= \{1+(\sigma-\sigma^2)(1-\tau)\}^2 \\
&= 1+2(\sigma-\sigma^2)(1-\tau)+(\sigma-\sigma^2)(1-\tau)(\sigma-\sigma^2)(1-\tau) \\
&= 1+2(\sigma-\sigma^2)(1-\tau) = 1-(\sigma-\sigma^2)(1-\tau)
\end{aligned}
$$

Now,

$$
\begin{aligned}
u_1^4u_2^2 &= \sigma^2(1+\sigma-\sigma^2) = \sigma^2+1-\sigma = 1-\sigma+\sigma^2, \\
u_1^4u_2^2u_3^4 &= (1-\sigma+\sigma^2)\{1-(1+\sigma+\sigma^2)(1-\tau)\} \\
&= (1-\sigma+\sigma^2)-(1-\sigma+\sigma^2)(1+\sigma+\sigma^2)(1-\tau) \\
&= (1-\sigma+\sigma^2)-(1+\sigma+\sigma^2)(1-\tau), \\
u_1^4u_2^2u_3^4u_4^2 &= \{(1-\sigma+\sigma^2)-(1+\sigma+\sigma^2)(1-\tau)\}\{1-(\sigma-\sigma^2)(1-\tau)\} \\
&= (1-\sigma+\sigma^2)-(1-\sigma+\sigma^2)(\sigma-\sigma^2)(1-\tau)-(1+\sigma+\sigma^2)(1-\tau) \\
&\quad +(1+\sigma+\sigma^2)(1-\tau)(\sigma-\sigma^2)(1-\tau).
\end{aligned}
$$

Since, $(1-\tau)(\sigma-\sigma^2) = (\sigma-\sigma^2)(1+\tau)$, we get $(1+\sigma+\sigma^2)(1-\tau)(\sigma-\sigma^2)(1-\tau) = 0$. Further $(1-\sigma+\sigma^2)(\sigma-\sigma^2) = -1+\sigma^2$.

Combining, we get

$$
\begin{aligned}
u_1^4u_2^2u_3^4u_4^2 &= (1-\sigma+\sigma^2)-(-1+\sigma^2)(1-\tau)-(1+\sigma+\sigma^2)(1-\tau) \\
&= (1-\sigma+\sigma^2)-(\sigma-\sigma^2)(1-\tau) \\
&= 1-2(\sigma-\sigma^2)+(\sigma-\sigma^2)\tau \\
&= 1+(\sigma-\sigma^2)+(\sigma-\sigma^2)\tau = 1+(\sigma-\sigma^2)(1+\tau) \\
&= \omega_1.
\end{aligned}
$$

Hence, $u_1^4u_2^2u_3^4u_4^2 = \omega_1$.

This proves the first relation, namely $u_1^4u_2^2u_3^4u_4^2 = \omega_1$. Similarly, other relations can be proved. Hence, $\mathscr{U}(\mathbb{Z}_3S_3) \subseteq \langle u_1, u_2, u_3, u_4\rangle$.

Further the following relations can be shown to hold among $u_i's$:

$$u_1^6 = u_3^6 = u_2^3 = u_4^3 = 1,\ u_1u_2 = u_2u_1,\ u_3u_1 = u_1u_2^2u_3^5,$$
$$u_3u_2 = u_2^2u_3^3,\ u_4u_3 = u_1^2u_2^2u_3^5u_4^2,\ u_4u_1 = u_1^5u_2u_3^2u_4,\ u_4u_2 = u_1^2u_3^4u_4$$

and that u_1^3, u_3^2 commute with each u_i. The group $\langle u_1, u_2, u_3, u_4\rangle$ is obviously contained in $\mathscr{U}(\mathbb{Z}_3S_3)$. We have obtained canonical form presentation of the unit group $\mathscr{U}(\mathbb{Z}_3S_3)$ as follows:

Theorem 3.7. $\mathscr{U}(\mathbb{Z}_3S_3) = \{u_1^iu_2^ju_3^ku_4^l \mid 0 \le i,k \le 5, 0 \le j,l \le 2\}$, *where* $u_1 = -\sigma^2, u_2 = -(1+\sigma^2), u_3 = 1-\sigma^2+\tau, u_4 = 1+(\sigma-\sigma^2)(1-\tau)$ *and they satisfy the following relations:*

$$\begin{aligned}
&u_1^6 = u_3^6 = u_2^3 = u_4^3 = 1,\\
&u_1u_2 = u_2u_1,\ u_3u_1 = u_1u_2^2u_3^5,\\
&u_3u_2 = u_2^2u_3^3,\ u_4u_3 = u_1^2u_2^2u_3^5u_4^2,\\
&u_4u_1 = u_1^5u_2u_3^2u_4,\ u_4u_2 = u_1^2u_3^4u_4\\
&\quad \textit{and} \quad u_1^3, u_3^2 \textit{ commute with each } u_i.
\end{aligned}$$

We can also write a presentation of the unit group in terms of 3- generators as follows:

Theorem 3.8. *The unit group*

$$\begin{aligned}
\mathscr{U}(\mathbb{Z}_3S_3) = \langle v_1, v_2, v_3 \mid\ & v_1^6 = v_2^6 = v_3^3 = 1,\ v_3v_2 = v_1v_2v_1v_3^2,\\
& v_3v_1 = v_2v_1^5v_2^5v_3,\ v_2v_1 = v_1^2v_2v_1^2v_2v_1v_2^{-1}v_1^2,\\
& v_1^3 \textit{ and } v_2^2 \textit{ commute with each } v_i\rangle.
\end{aligned}$$

This can be done by taking $v_1 = u_1, v_2 = u_3, v_3 = u_4$ in the presentation given in the earlier theorem.

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