Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 309–311
www.emis.de/journals
ISSN 1786-0091

ON HADAMARD PROPERTY OF 2-GROUPS WITH SPECIAL CONDITIONS ON NORMAL SUBGROUPS

Z. MOSTAGHIM

Abstract. In this paper we investigate Hadamard property of $2-$ groups satisfy the strong and the weak conditions on normal subgroups. Also, we show some classes of groups are not Hadamard groups.

1. Introduction and Preliminaries

Let G be a finite group of order $4n$ containing a central involution e^*, and T a transversal of G with respect to $< e^* >$. If T and Tr, where r is any element of G outside $< e^* >$, intersect in n elements, then T and G are called an Hadamard subset and an Hadamard group (with respect to $< e^* >$) respectively. The notion of an Hadamard group was introduced for the first time by Ito, [2].

In [1] Fernandez-Alcober and Moreto studied p-groups satisfying what they called the *strong condition* and the *weak condition* on normal subgroups. By [1] G is said to satisfy the strong condition on normal subgroups if, for any $N \lhd G$, either $G' \leq N$ or $N \leq Z(G)$. Similarly, G is said to satisfy the weak condition on normal subgroups when, for any $N \lhd G$, either $G' \leq N$ or $|NZ(G) : Z(G)| \leq p$.

However, we are only interested in the case where $p = 2$, and hence we assume that $p = 2$.

The following Lemmas and Theorems collect some results will be used later.

Lemma 1.1 ([2]). *If G is an Hadamard group of order $2n$ with $n > 2$, then n is a multiple of 4.*

Lemma 1.2 ([2]). *Let G be an Hadamard group of order $2n$ such that $G = N\times < e^* >$, where N is normal subgroup of G of index 2. Then the order of N is a square.*

2000 *Mathematics Subject Classification.* 20D15.

Key words and phrases. Hadamard groups; strong and weak conditions; p-groups.

Lemma 1.3 ([2]). *Let G_i be an Hadamard group of order $2n_i$ with prescribed subset D_i and element $e_i^*(i = 1, 2)$. Then $G = G_1 \times G_2 / < e_1^* e_2^* >$ is an Hadamard group of order $2n_1 n_2$ with prescribed subset $D = D_1 D_2 < e_1^* e_2^* >$ considered as the set of cosets and element $e^* = e_1^* < e_1^* e_2^* >$.*

Lemma 1.4 ([2]). *Let G be an Hadamard group of order $2n = 2^k m$ with m odd and S a Sylow 2-subgroup of G. Then S is not a dihedral or a cyclic group.*

Theorem 1.5 ([1]). *(i) If G satisfies the strong condition on normal subgroups, then it has nilpotency class $c \leq 3$, and if $c = 3$ then $|G : Z(G)| = p^4$.*

(ii) If G satisfies the weak condition on normal subgroups, then it has nilpotency class $c \leq 4$. If $c = 4$ then $|G : Z(G)| = p^4$, whereas for $c = 3$ we have $|G : Z(G)| = p^3, p^4$ or p^6 for odd p and $|G : Z(G)| = 2^3$ or 2^4 when $p = 2$.

Theorem 1.6 ([1]). *Let G be a p-group.*

(i) If G satisfies the strong condition on normal subgroups and has nilpotency class 3, then $|G| \leq p^5$. Furthermore, if $p = 2$ then $|G| = 2^4$, that is, G has maximal class.

(ii) If G satisfies the weak condition on normal subgroups and has nilpotency class 4, then $|G| \leq p^6$. Furthermore, if $p = 2$ then $|G| = 2^5$, that is, G has maximal class.

(iii) If G satisfies the weak condition on normal subgroups, has nilpotency class 3 and $|G : Z(G)| = p^6$, then G has bounded order. In fact, $|G| \leq p^{18}$.

2. Main results

Theorem 2.1. *(a) Let G be an elementary abelian 2-group of non-square order. Then G is an Hadamard group.*

(b) Let G be an elementary abelian 2-group of square order. Then G is not an Hadamard group.

Proof. (a)We show by induction that G is an Hadamard group. It is obvious for elementary abelian 2-groups of orders 2 and 8. We describe D and e^* for this groups in section 3. Now let G be an elementary abelian 2-group of order 2^{2k+1} and G be an Hadamard group, with prescribed subset D_1 and element e_1^*. Also H be an elementary abelian 2-group of order 8, with prescribed subset D_2 and element e_2^*. By Lemma 1.3, $G \times H / < e_1^* e_2^* >$ is an elementary abelian Hadamard group of order 2^{2k+3}. It is easy to see that the isomorphic image of an Hadamard group is an Hadamard group. This completes the proof.

(b) Assume on the contrary that G is an Hadamard group. We can consider $|G| = 2^{2n}$ and $G = N \times < e^* >$, where $N \triangleright G$, $|N| = 2^{2n-1}$. By Lemma 1.2, this is impossible. This shows that G is not an Hadamard group. $\square$

Corollary 2.2. *The elementary abelian 2-group of non-square order is an Hadamard group with respect to every element of order 2.*

Corollary 2.3. *Non-abelian 2-groups of square order that all of subgroups are normal are Hadamard groups.*

Corollary 2.4. *A non-cyclic finite 2-group with exactly one subgroup of order 2 is an Hadamard group.*

By Theorems 1.5 and 1.6 we have

Theorem 2.5. *Let G be a 2-group. (a) If G satisfies the strong condition on normal subgroups and has nilpotency class 3 and G is not a semidihedral group, then G is an Hadamard group.*

(b)If G satisfies the weak condition on normal subgroups and has nilpotency class 4 and G is not a semidihedral group, then G is an Hadamard group.

Now we consider non-abelian finite groups that are not 2-groups and all of proper subgroups are abelian.

Theorem 2.6. *Let G be a non-abelian finite group which is not a 2-group and all of proper subgroups of G are abelian. Then G is not an Hadamard group.*

Proof. It is easy to see that G is a p-group or $|G| = p^a q^b$ where p and q are distinct primes. In the latter, one of Sylow subgroups of G is cyclic and another Sylow subgroup is normal and elementary abelian.

If G be a p-group, it is obvious that G is not an Hadamard group. Let $|G| = p^a q^b$ and $p < q$.

We consider the following cases.

Case 1. If $p \neq 2, q \neq 2$, then by Lemma 1.1 G is not an Hadamard group.

Case 2. Let $p = 2$

(a) If 2-Sylow subgroup is cyclic, then by Lemma 1.4 G is not an Hadamard group.

(b) Let P be 2-Sylow subgroup of G. Then $P \rhd G$ and elementary abelian. Therefore q-Sylow subgroup of G is cyclic. Since all of Sylow subgroups of G is abelian, then $G' \cap Z(G) = 1$. Let $2||Z(G)|$, then $Z(G) \subseteq P$. Since $P \rhd G$, then $[P, G] < P$ and therefore $[P, G] = 1$. So we have $P = Z(G)$. Then for all of q-Sylow subgroups of G, we have $P \leq C(Q)$ where $Q \in syl_q(G)$. This is impossible. So $2 \nmid |Z(G)|$. Then G is not an Hadamard group. $\square$

References

[1] G. A. Fernández-Alcober and A. Moretó. Groups with two extreme character degrees and their normal subgroups. *Trans. Amer. Math. Soc.*, 353(6):2171–2192 (electronic), 2001.

[2] N. Ito. On Hadamard groups. *J. Algebra*, 168(3):981–987, 1994.

Department of Mathematics,
Iran University of Science and Technology,
Tehran, Iran
E-mail address: mostaghim@iust.ac.ir