Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
27 (2011), 1–7
www.emis.de/journals
ISSN 1786-0091

ORE EXTENSIONS OVER NEAR PSEUDO-VALUATION RINGS AND NOETHERIAN RINGS

V. K. BHAT

Abstract. We recall that a ring R is called near pseudo-valuation ring if every minimal prime ideal is a strongly prime ideal.

Let R be a commutative ring, σ an automorphism of R and δ a σ-derivation of R. We recall that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-invariant and δ-invariant ideal I (i.e. $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$) of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided. A ring R is said to be almost δ-divided ring if every minimal prime ideal of R is δ-divided.

Recall that an endomorphism σ of a ring R is called Min.Spec-type if $\sigma(U) \subseteq U$ for all minimal prime ideals U of R and R is a Min.Spec-type ring (if there exists a Min.Spec-type endomorphism of R). With this we prove the following.

Let R be a commutative Noetherian $\mathbb{Q}$-algebra ($\mathbb{Q}$ is the field of rational numbers), σ a Min.Spec-type automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let any strongly prime ideal U of R with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ implies that $U[x;\sigma,\delta]$ is a strongly prime ideal of $R[x;\sigma,\delta]$. Then

(1) R is a near pseudo valuation ring implies that $R[x;\sigma,\delta]$ is a near pseudo valuation ring
(2) R is an almost δ-divided ring if and only if $R[x;\sigma,\delta]$ is an almost δ-divided ring.

1. Introduction

We follow the notation as in Bhat [14]. All rings are associative with identity. Throughout this paper R denotes a commutative ring with identity $1 \neq 0$. The set of prime ideals of R is denoted by $\mathrm{Spec}(R)$, the set of minimal prime ideals of R is denoted by $\mathrm{Min.\,Spec}(R)$, and the set of strongly prime ideals is denoted

2000 *Mathematics Subject Classification.* Primary 16N40; Secondary 16P40, 16S36.

Key words and phrases. Ore extension, Min.Spec-type automorphism, derivation, divided prime, almost divided prime, pseudo-valuation ring, near pseudo-valuation ring.

The author would like to express his sincere thanks to the referee for the remarks and suggestions.

by S. Spec(R). The fields of rational numbers and real numbers are denoted by $\mathbb{Q}$ and $\mathbb{R}$ respectively unless otherwise stated.

We recall that as in Hedstrom and Houston [16], an integral domain R with quotient field F, is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). For a survey article on pseudo-valuation domains, the reader is referred to Badawi [6]

In Badawi, Anderson and Dobbs [8], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way. A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion; i.e. $aP \subseteq bR$ or $bR \subseteq aP$) for all $a, b \in R$. A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. For more details on pseudo-valuation rings, the reader is referred to Badawi [7].

The concept of pseudo-valuation domain is generalized to the context of rings with zero divisors as in [8, 1, 3, 4, 5].

This article concerns the study of skew polynomial rings over PVDs. Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R ($\delta\colon R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$). In case σ is identity, δ is just called a derivation. For example let $R = F[x]$, F a field. Then $\sigma\colon R \to R$ defined by $\sigma(f(x)) = f(0)$ is an endomorphism of R. Also let $K = \mathbb{R} \times \mathbb{R}$. Then $g\colon K \to K$ by $g(a, b) = (b, a)$ is an automorphism of K.

Let σ be an automorphism of a ring R and $\delta\colon R \to R$ any map. Let $\phi\colon R \to M_2(R)$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R if and only if ϕ is a homomorphism.

Also let $R = F[x]$, F a field. Then the usual differential operator $\frac{d}{dx}$ is a derivation of R.

We denote the Ore extension $R[x; \sigma, \delta]$ by $O(R)$. If I is an ideal of R such that I is σ-invariant; i.e. $\sigma(I) \subseteq I$ and I is δ-invariant; i.e. $\delta(I) \subseteq I$, then we denote $I[x; \sigma, \delta]$ by $O(I)$. We would like to mention that $R[x; \sigma, \delta]$ is the usual set of polynomials with coefficients in R, i.e. $\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R\}$ in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$.

In case δ is the zero map, we denote the skew polynomial ring $R[x; \sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) \subseteq I$, we denote $I[x; \sigma]$ by $S(I)$. In case σ is the identity map, we denote the differential operator ring $R[x; \delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x; \delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example see [10, 11, 12, 13, 14, 15].

Near Pseudo-valuation rings. Recall that a ring R is called a near pseudo-valuation ring (NPVR) if each minimal prime ideal P of R is strongly prime (Bhat [13]). For example a reduced ring is NPVR. Here the term near may not be interpreted as near ring (Bell and Mason [9]). We note that a near pseudo-valuation ring (NPVR) is a pseudo-valuation ring (PVR), but the converse is not true. For example a reduced ring is a NPVR, but need not be a PVR.

Divided rings. We recall that a prime ideal P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided (Badawi [2]). It is known (Lemma (1) of Badawi, Anderson and Dobbs [8]) that a pseudo-valuation ring is a divided ring. Recall that a ring R is called an almost divided ring if every minimal prime ideal of R is divided (Bhat [13]).

δ-divided rings. A prime ideal P of R is said to be δ-divided (where δ is a σ-derivation of R) if it is comparable (under inclusion) to every σ-invariant and δ-invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided (Bhat [11]). A ring R is said to be almost δ-divided ring if every minimal prime ideal of R is δ-divided (Bhat [13]). For more details on near pseudo-valuation rings, δ-divided rings and almost δ-divided rings the reader is referred to [11, 13, 14].

The author of this paper has proved the following in [14] concerning strongly prime ideals of Ore extensions.

Theorem B (Bhat [14])**.** *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let δ be a derivation of R. Further let any $U \in \mathrm{S.Spec}(R)$ with $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R)$. Then*

(1) *R is a near pseudo-valuation ring implies that $D(R)$ is a near pseudo-valuation ring*
(2) *R is an almost δ-divided ring if and only if $D(R)$ is an almost δ-divided ring.*

Theorem BB (Bhat [14])**.** *Let R be a Noetherian ring. Let σ be a Min.Spec-type automorphism of R. Further let any $U \in \mathrm{S.Spec}(R)$ with $\sigma(U) = U$ implies that $O(U) \in \mathrm{S.Spec}(O(R)$. Then*

(1) *R is a near pseudo-valuation ring implies that $S(R)$ is a near pseudo-valuation ring*
(2) *R is an almost σ-divided ring if and only if $S(R)$ is an almost σ-divided ring.*

In this paper we generalize the above results of [14] and answer the problem posed in [14].

Theorem A. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ a Min.Spec-type automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let any $U \in \mathrm{S.Spec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R))$. Then*

(1) *R is a near pseudo valuation ring implies that $R[x; \sigma, \delta]$ is a near pseudo valuation ring*
(2) *R is an almost δ-divided ring if and only if $R[x; \sigma, \delta]$ is an almost δ-divided ring.*

This is proved in Theorem (2.5), but before that, we have the following definition.

Definition 1.1 (see [14])**.** Let R be a ring. We say that an endomorphism σ of R is Min.Spec-type if $\sigma(U) \subseteq U$ for all minimal prime ideals U of R. We say that a ring R is Min.Spec-type ring if there exists a Min.Spec-type endomorphism of R.

Example 1.2 (see [14]). Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Let $\sigma \colon R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is a Min.Spec-type endomorphism of R, and therefore, R is a Min.Spec-type ring.

2. Proof of Main Theorem

Theorem 2.1. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be a Min.Spec-type automorphism of R and δ a σ-derivation of R. Then $\delta(U) \subseteq U$ for all $U \in \operatorname{Min.Spec}(R)$.*

Proof. Let $U \in \operatorname{Min.Spec}(R)$. We have $\sigma(U) \subseteq U$. Consider the set

$$T = \{a \in U \mid \text{ such that } \delta^k(a) \in U \text{ for all integers } k \geq 1\}.$$

First of all, we will show that T is an ideal of R. Let $a, b \in T$. Then $\delta^k(a) \in U$ and $\delta^k(b) \in U$ for all integers $k \geq 1$. Now $\delta^k(a-b) = \delta^k(a) - \delta^k(b) \in U$ for all $k \geq 1\}$. Therefore $a - b \in T$. Therefore T is a δ-invariant ideal of R.

We will now show that $T \in \operatorname{Spec}(R)$. Suppose $T \notin \operatorname{Spec}(R)$. Let $a \notin T$, $b \notin T$ be such that $aRb \subseteq T$. Let t, s be least such that $\delta^t(a) \notin U$ and $\delta^s(b) \notin U$. Now there exists $c \in R$ such that $\delta^t(a)c\sigma^t(\delta^s(b)) \notin U$. Let $d = \sigma^{-t}(c)$. Now $\delta^{t+s}(adb) \in U$ as $aRb \subseteq T$. This implies on simplification that

$$\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) + u \in U,$$

where u is sum of terms involving $\delta^l(a)$ or $\delta^m(b)$, where $l < t$ and $m < s$. Therefore by assumption $u \in U$ which implies that $\delta^t(a)\sigma^t(d)\sigma^t(\delta^s(b)) \in U$. This is a contradiction. Therefore, our supposition must be wrong. Hence $T \in \operatorname{Spec}(R)$. Now $T \subseteq U$, so $T = U$ as $U \in \operatorname{Min.Spec}(R)$. Hence, $\delta(U) \subseteq U$. $\square$

Lemma 2.2. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be a Min.Spec-type automorphism of R and δ a σ-derivation of R. Then*

1. *if U is a minimal prime ideal of R, then $O(U)$ is a minimal prime ideal of of $O(R)$ and $O(U) \cap R = U$*
2. *if P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R.*

Proof. (1) Let U be a minimal prime ideal of R. Now $\sigma(U) \subseteq U$ and by Theorem (2.1) $\delta(U) \subseteq U$. Now, on the same lines as in Theorem (2.22) of

Goodearl and Warfield [15] we have $O(U) \in \text{Spec}(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R, a contradiction. Therefore $O(U) \in \text{Min.Spec}(O(R))$. Now it is easy to see that $O(U) \cap R = U$.

(2) We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Now, the proof follows on the same lines as in Theorem (2.22) of Goodearl and Warfield [15] using Lemma (2.1) and Lemma (2.2) of Bhat [11] and Theorem (2.1). □

Theorem 2.3. *Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the ore extension $O(R) = R[x;\sigma,\delta]$ is right/left Noetherian.*

Proof. See Theorem (1.12) of Goodearl and Warfield [15]. □

Remark 2.4. Let σ be an endomorphism of a ring R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma,\delta]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. Also δ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x;\sigma,\delta]$ by $\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i)$.

We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$ for all $a \in R$, then the above does not hold. For example let $f(x) = xa$ and $g(x) = xb$, $a, b \in R$. Then

$$\overline{\delta}(f(x)g(x)) = x^2\{\delta(\sigma(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\},$$

but

$$\overline{\delta}(f(x))\overline{\sigma}(g(x)) + f(x)\overline{\delta}(g(x)) = \\ = x^2\{\sigma(\delta(a))\sigma(b) + \sigma(a)\delta(b)\} + x\{\delta^2(a)\sigma(b) + \delta(a)\sigma(b)\}.$$

We are now in a position to prove Theorem A as follows.

Theorem 2.5. *Let R be a commutative Noetherian $\mathbb{Q}$-algebra, σ a Min.Spec-type automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further let any $U \in \text{S.Spec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ implies that $O(U) \in \text{S.Spec}(O(R)$. Then*

1. *R is a near pseudo valuation ring implies that $R[x;\sigma,\delta]$ is a near pseudo valuation ring*
2. *R is an almost δ-divided ring if and only if $R[x;\sigma,\delta]$ is an almost δ-divided ring.*

Proof. (1) Let R be a near pseudo-valuation ring which is also an algebra over $\mathbb{Q}$. Now $O(R)$ is Noetherian by Theorem (2.3). Let $J \in \text{Min.Spec}(O(R))$. Then by Lemma (2.2) $J \cap R \in \text{Min.Spec}(R)$. Now R is a near pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in \text{S.Spec}(R)$. Also $\delta(J \cap R) \subseteq J \cap R$ by Theorem (2.1). Now Lemma (2.2) implies that $O(J \cap R) = J$, and by hypothesis $O(J \cap R) \in \text{S.Spec}(O(R))$. Therefore, $J \in \text{S.Spec}(O(R))$. Hence $O(R)$ is a near pseudo-valuation ring.

(2) Let R be an almost δ-divided which is also an algebra over $\mathbb{Q}$. Now $O(R)$ is Noetherian by Theorem (2.3). Let $J \in \text{Min.Spec}(O(R))$ and K be an ideal

of $O(R)$ such that $\sigma(K) \subseteq K$ and $\delta(K) \subseteq K$. Note that σ can be extended to an automorphism of $O(R)$ and δ can be extended to a σ-derivation of $O(R)$ by Remark (2.4). Now by Lemma (2.2) $J \cap R \in \mathrm{Min.Spec}(R)$. Now R is an almost δ-divided commutative Noetherian $\mathbb{Q}$-algebra, therefore $J \cap R$ and $K \cap R$ are comparable (under inclusion), say $J \cap R \subseteq K \cap R$. Now $\delta(K \cap R) \subseteq K \cap R$. Therefore, $O(K \cap R)$ is an ideal of $O(R)$ and so $O(J \cap R) \subseteq O(K \cap R)$. This implies that $J \subseteq K$. Hence $O(R)$ is an almost δ-divided ring.

Conversely suppose that $O(R)$ is almost δ-divided. Let $U \in \mathrm{Min.Spec}(R)$ and V be an ideal of R such that $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$. Now by Theorem (2.1) $\delta(U) \subseteq U$, and Lemma (2.2) implies that $O(U) \in \mathrm{Min.Spec}(O(R))$. Now $O(R)$ is an almost δ-divided ring, therefore $O(U)$ and $O(V)$ are comparable (under inclusion), say $O(U) \subseteq O(V)$. Therefore, $O(U) \cap R \subseteq O(V) \cap R$; i.e. $U \subseteq V$. Hence R is an almost δ-divided ring. □

We note that in above Theorem the hypothesis that any $U \in \mathrm{S.Spec}(R)$ with $\delta(U) \subseteq U$ implies that $O(U) \in \mathrm{S.Spec}(O(R))$ can not be deleted as extension of a strongly prime ideal of R need not be a strongly prime ideal of $O(R)$.

Example 2.6 (see [14]). $R = \mathbb{Z}_{(p)}$. This is in fact a discrete valuation domain, and therefore, its maximal ideal $P = pR$ is strongly prime. But, $pR[x]$ is not strongly prime in $R[x]$ because it is not comparable with $xR[x]$ (so the condition of being strongly prime in $R[x]$ fails for $a = 1$ and $b = x$).

Question 2.7. Let R be a NPVR. Let σ be an automorphism of R and δ a σ-derivation of R. Is $O(R) = R[x; \sigma, \delta]$ a NPVR?

References

[1] D. F. Anderson, A. Badawi, and D. E. Dobbs. Pseudo-valuation rings. II. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 3(2):535–545, 2000.

[2] A. Badawi. On divided commutative rings. *Comm. Algebra*, 27(3):1465–1474, 1999.

[3] A. Badawi. On ϕ-pseudo-valuation rings. In *Advances in commutative ring theory (Fez, 1997)*, volume 205 of *Lecture Notes in Pure and Appl. Math.*, pages 101–110. Dekker, New York, 1999.

[4] A. Badawi. On Φ-pseudo-valuation rings. II. *Houston J. Math.*, 26(3):473–480, 2000.

[5] A. Badawi. On ϕ-chained rings and ϕ-pseudo-valuation rings. *Houston J. Math.*, 27(4):725–736, 2001.

[6] A. Badawi. Pseudo-valuation domains: a survey. In *Mathematics & mathematics education (Bethlehem, 2000)*, pages 38–59. World Sci. Publ., River Edge, NJ, 2002.

[7] A. Badawi. On pseudo-almost valuation domains. *Comm. Algebra*, 35(4):1167–1181, 2007.

[8] A. Badawi, D. F. Anderson, and D. E. Dobbs. Pseudo-valuation rings. In *Commutative ring theory (Fès, 1995)*, volume 185 of *Lecture Notes in Pure and Appl. Math.*, pages 57–67. Dekker, New York, 1997.

[9] H. E. Bell and G. Mason. On derivations in near-rings and rings. *Math. J. Okayama Univ.*, 34:135–144 (1994), 1992.

[10] V. K. Bhat. On 2-primal Ore extensions. *Ukr. Mat. Visn.*, 4(2):173–179, 2007.

[11] V. K. Bhat. Polynomial rings over pseudovaluation rings. *Int. J. Math. Math. Sci.*, pages Art. ID 20138, 6, 2007.
[12] V. K. Bhat. Associated prime ideals of skew polynomial rings. *Beiträge Algebra Geom.*, 49(1):277–283, 2008.
[13] V. K. Bhat. On near pseudo-valuation rings and their extensions. *Int. Electron. J. Algebra*, 5:70–77, 2009.
[14] V. K. Bhat. Ore extensions over near pseudo-valuation rings. *Acta Math. Acad. Paedagog. Nyházi. (N.S.)*, 26(1):45–53, 2010.
[15] K. R. Goodearl and R. B. Warfield, Jr. *An introduction to noncommutative Noetherian rings*, volume 16 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1989.
[16] J. R. Hedstrom and E. G. Houston. Pseudo-valuation domains. *Pacific J. Math.*, 75(1):137–147, 1978.

Received May 17, 2010.

School of Mathematics,
Shri Mata Vaishno Devi University,
Sub-Post Office, Katra, Jammu and Kashmir - 182320,
India
E-mail address: vijaykumarbhat2000@yahoo.com