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CONVERGENCE ALMOST EVERYWHERE OF FOURIER SERIES OF THE FUNCTIONS OF BOUNDED VARIATION

LARRY GOGOLADZE AND VAKHTANG TSAGAREISHVILI

Abstract. Let $(\varphi_n(x))$ be an orthonormal system on $[0,1]$ (ONS). If $f(x)$ is a function of bounded variation $(f(x) \in V)$, then the convergence of Fourier series of $f(x)$ generally does not occur (see [2]).

In the present paper we prove that if ONS $(\varphi_n(x))$ satisfies some conditions, then $\lim_{n\to\infty} S_n(f,x)$ exists a.e. on $[0,1]$, where $f(x) \in V$ is an arbitrary function.

1. Some definitions and theorems

Let V be a class of a functions of bounded variation and

$$\|f\|_V = \int_0^1 |f'(x)|\,dx + \sup_{x\in[0,1]} |f(x)|$$

be a norm. The sum

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n(f)\varphi_n(x)$$

is the Fourier series of the function $f(x) \in L(0,1)$.

$$\sigma_n^{\alpha}(f,x) = \frac{1}{A_n^{\alpha}} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k(f;x) \quad (\alpha > 0)$$

is the Césaro mean of the sum $S_n(f;x)$. Let

$$P_n(x) = \sum_{k=2^n}^{2^{n+1}-1} c_k \varphi_k(x), \tag{1.1}$$

where (c_k) are arbitrary numbers, and let

$$A(P_n) = \max_{1\le i<2^n} \left| \int_0^{i/2^n} P_n(x)\,dx \right| = \left| \int_0^{i_n/2^n} P_n(x)\,dx \right|. \tag{1.2}$$

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The following theorems are valid (see [1, pp. 87, 132]).

Theorem A. *If* $(\varphi_n(x))$ *is ONS on* $[0,1]$ *and*

$$\sum_{n=1}^{\infty} c_n^2 \log^2 n < +\infty,$$

then

$$\sum_{n=1}^{\infty} c_n \varphi_n(x) \tag{1.3}$$

converges a.e. on $[0,1]$.

Theorem B. *If* $(\varphi_n(x))$ *is ONS on* $[0,1]$ *and*

$$\sum_{n=1}^{\infty} c_n^2 (\log\log n)^2 < +\infty,$$

then $\lim\limits_{n\to\infty} \sigma_n^{\alpha}(f,x)$ *exists almost everywhere on* $[0,1]$ *for any* $\alpha > 0$.

Lemma 1.1. *If* $\Phi(x) \in L_2(0,1)$ *and* $f(x) \in L_2(0,1)$ *is a function finite in every point of* $[0,1]$, *then*

$$\int_0^1 f(x)\Phi(x)\,dx = \sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right)\int_0^{i/n}\Phi(x)\,dx + \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(f(x) - f\left(\frac{i}{n}\right)\right)\Phi(x)\,dx + f(1)\int_0^1\Phi(x)\,dx. \tag{1.4}$$

Proof. It is evident that

$$\int_0^1 f(x)\Phi(x)\,dx = \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x)\Phi(x)\,dx. \tag{1.5}$$

Applying Abelian transformation we have

$$\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\int_{\frac{i-1}{n}}^{\frac{i}{n}}\Phi(x)\,dx = \sum_{i=1}^{n-1}\left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right)\right)\int_0^{i/n}\Phi(x)\,dx + f(1)\int_0^1\Phi(x)\,dx. \tag{1.6}$$

Combining (1.5) and (1.6) we get (1.4). □

2. The main results

Theorem 2.1. *Let $(\varphi_n(x))$ be ONS on $[0,1]$ satisfying the following conditions:*

1) $\sum\limits_{n=1}^{\infty} \widehat{\varphi}_n^2(1)\log^2 n < +\infty \ \left(\widehat{\varphi}_n(1) = \int_0^1 \varphi_n(x)\,dx\right)$;
2) *for every sequence of polynomials $(P_n(x))$ (see (1.1) and (1.2))*

$$\sum_{n=1}^{\infty} \left(n\, \frac{A(P_n)}{\|P_n\|_{L_2}} \right)^2 < +\infty.$$

(Here and below if $A(P_n) = \|P_n\|_{L_2} = 0$, then we assume that $\frac{A(P_n)}{\|P_n\|_{L_2}} = 0$.)

Then for any function $f(x) \in V$

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f)\log^2 n < +\infty.$$

Proof. Suppose that

$$P_n(x) = \sum_{k=2^n}^{2^{n+1}-1} (\widehat{\varphi}_k(f)\log^2 k)\varphi_k(x),$$

then

$$\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f)\log^2 k = \int_0^1 f(x) \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f)\log^2 k\varphi_k(x)\,dx = \int_0^1 f(x)P_n(x)\,dx. \tag{2.1}$$

Considering equality (1.4) with $f(x) \in V$ and $\Phi(x) = P_n(x)$, we have

$$\int_0^1 f(x)P_n(x)\,dx = \sum_{i=1}^{2^n-1} \left(f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right) \int_0^{\frac{i}{2^n}} P_n(x)\,dx + \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left(f(x) - f\left(\frac{i}{2^n}\right) \right) P_n(x)\,dx + f(1)\int_0^1 P_n(x)\,dx. \tag{2.2}$$

If $f(x) \in V$, then

$$\left| \sum_{i=1}^{2^n-1} \left(f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right) \int_0^{\frac{i}{2^n}} P_n(x)\,dx \right| \le \le \max_{1\le i<2^n} \left| \int_0^{\frac{i}{2^n}} P_n(x)\,dx \right| \sum_{i=1}^{2^n-1} \left| f\left(\frac{i}{2^n}\right) - f\left(\frac{i+1}{2^n}\right) \right| < \|f\|_V \cdot A(P_n) \tag{2.3}$$

$$= \|f\|_V \cdot \frac{A(P_n)}{\|P_n\|_{L_2}} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^4 k \Big)^{\frac{1}{2}}$$
$$\leq 2\|f\|_V \frac{nA(P_n)}{\|P_n\|_{L_2}} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \Big)^{\frac{1}{2}}.$$

On the other hand,

$$\Big| \sum_{i=1}^{2^n} \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} \left(f(x) - f\left(\frac{i}{2^n}\right)\right) P_n(x)\,dx \Big| \leq \tag{2.4}$$
$$\leq \sum_{i=1}^{2^n} \sup_{x\in[\frac{i-1}{2^n},\frac{i}{2^n}]} \left| f(x) - f\left(\frac{i}{2^n}\right)\right| \int_{\frac{i-1}{2^n}}^{\frac{i}{2^n}} |P_n(x)|\,dx$$
$$\leq \Big(\int_0^1 P_n^2(x)\,dx \Big)^{\frac{1}{2}} \sum_{i=1}^{2^n} V_{(i-1)2^{-n}}^{2i-n}(f) \cdot 2^{-\frac{n}{2}}$$
$$\leq \|f\|_V \cdot 2^{-\frac{n}{2}} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^4 k \Big)^{\frac{1}{2}}$$
$$\leq 2\|f\|_V n 2^{-\frac{n}{2}} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \Big)^{\frac{1}{2}}$$

and

$$\Big| \int_0^1 P_n(x)\,dx \Big| = \Big| \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f) \log^2 k \int_0^1 \varphi_k(x)\,dx \Big|$$
$$\leq \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \Big)^{\frac{1}{2}} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \Big)^{\frac{1}{2}}.$$

Using (2.3) and (2.4) from (2.2) we get

$$\Big| \int_0^1 f(x) P_n(x)\,dx \Big| \leq \tag{2.5}$$
$$\leq 2n\|f\|_V \Big(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{n} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \Big)^{\frac{1}{2}} \Big) \times$$
$$\times \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \Big)^{\frac{1}{2}}.$$

In such a way (2.1) implies

$$\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f) \log^2 k \leq 2\|f\|_V n \Big(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{n} \Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1) \log^2 k \Big)^{\frac{1}{2}} \Big) \times$$

$$\times\Big(\sum_{k=2^n}^{2^{n+1}-1}\widehat{\varphi}_k^2(f)\log^2 k\Big)^{\frac{1}{2}}.$$

Hence,

$$\sum_{k=2^n}^{2^{n+1}-1}\widehat{\varphi}_k^2(f)\log^2 k \le 4\|f\|_V^2 n^2\bigg(\frac{A(P_n)}{\|P_n\|_{L_2}}+2^{-\frac{n}{2}}+\frac{1}{n}\Big(\sum_{k=2^n}^{2^{n+1}-1}\widehat{\varphi}_k^2(1)\log^2 k\Big)^{\frac{1}{2}}\bigg)^2.$$

Since

$$\sum_{n=1}^{\infty}n^2 2^{-n}<+\infty,\quad \sum_{n=1}^{\infty}\Big(\frac{nA(P_n)}{\|P_n\|_{L_2}}\Big)^2<+\infty \quad\text{and}\quad \sum_{n=1}^{\infty}\widehat{\varphi}_n^2(1)\log^2 n<+\infty,$$

we have

$$\sum_{n=1}^{\infty}\widehat{\varphi}_n^2(f)\log^2 n<+\infty.$$

Theorem 2.1 is proved. □

Theorem 2.2. *Let $(\varphi_n(x))$ be ONS on $[0,1]$ satisfying the conditions of Theorem 2.1, then the Fourier series of any function $f(x)\in A$ converges almost everywhere on $[0,1]$.*

The validity of Theorem 2.2 follows directly from Theorems A and 2.1.

Theorem 2.3. *Let $(\varphi_n(x))$ be ONS on $[0,1]$ satisfying conditions:*

1) $\sum\limits_{n=1}^{\infty}\widehat{\varphi}_n^2(1)(\log\log n)^2<+\infty,$
2) *for every sequence of polynomials $(P_n(x))$ (see (1.1) and (1.2))*

$$\sum_{n=1}^{\infty}\Big(\frac{A(P_n)}{\|P_n\|_{L_2}}\log n\Big)^2<+\infty.$$

Then for any $f(x)\in V$

$$\sum_{n=1}^{\infty}\widehat{\varphi}_n^2(f)(\log\log n)^2<+\infty.$$

Proof. Suppose that

$$P_n(x)=\sum_{k=2^n}^{2^{n+1}-1}\widehat{\varphi}_k(f)(\log\log k)^2\varphi_k(x).$$

Analogously to Theorem 2.1 we have (see (2.5))

$$\Big|\int_0^1 f(x)P_n(x)\,dx\Big| \le$$

$$\le 2\|f\|_V \log n\bigg(\frac{A(P_n)}{\|P_n\|_{L_2}} + 2^{-\frac{n}{2}} + \frac{1}{\log n}\bigg(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(1)(\log\log k)^2\bigg)^{\frac12}\bigg)\times$$

$$\times\bigg(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f)(\log\log k)^2\bigg)^{\frac12},$$

According to the equality

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f)(\log\log k)^2 &= \int_0^1 f(x) \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k(f)(\log\log k)^2\varphi_k(x)\,dx \\ &= \int_0^1 f(x)P_n(x)\,dx\end{aligned}$$

we conclude that

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f)(\log\log n)^2 < +\infty.$$

Theorem 2.3 is proved. □

Theorem 2.4. *Let $(\varphi_n(x))$ be ONS on $[0,1]$ satisfying conditions of Theorem 2.3. Then for any $f(x)\in A$ and $\alpha>0$*

$$\lim_{n\to\infty} \sigma_n^{\alpha}(f,x)$$

exists a.e. on $[0,1]$.

The validity of Theorem 2.4 follows from Theorems 2.3 and Theorem B.

Theorem 2.5. *Suppose that for some ONS $(\varphi_n(x))$ the following conditions are satisfied:*

1) $\int_0^1 \varphi_n(x)\,dx = 0$, $n=1,2,\dots$;
2) *there exist a sequence of polynomials $(P_n(x))$ (see (1.1) and (1.2)) such that*

$$\overline{\lim_{n\to\infty}} \frac{nA(P_n)}{\|P_n\|_{L_2}} = +\infty.$$

Then there exists a function $f_0(x)\in V$ such that

$$\sum_{n=1}^{\infty} \widehat{\varphi}_k^2(f_0)\log^2 n = +\infty.$$

Proof. Consider the sequence of linear functionals

$$C_n(f) = \frac{n}{\|P_n\|_{L_2}} \int_0^1 f(x)P_n(x)\,dx, \tag{2.6}$$

where $P_n(x) = \sum_{l=2^n}^{2^{n+1}-1} c_k\varphi_k(x)$ and $(c_k) \in \ell_2$ satisfies condition 2) of Theorem 2.5.

Let

$$f_n(x) = \begin{cases} 0 \text{ for } x \in \left[0, \frac{i_n}{2^n}\right], \\ 1 \text{ for } x \in \left[\frac{i_n+1}{2^n}, 1\right], \\ \text{linear and continuous on } \left[\frac{i_n}{2^n}, \frac{i_n+1}{2^n}\right] \end{cases} \tag{2.7}$$

be the sequence of functions from V. Obviously,

$$\|f_n\|_V = 2, \quad n = 1, 2, \dots .$$

Now, consider (2.2) with $f(x) = f_n(x)$. One can find

$$\int_0^1 f_n(x)P_n(x)\,dx = -\int_0^{\frac{i_n}{2^n}} P_n(x)\,dx + \int_{\frac{i_n}{2^n}}^{\frac{i_n+1}{2^n}} \left(f_n(x) - f_n\left(\frac{i_n+1}{2^n}\right)\right) P_n(x)\,dx. \tag{2.8}$$

Since (see (2.7))

$$\left|\int_{\frac{i_n}{2^n}}^{\frac{i_n+1}{2^n}} \left(f_n(x) - f_n\left(\frac{i_n+1}{2^n}\right)\right) P_n(x)\,dx\right| \le \le 2^{-\frac{n}{2}}\left(\int_0^1 P_n^2(x)\,dx\right)^{\frac{1}{2}} = 2^{-\frac{n}{2}}\|P_n\|_{L_2}, \tag{2.9}$$

from (2.8) we get

$$\left|\int_0^1 f_n(x)P_n(x)\,dx\right| \ge A(P_n) - 2^{-\frac{n}{6}}\|P_n\|_{L_2}.$$

Taking into account condition 2) of Theorem 2.5 we obtain

$$\frac{n}{\|P_n\|_{L_2}}\left|\int_0^1 f_n(x)P_n(x)\,dx\right| \ge \frac{nA(P_n)}{\|P_n\|_{L_2}}\,A(P_n) - n2^{-\frac{n}{2}}.$$

Consequently (see (2.6)),

$$\overline{\lim_{n\to\infty}}\,|C_n(f_n)| = +\infty. \tag{2.10}$$

From (2.10) and Banach-Steinhaus theorem it follows that there exists a function $f_0(x) \in A$ such that

$$\overline{\lim_{n\to\infty}}\,|C_n(f_0)| = +\infty. \tag{2.11}$$

Next, by virtue of Hölder's inequality we get

$$\begin{aligned}\Big|\int_0^1 f_0(x)P_n(x)\,dx\Big| &= \Big|\sum_{k=2^n}^{2^{n+1}-1} c_k \int_0^1 f_0(x)\varphi_k(x)\,dx\Big| \\ &= \Big|\sum_{k=2^n}^{2^{n+1}-1} c_k\widehat{\varphi}_k(f_0)\Big| \le \Big(\sum_{k=2^n}^{2^{n+1}-1} c_k^2\Big)^{\frac12}\Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0)\Big)^{\frac12} \\ &\le \frac{\|P_n\|_{L_2}}{n}\Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0)\log^2 k\Big)^{\frac12}.\end{aligned}$$

Therefore,

$$\Big(\sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0)\log^2 k\Big)^{\frac12} \ge \frac{n}{\|P_n\|_{L_2}}\Big|\int_0^1 f_0(x)P_n(x)\,dx\Big| = C_n(f_0).$$

Then from (2.11) we have

$$\overline{\lim_{n\to\infty}} \sum_{k=2^n}^{2^{n+1}-1} \widehat{\varphi}_k^2(f_0)\log^2 k = +\infty.$$

Theorem 2.5 is completely proved. □

Theorem 2.6. *If $(\varphi_n(x))$ is ONS on $[0,1]$ satisfying conditions 1) and 2) of Theorem 2.5, then there exists a function $f_0(x) \in A$ such that*

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(f_0)(\log\log n)^2 = +\infty.$$

Theorem 2.6 is proved analogously to Theorem 2.5.

Remark 2.1. If in Theorem 2.5 we change condition $\int_0^1 \varphi_n(x)dx = 0$ by the inequality $\sum_{n=1}^{\infty} \widehat{\varphi}_n^2(1)\log^2 n < +\infty$, then Theorem 2.5 will also be valid.

3. Efficiency of Theorem 2.1

Consider trigonometric system $(\sqrt{2}\cos 2\pi nx, \sqrt{2}\sin 2\pi nx)$ on $[0,1]$ for every sequences (a_k) and (b_k). Then

$$\begin{aligned}A(P_n) &= \Big|\int_0^{\frac{i_n}{2^n}} \sum_{k=2^n}^{2^{n+1}-1} \sqrt{2}(a_k\sin 2\pi kx + b_k\cos 2\pi kx)\,dx\Big| \\ &\le 2\sum_{k=2^n}^{2^{n+1}-1} \frac{|a_k|+|b_k|}{k} \le 4\Big(\sum_{k=2^n}^{2^{n+1}-1}(a_k^2+b_k^2)\Big)^{\frac12}\Big(\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^2}\Big)^{\frac12} \\ &\le 2^{-\frac{n}{2}+2}\|P_n\|_{L_2}.\end{aligned}$$

Consequently, for the trigonometric system conditions of Theorem 2.1 are valid.

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I. Javakhishvili Tbilisi State University
2 University St.,
Tbilisi 0186,
Georgia
E-mail address: lgogoladze1@hotmail.com
E-mail address: cagare@ymail.com