Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
31 (2015), 61–69
www.emis.de/journals
ISSN 1786-0091

GENERALIZED QUASI-EINSTEIN MANIFOLDS ADMITTING SPECIAL VECTOR FIELDS

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. The purpose of this paper is to study the properties of some special vector fields on generalized quasi-Einstein manifolds. We determine the conditions for a generalized quasi-Einstein manifold admitting special vector fields when the Ricci tensor of the manifold satisfies some conditions.

1. Introduction

A non-flat n-dimensional Riemannian manifold (M, g) $(n > 2)$ is said to be an *Einstein manifold* if the condition

$$S = \frac{r}{n} g \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M, g), respectively. Einstein manifolds play an important role in Riemannian geometry, as well as in general relativity. For this reason, these manifolds have been studied by many authors.

A non-flat n-dimensional Riemannian manifold (M, g) $(n > 2)$ is called a *quasi-Einstein manifold* if its type $(0, 2)$ Ricci tensor S is not identically zero and satisfies the condition

$$S = ag + b\phi \otimes \phi \tag{1.2}$$

where a and b are non-zero real numbers and ϕ is a nowhere vanishing 1-form on M, mentioned as the *associated 1-form.* The unit vector field U, metrically equivalent to ϕ, i.e., specified by

$$g(X, U) = \phi(X) \tag{1.3}$$

for all $X \in \chi(M)$, is called the *generator* of the manifold. This manifold is denoted by $(QE)_n$.

2010 *Mathematics Subject Classification.* 53B20; 53C25.
Key words and phrases. Generalized quasi-Einstein manifold, torse-forming vector field, φ(Ric)-vector field, Codazzi tensor.

Quasi-Einstein manifolds were examined by R. Deszcz et al. in [8]. We note that this term is used also in a different sense, see [2]. The notion was generalized by M. C. Chaki and R. K. Maity [5], replacing the real numbers a, b by non-constant real-valued functions on M and taking U as a unit vector field. Further generalizations are also known: generalized quasi-Einstein manifolds [3, 6], super quasi-Einstein manifolds [4], mixed generalized quasi-Einstein manifolds [1], pseudo generalized quasi-Einstein manifolds [12], and many others.

Following De and Ghosh [6], by a *generalized quasi-Einstein manifold* we mean a non-flat n-dimensional Riemannian manifold (M, g) $(n > 2)$, whose type $(0, 2)$ Ricci tensor S is of the form

$$S = ag + b\phi \otimes \phi + c\psi \otimes \psi, \tag{1.4}$$

where a, b, $c \in C^\infty(M)$ the *associated scalars* with nowhere vanishing b and c; ϕ and ψ are nowhere zero 1-forms on M such that the unit vector fields U and V metrically equivalent to ϕ and ψ, respectively, are orthogonal, i.e., we have

$$g(U, V) = 0, \quad g(U, U) = g(V, V) = 1. \tag{1.5}$$

These vector fields are called the *generators* of the manifold. Using the sharp operator $\sharp$, we can write $U = \phi^\sharp$, $V = \psi^\sharp$. Such a manifold will be denoted by $G(QE)_n$. If $c = 0$, then this manifold reduces to a quasi-Einstein manifold. Examples of generalized quasi-Einstein manifolds can be found in [7, 11].

2. Special Vector Fields On Generalized Quasi-Einstein Manifolds

In this section, we examine special vector fields on $G(QE)_n$ satisfying some conditions upon the Ricci tensor. Throughout in the following, ∇ is the Levi-Civita connection of (M, g).

Let $(E_i)_{i=1}^n$ be an orthonormal frame field on M. Then from (1.4) and (1.5) we obtain

$$r = \sum_{i=1}^{n} S(E_i, E_i) = na + b + c. \tag{2.1}$$

Similarly,

$$S(U, U) = a + b, \tag{2.2}$$

$$S(V, V) = a + c. \tag{2.3}$$

Definition 2.1 ([13])**.** A vector field ξ in a Riemannian manifold M is called *torse-forming* if it satisfies the condition

$$\nabla_X \xi = \rho X + \lambda(X)\xi, \tag{2.4}$$

where $X \in \chi(M)$, λ is a 1-form and ρ is a smooth function on M.

In the local transcription, this reads

$$\nabla_i \xi^h = \rho\, \delta_i^h + \xi^h \lambda_i \tag{2.5}$$

where ξ^h and λ_i are the components of ξ and λ, and δ_i^h is the Kronecker symbol.

A torse-forming vector field ξ is called *recurrent* if $\rho = 0$, and hence

$$\nabla_X \xi = \lambda(X)\xi \tag{2.6}$$

for all $X \in \chi(M)$.

If the 1-form λ in (2.4) is exact, we speak of a *concircular vector field*; if $\lambda = 0$, then ξ is called a *special concircular vector field*. In the latter case, (2.4) reduces to

$$\nabla_X \xi = \rho X, \quad X \in \chi(M). \tag{2.7}$$

A vector field ξ on M is a special concircular vector field if, and only if,

$$\nabla \xi^\flat = \rho g \quad (\rho \in C^\infty(M)), \tag{2.7*}$$

where $\xi^\flat$ is the 1-form metrically equivalent to ξ. Indeed, since ∇ is the Levi-Civita connection, for any $X, Y \in \chi(M)$ we have

$$\begin{aligned}\nabla \xi^\flat(X,Y) &= (\nabla_X \xi^\flat)(Y) = X\xi^\flat(Y) - \xi^\flat(\nabla_X Y)\\ &= Xg(\xi, Y) - g(\xi, \nabla_X Y) = g(\nabla_X \xi, Y),\end{aligned}$$

so the conditions (2.7) and (2.7*) are equivalent.

Definition 2.2. A symmetric tensor field T of type $(0,2)$ on a Riemannian manifold (M,g) is said to be a Codazzi tensor if it satisfies the condition

$$(\nabla_X T)(Y,Z) = (\nabla_Y T)(X,Z); \quad X, Y, Z \in \chi(M). \tag{2.8}$$

Definition 2.3. The Ricci tensor S is called *cyclic parallel* if it satisfies the condition

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{2.9}$$

for any vector fields X, Y, Z on M.

Definition 2.4 ([10])**.** A $\varphi(\mathrm{Ric})$-*vector field* is a vector field φ on a Riemannian manifold (M,g), satisfying the condition

$$\nabla \varphi = \mu \mathrm{Ric} \tag{2.10}$$

where μ is a constant and Ric is the type $(1,1)$ Ricci tensor.

When (M,g) is an Einstein space, the vector field φ is concircular. If $\mu \neq 0$, then we call that the vector field φ is proper $\varphi(\mathrm{Ric})$-vector field. Moreover, when $\mu = 0$, the vector field φ is covariantly constant.

Einstein spaces are characterized by the proportionally of the Ricci tensor S to the metric tensor, so in these spaces the special concircular vector fields can also be defined by $\nabla \xi = \varrho \mathrm{Ric}$. This suggests a general investigation of vector fields satisfying the latter relation and the conditions for their existence in general (i.e. non-Einstein) Riemannian spaces, with the specialization $\varrho = \mu =$ const. φ(Ric)-vector fields are closely related to Ricci flows, introduced by Hamilton [9].

Now, we can state the following theorems and corollaries.

Theorem 2.1. *In a $G(QE)_n$ manifold, both of the generators cannot be torse-forming vector fields.*

Proof. Suppose that the generators U and V of a $G(QE)_n$ manifold corresponding to the 1-forms ϕ and ψ, respectively, are torse-forming vector fields. In this case, using conditions $g(U,U) = 1$, $g(V,V) = 1$ and (2.4), we obtain

$$(\nabla_X \phi)(Y) = \rho(g(X,Y) - \phi(X)\phi(Y)), \tag{2.11}$$

$$(\nabla_X \psi)(Y) = \sigma(g(X,Y) - \psi(X)\psi(Y)), \tag{2.12}$$

where ρ and σ are scalar functions.

Taking the covariant derivative of the equality $g(U,V) = 0$, using (2.11) and (2.12) we get

$$g(\nabla_X U, V) + g(U, \nabla_X V) = \rho\psi(X) + \sigma\phi(X) = 0. \tag{2.13}$$

Putting $X = U$ and $X = V$ in (2.13) and using (1.5), it follows that ρ and σ must be zero, which contradicts to the fact that they are non-zero functions. This concludes the proof. □

Theorem 2.2. *Let the Ricci tensor of a $G(QE)_n$ be a Codazzi tensor, and suppose that one of the generator vector fields $\phi^\sharp$, $\psi^\sharp$ is torse-forming, and the other is not. Then we have*

$$\rho = \frac{1}{b} a_k \phi^k \quad \text{or} \quad \rho = \frac{1}{c} a_k \psi^k,$$

where $a_k := \partial_k a$ (summation convention in force).

Proof. In local coordinates, we have from (2.8)

$$\nabla_k S_{ij} = \nabla_j S_{ik}. \tag{2.14}$$

Taking the covariant derivative of (1.4), we get

$$\begin{aligned} \nabla_k S_{ij} = a_k g_{ij} + b_k \phi_i \phi_j + b(\nabla_k \phi_i)\phi_j + b(\nabla_k \phi_j)\phi_i + c_k \psi_i \psi_j \\ + c(\nabla_k \psi_i)\psi_j + c(\nabla_k \psi_j)\psi_i \end{aligned} \tag{2.15}$$

where a, b, c are the associated scalars of the manifold; $a_k = \partial_k a$, $b_k = \partial_k b$ and $c_k = \partial_k c$.

If $U = \phi^\sharp$ is torse-forming and $V = \psi^\sharp$ is not, then using (1.5) and (2.4) we obtain

$$\nabla_k \phi_i = \rho(g_{ik} - \phi_i \phi_k) \tag{2.16}$$

where ρ is a scalar function.

Using (2.15) and (2.16), we obtain

$$\nabla_k S_{ij} = a_k g_{ij} + b_k \phi_i \phi_j + c_k \psi_i \psi_j + b\rho\phi_i (g_{jk} - \phi_j \phi_k) + b\rho\phi_j (g_{ik} - \phi_i \phi_k) + c(\nabla_k \psi_i)\psi_j + c(\nabla_k \psi_j)\psi_i. \tag{2.17}$$

From (2.14) and (2.17), we get

$$a_k g_{ij} - a_j g_{ik} + b_k \phi_i \phi_j - b_j \phi_i \phi_k + c_k \psi_i \psi_j - c_j \psi_i \psi_k + b\rho(g_{ik}\phi_j - g_{ij}\phi_k) + c(\psi_j \nabla_k \psi_i + \psi_i \nabla_k \psi_j - \psi_k \nabla_j \psi_i - \psi_i \nabla_j \psi_k) = 0. \tag{2.18}$$

Multiplying (2.18) by g^{ij}, and the last equation by ϕ^k, and using (1.5), we find

$$\big((n-1)a_k + c_k\big)\phi^k + [(1-n)b + c]\rho = 0 \tag{2.19}$$

where $k = 1, 2, \ldots, n$.

Moreover, multiplying (2.18) by g^{ij}, and the last equation by ψ^k, and using (1.5), we get

$$\big((n-1)a_k + b_k\big)\psi^k - c\nabla_k \psi^k = 0. \tag{2.20}$$

On the other hand, multiplying (2.18) by $\phi^i\phi^j$ and using (1.5), we obtain

$$a_k + b_k - (a_j\phi^j + b_j\phi^j)\phi_k = 0, \tag{2.21}$$

where $j = 1, 2, \ldots, n$. If we multiply (2.21) by ψ^k, we get

$$(a_k + b_k)\psi^k = 0. \tag{2.22}$$

Similarly, multiplying (2.18) by $\psi^i\psi^j$ and using (1.5), we get

$$a_k + c_k - b\rho\phi_k - c\psi^j \nabla_j \psi_k - (a_j + c_j)\psi^j \psi_k = 0. \tag{2.23}$$

Multiplying (2.23) by ϕ^k and taking into account (1.5), we have

$$(a_k + c_k)\phi^k - (b - c)\rho = 0. \tag{2.24}$$

Combining (2.19) and (2.24), we get

$$\rho = \frac{1}{b} a_k \phi^k. \tag{2.25}$$

The other case ($V = \psi^\sharp$ is torse-forming, $U = \phi^\sharp$ is not) is completely analogous. □

Corollary 2.1. *Let the Ricci tensor of $G(QE)_n$ be a Codazzi tensor. If one of the generators of a $G(QE)_n$ is a torse-forming vector field and the other is not, then the associated scalar a cannot be constant.*

Proof. From (2.25), the proof is immediate. □

Theorem 2.3. *Let the Ricci tensor of $G(QE)_n$ be a Codazzi tensor. Suppose that one of the generator vector fields $\phi^\sharp$ and $\psi^\sharp$ is torse-forming, and the second is not. Then $\phi^\sharp$ (or $\psi^\sharp$) is divergence-free if, and only if, $\phi^\sharp$ (or $\psi^\sharp$) is orthogonal to grad(a) and grad(c) (or grad(a) and grad(b)), where a, b and c are the associated scalars.*

Proof. Suppose that $\phi^\sharp$ is torse-forming and $\psi^\sharp$ is not. Using local coordinates, from (2.20) and (2.22) we obtain

$$\nabla_k \psi^k = \frac{1}{c}(n-2)a_k\psi^k. \tag{2.26}$$

If $\psi^\sharp$ is divergence-free, then from (2.26) and (2.22) we get, $a_k\psi^k{=}0$ and $b_k\psi^k{=}0$. Thus $\psi^\sharp$ is orthogonal to $grad(a)$ and $grad(b)$. The converse is also true. The other case can be treated in the same way. □

Theorem 2.4. *Let the Ricci tensor of $G(QE)_n$ be cyclic parallel. If one the generator vector fields $\phi^\sharp$ and $\psi^\sharp$ is torse-forming and the other is not, then*

$$\rho = \frac{1}{2b}b_k\phi^k \quad \text{or} \quad \rho = \frac{1}{2c}c_k\psi^k, \tag{2.27}$$

where b and c are the associated scalars of the manifold; $b_k := \partial_k b$, $c_k := \partial_k c$.

Proof. Our result is formulated in terms of local coordinates, so our next calculations are also local. Permuting cyclically the indices in (2.17), adding the three equalities obtained, and using (2.9) we find

$$\begin{aligned} &a_k g_{ij} + a_i g_{jk} + a_j g_{ik} + b_k\phi_i\phi_j + b_i\phi_j\phi_k + b_j\phi_i\phi_k \\ &\quad + c_k\psi_i\psi_j + c_i\psi_j\psi_k + c_j\psi_i\psi_k + 2b\rho(\phi_i g_{jk} + \phi_j g_{ik} + \phi_k g_{ij}) \\ &\quad - 6b\rho\phi_i\phi_j\phi_k + c(\psi_j\nabla_k\psi_i + \psi_i\nabla_k\psi_j + \psi_k\nabla_i\psi_j + \psi_j\nabla_i\psi_k \\ &\qquad + \psi_i\nabla_j\psi_k + \psi_k\nabla_j\psi_i) = 0. \end{aligned} \tag{2.28}$$

Multiplying (2.28) by g^{ij} and using (1.5), we obtain

$$\begin{aligned} &(n+2)a_k + b_k + 2b_i\phi^i\phi_k + c_k + 2c_i\psi^i\psi_k \\ &\qquad + 2(n-1)b\rho\phi_k + 2c(\nabla_i\psi^i)\psi_k + 2c\psi^i\nabla_i\psi_k = 0. \end{aligned} \tag{2.29}$$

From this, multiplying (2.29) by ϕ^k and ψ^k, respectively, and using (1.5), we get

$$[(n+2)a_k + 3b_k + c_k]\phi^k + [2(n-1)b - 2c]\rho = 0, \tag{2.30}$$

$$[(n+2)a_k + b_k + 3c_k]\psi^k + 2c\nabla_i\psi^i = 0. \tag{2.31}$$

On the other hand, multiplying (2.28) by $\phi^i\phi^j$, it follows that

$$a_k + b_k + 2(a_i + b_i)\phi^i\phi_k = 0. \tag{2.32}$$

Moreover, multiplying (2.32) by ϕ^k and ψ^k, respectively, and using (1.5), we get

$$(a_k + b_k)\phi^k = 0, \tag{2.33}$$

$$(a_k + b_k)\psi^k = 0. \tag{2.34}$$

Similarly, if we multiply (2.28) by $\psi^i\psi^j$, we get

$$a_k + c_k + 2(a_i + c_i)\psi^i\psi_k + 2b\rho\phi_k + 2c\psi^i\nabla_i\psi_k = 0. \tag{2.35}$$

Again, multiplying (2.35) by ϕ^k and ψ^k, respectively, and using (1.5), we find

$$(a_k + c_k)\phi^k + 2(b - c)\rho = 0, \tag{2.36}$$

$$(a_k + c_k)\psi^k = 0. \tag{2.37}$$

Thus, using (2.30), (2.33) and (2.36), it follows that

$$\rho = \frac{1}{2b} b_k \phi^k,$$

as was to be shown. The other case is completely analogous. □

Corollary 2.2. *If one of the generator vector fields of $G(QE)_n$ is a torse-forming vector field and the Ricci tensor of $G(QE)_n$ is cyclic parallel tensor, then the associated scalars a, b and c cannot be constant at the same time.*

Proof. If we suppose that one of the associated scalars is constant, then from (2.27), (2.33), (2.34) and (2.37) we see that ρ must be zero, which contradicts our assumption. □

Theorem 2.5. *Let the Ricci tensor of $G(QE)_n$ be cyclic parallel. If the generator vector field $\phi^\sharp$ is torse-forming while $\psi^\sharp$ is not, then $\psi^\sharp$ is divergence-free, and vice versa.*

Proof. Using (2.9) and the second Bianchi identity, we find that the scalar curvature is constant. Then, taking the covariant derivative of (2.1), using (2.31) and (2.37) yield the desired relation $\nabla_i\psi^i = 0$. □

Next, we assume that the vector fields generated by the 1-forms ϕ and ψ are $\phi(\mathrm{Ric})$ and $\psi(\mathrm{Ric})$ vector fields, respectively. Then we can state the following theorem.

Theorem 2.6. *If the generator $\phi^\sharp$ (or $\psi^\sharp$) of $G(QE)_n$ is $\phi(\mathrm{Ric})$ (or $\psi(\mathrm{Ric})$) vector field, then this vector field must be covariantly constant.*

Proof. For a $G(QE)_n$, relation (2.15) is valid. If $\phi^\sharp$ is $\phi(\mathrm{Ric})$ vector field, then we have

$$\nabla_j\phi_i = \mu S_{ij}, \tag{2.38}$$

where μ is a constant.

Multiplying (2.38) by ϕ^i and using (1.4) and (1.5) we obtain

$$\mu S_{ij}\phi^i = \mu(a + b)\phi_j = 0. \tag{2.39}$$

Suppose that μ is a non-zero constant. In this case, from (2.39), we find

$$a = -b. \tag{2.40}$$

From (1.4) and (2.40), it is found that

$$S_{ij} = a(g_{ij} - \phi_i\phi_j) + c\psi_i\psi_j. \tag{2.41}$$

Otherwise taking the covariant derivative of the expression $S_{ij}\phi^i$ and using (2.38) and (2.39), we obtain

$$(\nabla_k S_{ij})\phi^i + \mu S_{ij}S^i_k = 0 \tag{2.42}$$

where $S^i_k = g^{im}S_{mk}$.

Multiplying (2.42) by g^{jk}, we get

$$(\nabla_k S^k_i)\phi^i + \mu S_{ij}S^{ij} = 0. \tag{2.43}$$

It was shown, [10], that Riemannian spaces with a ϕ(Ric) vector field of constant length have constant scalar curvature. Since the generator $\phi^\sharp$ is a unit vector field and it is also a ϕ(Ric) vector field, the scalar curvature of the manifold is constant. In this case, using the contracted second Bianchi identity and considering that the scalar curvature of the manifold is constant, it is obtained that

$$\nabla_k S^k_i = \frac{1}{2}\nabla_i r = 0. \tag{2.44}$$

By the aid of (2.43) and (2.44) and supposing that μ is a non-zero constant, we find

$$S_{ij}S^{ij} = 0. \tag{2.45}$$

Using (1.5) and (2.41) in (2.45), it follows that

$$(n-1)a^2 + 2ac + c^2 = 0, \tag{2.46}$$

and so,

$$(n-2)a^2 + (a+c)^2 = 0. \tag{2.47}$$

From (2.47), it is seen that a and $a+c$ must be zero, that is, $a = c = 0$. But, in this case, from (2.41) we get that the Ricci tensor vanishes which is a contradiction to the hypothesis. Thus, the constant μ must be zero which means that the vector field $\phi^\sharp$ is covariantly constant. The other case (assuming that $\psi^\sharp$ is ψ(Ric) vector field) is completely analogous. □

Corollary 2.3. *In a $G(QE)_n$ manifold, the generators cannot be proper ϕ(Ric) and ψ(Ric) vector fields.*

Proof. From the Theorem 2.6., the proof is immediate. □

Acknowledgements

The authors express their sincere thanks to the referees for their valuable suggestions towards the improvement of the presentation.

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Received December 3, 2013.

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