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ON PROPERTIES OF THE INTRINSIC GEOMETRY OF SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD

ANATOLY P. KOPYLOV AND MIKHAIL V. KOROBKOV

Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. Assume that (X, g) is an n-dimensional smooth connected Riemannian manifold without boundary and Y is an n-dimensional compact connected C^0-submanifold in X with nonempty boundary ∂Y ($n \geq 2$). We consider the metric function $\rho_Y(x, y)$ generated by the intrinsic metric of the interior $\operatorname{Int} Y$ of Y in the following natural way: $\rho_Y(x, y) = \liminf_{x'\to x,\, y'\to x;\ x',y'\in \operatorname{Int} Y} \{\inf[l(\gamma_{x',y',\operatorname{Int} Y})]\}$, where $\inf[l(\gamma_{x',y',\operatorname{Int} Y})]$ is the infimum of the lengths of smooth paths joining x' and y' in the interior $\operatorname{Int} Y$ of Y. We study conditions under which ρ_Y is a metric and also the question about the existence of geodesics in the metric ρ_Y and its relationship with the classical intrinsic metric of the hypersurface ∂Y.

Let (X, g) be an n-dimensional smooth connected Riemannian manifold without boundary and let Y be an n-dimensional compact connected C^0-submanifold in X with nonempty boundary ∂Y ($n \geq 2$). A classical object of investigations (see, for example, [1]) is given by the intrinsic metric $\rho_{\partial Y}$ on the hypersurface ∂Y defined for $x, y \in \partial Y$ as the infimum of the lengths of curves $\nu \subset \partial Y$ joining x and y. In the recent decades, an alternative approach arose in the rigidity theory for submanifolds of Riemannian manifolds (see, for instance, the recent articles [2, 3, 4], which also contain a historical survey of works on the topic). In accordance with this approach, the metric on ∂Y

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is induced by the intrinsic metric of the interior $\operatorname{Int} Y$ of the submanifold Y. Namely, suppose that Y satisfies the following condition[1]:

(i) if $x, y \in Y$, then

$$\rho_Y(x,y) = \liminf_{x'\to x,\, y'\to y;\, x',\, y'\in \operatorname{Int} Y} \{\inf [l(\gamma_{x',\, y',\, \operatorname{Int} Y})]\} < \infty, \tag{1}$$

where $\inf [l(\gamma_{x',\, y',\, \operatorname{Int} Y})]$ is the infimum of the lengths $l(\gamma_{x',\, y',\, \operatorname{Int} Y})$ of smooth paths $\gamma_{x',\, y',\, \operatorname{Int} Y}\colon [0,1] \to \operatorname{Int} Y$ joining x' and y' in the interior $\operatorname{Int} Y$ of Y.

Note that the intrinsic metric of convex hypersurfaces in $\mathbb{R}^n$ (i.e., a classical object) is an important particular case of a function ρ_Y. (To verify that, take as Y the complement of the convex hull of the hypersurface.) However, here there appear some new phenomena. The following question is of primary interest in our paper: Is the function ρ_Y defined by (1) a metric on Y? If $n = 2$ then the answer is 'yes' (see Theorem 1 below) and if $n > 2$ then it is 'no' (see Theorem 2). Moreover, we prove that if ρ_Y is a metric (for an arbitrary dimension $n \ge 2$) then any two points $x, y \in Y$ may be joined by a shortest curve (geodesic) whose length in the metric ρ_Y coincides with $\rho_Y(x,y)$ (Theorem 3).

We will begin with the following result.

Theorem 1. *Let $n = 2$. Then, under condition* (i), *ρ_Y is a metric on Y.*

Proof. It suffices to prove that ρ_Y satisfies the triangle inequality. Let A, O, and D be three points on the boundary of Y (note that this case is basic because the other cases are simpler). Consider $\varepsilon > 0$ and assume that $\gamma_{A_\varepsilon O^1_\varepsilon}\colon [0,1] \to \operatorname{Int} Y$ and $\gamma_{O^2_\varepsilon D_\varepsilon}\colon [2,3] \to \operatorname{Int} Y$ are smooth paths with the endpoints $A_\varepsilon = \gamma_{A_\varepsilon O^1_\varepsilon}(0)$, $O^1_\varepsilon = \gamma_{A_\varepsilon O^1_\varepsilon}(1)$ and $D_\varepsilon = \gamma_{O^2_\varepsilon D_\varepsilon}(3)$, $O^2_\varepsilon = \gamma_{O^2_\varepsilon D_\varepsilon}(2)$ satisfying the conditions $\rho_X(A_\varepsilon, A) \le \varepsilon$, $\rho_X(D_\varepsilon, D) \le \varepsilon$, $\rho_X(O^j_\varepsilon, O) \le \varepsilon$ $(j = 1;\ 2)$,

[1]Easy examples show that if X is an n-dimensional connected smooth Riemannian manifold without boundary then an n-dimensional compact connected C^0-submanifold in X with nonempty boundary may fail to satisfy condition (i). For $n = 2$, we have the following counterexample: Let (X, g) be the space $\mathbb{R}^2$ equipped with the Euclidean metric and let Y be a closed Jordan domain in $\mathbb{R}^2$ whose boundary is the union of the singleton $\{0\}$ consisting of the origin 0, the segment $\{(1-t)(e_1 + 2e_2) + t(e_1 + e_2) : 0 \le t \le 1\}$, and the segments of the following four types:

$$\left\{\frac{(1-t)(e_1+e_2)}{n} + \frac{te_1}{n+1} \ :\ 0 \le t \le 1\right\} \quad (n = 1, 2, \dots);$$

$$\left\{\frac{e_1 + (1-t)e_2}{n} \ :\ 0 \le t \le 1\right\} \quad (n = 2, 3, \dots);$$

$$\left\{\frac{(1-t)(e_1+2e_2)}{n} + \frac{2t(2e_1+e_2)}{4n+3} \ :\ 0 \le t \le 1\right\} \quad (n = 1, 2, \dots);$$

$$\left\{\frac{(1-t)(e_1+2e_2)}{n+1} + \frac{2t(2e_1+e_2)}{4n+3} \ :\ 0 \le t \le 1\right\} \quad (n = 1, 2, \dots).$$

Here e_1, e_2 is the canonical basis in $\mathbb{R}^2$. By the construction of Y, we have $\rho_Y(0,\ E) = \infty$ for every $E \in Y \setminus \{0\}$.

$|l(\gamma_{A_\varepsilon O^1_\varepsilon}) - \rho_Y(A,O)| \le \varepsilon$, and $|l(\gamma_{O^2_\varepsilon D_\varepsilon}) - \rho_Y(O,D)| \le \varepsilon$. Let (U,h) be a chart of the manifold X such that U is an open neighborhood of the point O in X, $h(U)$ is the unit disk $B(0,1) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ in $\mathbb{R}^2$, and $h(O) = 0$ ($0 = (0,0)$ is the origin in $\mathbb{R}^2$); moreover, $h\colon U \to h(U)$ is a diffeomorphism having the following property: there exists a chart (Z,ψ) of Y with $\psi(O) = 0$, $A, D \in U \setminus \mathrm{Cl}_X Z$ ($\mathrm{Cl}_X Z$ is the closure of Z in the space (X,g)) and $Z = \widetilde{U} \cap Y$ is the intersection of an open neighborhood $\widetilde{U}$ ($\subset U$) of O in X and Y whose image $\psi(Z)$ under ψ is the half-disk $B_+(0,1) = \{(x_1,x_2) \in B(0,1) : x_1 \ge 0\}$. Suppose that σ_r is an arc of the circle $\partial B(0,r)$ which is a connected component of the set $V \cap \partial B(0,r)$, where $V = h(Z)$ and $0 < r < r^* = \min\{|h(\psi^{-1}(x_1,x_2))| : x_1^2 + x_2^2 = 1/4,\ x_1 \ge 0\}$. Among these components, there is at least one (preserve the notation σ_r for it) whose ends belong to the sets $h(\psi^{-1}(\{-te_2 : 0 < t < 1\}))$ and $h(\psi^{-1}(\{te_2 : 0 < t < 1\}))$ respectively. Otherwise, the closure of the connected component of the set $V \cap B(0,r)$ whose boundary contains the origin would contain a point belonging to the arc $\{e^{i\theta}/2 : |\theta| \le \pi/2\}$ (here we use the complex notation $z = re^{i\theta}$ for points $z \in \mathbb{R}^2$ ($= \mathbb{C}$)). But this is impossible. Therefore, the above-mentioned arc σ_r exists.

It is easy to check that if ε is sufficiently small then the images of the paths $h \circ \gamma_{A_\varepsilon O^1_\varepsilon}$ and $h \circ \gamma_{O^2_\varepsilon D_\varepsilon}$, also intersect σ_r, i.e., there are $t_1 \in]0,1[$, $t_2 \in]2,3[$ such that $\gamma_{A_\varepsilon O^1_\varepsilon}(t_1) = x^1 \in Z$, $\gamma_{O^2_\varepsilon D_\varepsilon}(t_2) = x^2 \in Z$ and $h(x^j) \in \sigma_r$, $j = 1,2$. Let $\widetilde{\gamma}_r\colon [t_1,t_2] \to \sigma_r$ be a smooth parametrization of the corresponding subarc of σ_r, i.e., $\widetilde{\gamma}_r(t_j) = h(x^j)$, $j = 1,2$. Now we can define a mapping $\gamma_\varepsilon\colon [0,3] \to \mathrm{Int}\, Y$ by setting

$$\gamma_\varepsilon(t) = \begin{cases} \gamma_{A_\varepsilon O^1_\varepsilon}(t), & t \in [0,t_1]; \\ h^{-1}(\widetilde{\gamma}_r(t)), & t \in]t_1,t_2[; \\ \gamma_{O^2_\varepsilon D_\varepsilon}(t), & t \in [t_2,3]. \end{cases}$$

By construction, γ_ε is a piecewise smooth path joining the points $A_\varepsilon = \gamma_\varepsilon(0)$, $D_\varepsilon = \gamma_\varepsilon(3)$ in $\mathrm{Int}\, Y$; moreover,

$$l(\gamma_\varepsilon) \le l(\gamma_{A_\varepsilon O^1_\varepsilon}) + l(\gamma_{O^2_\varepsilon D_\varepsilon}) + l(h^{-1}(\sigma_r)).$$

By an appropriate choice of $\varepsilon > 0$, we can make $r > 0$ arbitrarily small, and since a piecewise smooth path can be approximated by smooth paths, we have $\rho_Y(A,D) \le \rho_Y(A,O) + \rho_Y(O,D)$. □

In connection with Theorem 1, there appears a natural question: Are there analogs of this theorem for $n \ge 3$? According to the following Theorem 2, the answer to this question is negative.

Theorem 2. *If $n \ge 3$ then there exists an n-dimensional compact connected C^0-manifold $Y \subset \mathbb{R}^n$ with nonempty boundary ∂Y such that condition (i) (where now $X = \mathbb{R}^n$) is fulfilled for Y but the function ρ_Y in this condition is not a metric on Y.*

Proof. It suffices to consider the case of $n = 3$. Suppose that A, O, D are points in $\mathbb{R}^3$, O is the origin in $\mathbb{R}^3$, $|A| = |D| = 1$, and the angle between the segments OA and OD is equal to $\frac{\pi}{6}$.

The manifold Y will be constructed so that $O \in \partial Y$, and $]O, A] \subset \operatorname{Int} Y$, $]O, D] \subset \operatorname{Int} Y$. Under these conditions, $\rho_Y(O, A) = \rho_Y(O, D) = 1$. However, the boundary of Y will create 'obstacles' between A and D such that the length of any curve joining A and D in $\operatorname{Int} Y$ will be greater than $\frac{12}{5}$ (this means the violation of the triangle inequality for ρ_Y).

Consider a countable collection of mutually disjoint segments $\{I_j^k\}_{j\in\mathbb{N},\, k=1,\dots,k_j}$ lying in the interior of the triangle $6\Delta AOD$ (which is obtained from the original triangle ΔAOD by dilation with coefficient 6) with the following properties:

(*) every segment $I_j^k = [x_j^k, y_j^k]$ lies on a ray starting at the origin, $y_j^k = 11x_j^k$, and $|x_j^k| = 2^{-j}$;

(**) For any curve γ with ends A, D whose interior points lie in the interior of the triangle $4\Delta AOD$ and belong to no segment I_j^k, the estimate $l(\gamma) \geq 6$ holds.

The existence of such a family of segments is certain: they must be situated chequerwise so that any curve disjoint from them be sawtooth, with the total length of its "teeth" greater than 6 (it can clearly be made greater than any prescribed positive number). However, below we exactly describe the construction.

It is easy to include the above-indicated family of segments in the boundary ∂Y of Y. Thus, it creates a desired 'obstacle' to joining A and D in the plane of ΔAOD. But it makes no obstacle to joining A and D in the space. The simplest way to create such a space obstacle is as follows: Rotate each segment I_j^k along a spiral around the axis OA. Make the number of coils so large that the length of this spiral be large and its pitch (i.e., the distance between the origin and the end of a coil) be sufficiently small. Then the set S_j^k obtained as the result of the rotation of the segment I_j^k is diffeomorphic to a plane rectangle, and it lies in a small neighborhood of the cone of revolution with axis AO containing the segment I_j^k. The last circumstance guarantees that the sets S_j^k are disjoint as before, and so (as above) it is easy to include them in the boundary ∂Y but, due to the properties of the I_j^k's and a large number of coils of the spirals S_j^k, any curve joining A, D and disjoint from each S_j^k has length $\geq \frac{12}{5}$.

We turn to an exact description of the constructions used. First describe the construction of the family of segments I_j^k. They are chosen on the basis of the following observation:

Let $\gamma\colon [0,1] \to 4\Delta AOD$ be any curve with ends $\gamma(0) = A$, $\gamma(1) = D$ whose interior points lie in the interior of the triangle $4\Delta AOD$. For $j \in \mathbb{N}$, put $R_j = \{x \in 4\Delta AOD : |x| \in [8 \cdot 2^{-j}, 4 \cdot 2^{-j}]\}$. It is clear that

$$4\Delta AOD \setminus \{O\} = \cup_{j\in\mathbb{N}} R_j.$$

Introduce the polar system of coordinates on the plane of the triangle ΔAOD with center O such that the coordinates of the points A, D are $r = 1$, $\varphi = 0$ and $r = 1$, $\varphi = \frac{\pi}{6}$, respectively. Given a point $x \in 6\Delta AOD$, let φ_x be the angular coordinate of x in $[0, \frac{\pi}{6}]$. Let $\Phi_j = \{\varphi_{\gamma(t)} : \gamma(t) \in R_j\}$. Obviously, there is $j_0 \in \mathbb{N}$ such that

$$\mathcal{H}^1(\Phi_{j_0}) \geq 2^{-j_0}\frac{\pi}{6}, \tag{2}$$

where $\mathcal{H}^1$ is the Hausdorff 1-measure. This means that, while in the layer R_{j_0}, the curve γ covers the angular distance $\geq 2^{-j_0}\frac{\pi}{6}$. The segments I_j^k must be chosen such that (2) together with the condition

$$\gamma(t) \cap I_j^k = \varnothing \quad \forall t \in [0,1]\ \forall j \in \mathbb{N}\ \forall k \in \{1, \dots, k_j\}$$

give the desired estimate $l(\gamma) \geq 6$. To this end, it suffices to take $k_j = [(2\pi)^j]$ (the integral part of $(2\pi)^j$) and

$$I_j^k = \{x \in 6\Delta AOD : \varphi_x = k(2\pi)^{-j}\frac{\pi}{6},\ |x| \in [11 \cdot 2^{-j}, 2^{-j}]\},$$

$k = 1, \dots, k_j$. Indeed, under this choice of the I_j^k's, estimate (2) implies that γ must intersect at least $(2\pi)^{j_0}2^{-j_0} = \pi^{j_0} > 3^{j_0}$ of the figures

$$U_k = \{x \in R_{j_0} : \varphi_x \in (k(2\pi)^{-j_0}\frac{\pi}{6}, (k+1)(2\pi)^{-j_0}\frac{\pi}{6})\}.$$

Since these figures are separated by the segments $I_{j_0}^k$ in the layer R_{j_0}, the curve γ must be disjoint from them each time in passing from one figure to another. The number of these passages must be at least $3^{j_0} - 1$, and a fragment of γ of length at least $2 \cdot 3 \cdot 2^{-j_0}$ is required for each passage (because the ends of the segments $I_{j_0}^k$ go beyond the boundary of the layer R_{j_0} containing the figures U_k at distance $3 \cdot 2^{-j_0}$). Thus, for all these passages, a section of γ is spent of length at least

$$6 \cdot 2^{-j_0}(3^{j_0} - 1) \geq 6.$$

Hence, the construction of the segments I_j^k with the properties $(*)$–$(**)$ is finished.

Let us now describe the construction of the above-mentioned space spirals.

For $x \in \mathbb{R}^3$, denote by Π_x the plane that passes through x and is perpendicular to the segment OA. On $\Pi_{x_j^k}$, consider the polar coordinates (ρ, ψ) with origin at the point of intersection $\Pi_{x_j^k} \cap [O, A]$ (in this system, the point x_j^k has coordinates $\rho = \rho_j^k$, $\psi = 0$). Suppose that a point $x(\psi) \in \Pi_{x_j^k}$ moves along an Archimedean spiral, namely, the polar coordinates of $x(\psi)$ are $\rho(\psi) = \rho_j^k - \varepsilon_j\psi$, $\psi \in [0, 2\pi M_j]$, where ε_j is a small parameter to be specified below, and $M_j \in \mathbb{N}$ is chosen so large that the length of any curve passing between all coils of the spiral is at least 10.

Describe the choice of M_j more exactly. To this end, consider the points $x(2\pi)$, $x(2\pi(M_j - 1))$, $x(2\pi M_j)$, which are the ends of the first, penultimate, and last coils of the spiral respectively (with $x(0) = x_j^k$ taken as the starting point of the spiral). Then M_j is chosen so large that the following condition hold:

($*_1$) The length of any curve on the plane $\Pi_{x_j^k}$, joining the segments $[x_j^k, x(2\pi)]$ and $[x(2\pi(M_j-1)), x(2\pi M_j)]$ and disjoint from the spiral $\{x(\psi) : \psi \in [0, 2\pi M_j]\}$, is at least 10.

Figuratively speaking, the constructed spiral bounds a "labyrinth", the mentioned segments are the entrance to and the exit from this labyrinth, and thus any path through the labyrinth has length ≥ 10.

Now, start rotating the entire segment I_j^k in space along the above-mentioned spiral, i.e., assume that $I_j^k(\psi) = \{y = \lambda x(\psi) : \lambda \in [1, 11]\}$. Thus, the segment $I_j^k(\psi)$ lies on the ray joining O with $x(\psi)$ and has the same length as the original segment $I_j^k = I_j^k(0)$. Define the surface $S_j^k = \cup_{\psi\in[0,2\pi M_j]} I_j^k(\psi)$. This surface is diffeomorphic to a plane rectangle (strip). Taking $\varepsilon_j > 0$ sufficiently small, we may assume without loss of generality that $2\pi M_j \varepsilon_j$ is substantially less than ρ_j^k; moreover, that the surfaces S_j^k are mutually disjoint (obviously, the smallness of ε_j does not affect property ($*_1$) which in fact depends on M_j).

Denote by $y(\psi) = 11x(\psi)$ the second end of the segment $I_j^k(\psi)$. Consider the trapezium P_j^k with vertices y_j^k, x_j^k, $x(2\pi M_j)$, $y(2\pi M_j)$ and sides I_j^k, $I_j^k(2\pi M_j)$, $[x_j^k, x(2\pi M_j)]$, and $[y_j^k, y(2\pi M_j)]$ (the last two sides are parallel since they are perpendicular to the segment AO). By construction, P_j^k lies on the plane AOD; moreover, taking ε_j sufficiently small, we can obtain the situation where the trapeziums P_j^k are mutually disjoint (since $P_j^k \to I_j^k$ under fixed M_j and $\varepsilon_j \to 0$). Take an arbitrary triangle whose vertices lie on P_j^k and such that one of these vertices is also a vertex at an acute angle in P_j^k. By construction, this acute angle is at least $\frac{\pi}{2} - \angle AOD = \frac{\pi}{3}$. Therefore, the ratio of the side of the triangle lying inside the trapezium P_j^k to the sum of the other two sides (lying on the corresponding sides of P_j^k) is at least $\frac{1}{2}\sin\frac{\pi}{3} > \frac{2}{5}$. If we consider the same ratio for the case of a triangle with a vertex at an obtuse angle of P_j^k then it is greater than $\frac{1}{2}$. Thus, we have the following property:

($*_2$) For arbitrary triangle whose vertices lie on the trapezium P_j^k and one of these vertices is also a vertex in P_j^k, the sum of lengths of the sides situated on the corresponding sides of P_j^k is less than $\frac{5}{2}$ of the length of the third side (lying inside P_j^k).

Let a point x lie inside the cone K formed by the rotation of the angle $\angle AOD$ around the ray OA. Denote by $\mathrm{Proj}_{\mathrm{rot}}\, x$ the point of the angle $\angle AOD$ which is the image of x under this rotation. Finally, let $K_{4\Delta AOD}$ stand for the corresponding truncated cone obtained by the rotation of the triangle $4\Delta AOD$, i.e., $K_{4\Delta AOD} = \{x \in K : \mathrm{Proj}_{\mathrm{rot}}\, x \in 4\Delta AOD\}$.

The key ingredient in the proof of our theorem is the following assertion:

($*_3$) For arbitrary space curve γ of length less than 10 joining the points A and D, contained in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$, and disjoint from each strip S_j^k, there exists a plane curve $\tilde{\gamma}$ contained in the triangle $4\Delta AOD \setminus \{O\}$, that joins A and D, is disjoint from all segments I_j^k and such that the length of $\tilde{\gamma}$ is less than $\frac{5}{2}$ of the length of $\mathrm{Proj_{rot}}\,\gamma$.

Prove $(*_3)$. Suppose that its hypotheses are fulfilled. In particular, assume that the inclusion $\mathrm{Proj_{rot}}\,\gamma \subset 4\Delta AOD\setminus\{O\}$ holds. We need to modify $\mathrm{Proj_{rot}}\,\gamma$ so that the new curve be contained in the same set but be disjoint from each of the I_j^k's. The construction splits into several steps.

Step 1. If $\mathrm{Proj_{rot}}\,\gamma$ intersects a segment I_j^k then it necessarily intersects also at least one of the shorter sides of P_j^k.

Recall that, by construction, $P_j^k = \mathrm{Proj_{rot}}\,S_j^k$; moreover, γ intersects no spiral strip S_j^k. If $\mathrm{Proj_{rot}}\,\gamma$ intersected P_j^k without intersecting its shorter sides then γ would pass through all coils of the corresponding spiral. Then, by $(*_1)$, the length of the corresponding fragment of γ would be ≥ 10 in contradiction to our assumptions. Thus, the assertion of step 1 is proved.

Step 2. Denote by $\gamma_{P_j^k}$ the fragment of the plane curve $\mathrm{Proj_{rot}}\,\gamma$ beginning at the first point of its entrance into the trapezium P_j^k to the point of its exit from P_j^k (i.e., to its last intersection point with P_j^k). Then this fragment $\gamma_{P_j^k}$ can be deformed without changing the first and the last points so that the corresponding fragment of the new curve lie entirely on the union of the sides of P_j^k; moreover, its length is at most $\frac{5}{2}$ of the length of $\gamma_{P_j^k}$.

The assertion of step 2 immediately follows from the assertions of step 1 and $(*_2)$.

The assertion of step 2 in turn directly implies the desired assertion $(*_3)$. The proof of $(*_3)$ is finished.

Now, we are ready to pass to the final part of the proof of Theorem 2.

$(*_4)$ The length of any space curve $\gamma \subset \mathbb{R}^3 \setminus \{O\}$ joining A and D and disjoint from each strip S_j^k is at least $\frac{12}{5}$.

Prove the last assertion. Without loss of generality, we may also assume that all interior points of γ are inside the cone K (otherwise the initial curve can be modified without any increase of its length so that it have property $(*_4)$). If γ is not included in the truncated cone $K_{4\Delta AOD}\setminus\{O\}$ then $\mathrm{Proj_{rot}}\,\gamma$ intersects the segment $[4A, 4D]$; consequently, the length of γ is at least $2(4\sin\angle OAD - 1) = 2(4\sin\frac{\pi}{3} - 1) = 2(2\sqrt{3} - 1) > 4$, and the desired estimate is fulfilled. Similarly, if the length of γ is at least 10 then the desired estimate is fulfilled automatically, and there is nothing to prove. Hence, we may further assume without loss of generality that γ is included in the truncated cone $K_{4\Delta AOD} \setminus \{O\}$ and its length is less than 10. Then, by $(*_3)$, there is a plane curve $\tilde{\gamma}$ contained in the triangle $4\Delta AOD \setminus \{O\}$, joining the points A and D, disjoint from each segment I_j^k, and such that the length of $\tilde{\gamma}$ is at most $\frac{5}{2}$ of the length of $\mathrm{Proj_{rot}}\,\gamma$. By property $(**)$ of the family of segments I_j^k, the length of $\tilde{\gamma}$ is at least 6.

Consequently, the length of $\mathrm{Proj}_{\mathrm{rot}}\,\gamma$ is at least $\frac{12}{5}$, which implies the desired estimate. Assertion $(*_4)$ is proved.

The just-proven property $(*_4)$ of the constructed objects implies Theorem 2. Indeed, since the strips S_j^k are mutually disjoint and, outside every neighborhood of the origin O, there are only finitely many of these strips, it is easy to construct a C^0-manifold $Y \subset \mathbb{R}^3$ that is homeomorphic to a closed ball (i.e., ∂Y is homeomorphic to a two-dimensional sphere) and has the following properties:

(I) $O \in \partial Y$, $[A, O[\cup [D, O[\subset \operatorname{Int} Y$;
(II) for every point $y \in (\partial Y) \setminus \{O\}$, there exists a neighborhood $U(y)$ such that $U(y) \cap \partial Y$ is C^1-diffeomorphic to the plane square $[0,1]^2$;
(III) $S_j^k \subset \partial Y$ for all $j \in \mathbb{N}$, $k = 1, \dots, k_j$.

The construction of Y with properties (I)–(III) can be carried out, for example, as follows: As the surface of the zeroth step, take a sphere containing O and such that A and D are inside the sphere. On the jth step, a small neighborhood of the point O of our surface is smoothly deformed so that the modified surface is still smooth, homeomorphic to a sphere, and contains all strips S_j^k, $k = 1, \dots, k_j$. Besides, we make sure that, at the each step, the so-obtained surface be disjoint from the half-intervals $[A, O[$ and $[D, O[$, and, as above, contain all strips S_i^k, $i \le j$, already included therein. Since the neighborhood we are deforming contracts to the point O as $j \to \infty$, the so-constructed sequence of surfaces converges (for example, in the Hausdorff metric) to a limit surface which is the boundary of a C^0-manifold Y with properties (I)–(III).

Property (I) guarantees that $\rho_Y(A, O) = \rho_Y(A, D) = 1$ and $\rho_Y(O, x) \le 1 + \rho_Y(A, x)$ for all $x \in Y$. Property (II) implies the estimate $\rho_Y(x, y) < \infty$ for all $x, y \in Y \setminus \{O\}$, which, granted the previous estimate, yields $\rho_Y(x, y) < \infty$ for all $x, y \in Y$. However, property (III) and the assertion $(*_4)$ imply that $\rho_Y(A, D) \ge \frac{12}{5} > 2 = \rho_Y(A, O) + \rho_Y(A, D)$. Theorem 2 is proved. □

In the case where ρ_Y is a metric (the dimension n (≥ 2) is arbitrary), the question of the existence of geodesics is solved in the following assertion, which implies that ρ_Y is an *intrinsic metric* (see, for example, §6 from [1]).

Theorem 3. *Assume that ρ_Y is a finite function and is a metric on Y. Then any two points $x, y \in Y$ can be joined in Y by a shortest curve $\gamma\colon [0, L] \to Y$ in the metric ρ_Y; i.e., $\gamma(0) = x$, $\gamma(L) = y$, and*

$$\rho_Y(\gamma(s), \gamma(t)) = t - s \quad \forall s, t \in [0, L], \quad s < t. \tag{3}$$

Proof. Fix a pair of distinct points $x, y \in Y$ and put $L = \rho_Y(x, y)$. Now, take a sequence of paths $\gamma_j\colon [0, L] \to Y$ such that $\gamma_j(0) = x_j$, $\gamma_j(L) = y_j$, $x_j \to x$, $y_j \to y$, and $l(\gamma_j) \to L$ as $j \to \infty$. Without loss of generality, we may also assume that the parametrizations of the curves γ_j are their natural parametrizations up to a factor (tending to 1) and the mappings γ_j converge uniformly to a mapping $\gamma\colon [0, L] \to Y$ with $\gamma(0) = x$, $\gamma(L) = y$. By these assumptions,

$$\lim_{j\to\infty} l(\gamma_j|_{[s,t]}) = t - s \quad \forall s, t \in [0, L], \quad s < t. \tag{4}$$

Take an arbitrary pair of numbers $s, t \in [0, L]$, $s < t$. By construction, we have the convergence $\gamma_j(s) \in \operatorname{Int} Y \to \gamma(s)$, $\gamma_j(t) \in \operatorname{Int} Y \to \gamma(t)$ as $j \to \infty$. From here and the definition of the metric $\rho_Y(\cdot, \cdot)$ it follows that

$$\rho_Y(\gamma(s), \gamma(t)) \le \lim_{j\to\infty} l(\gamma_j|_{[s,t]}).$$

By (4),

$$\rho_Y(\gamma(s), \gamma(t)) \le t - s \quad \forall s, t \in [0, L],\ s < t. \tag{5}$$

Prove that (5) is indeed an equality. Assume that

$$\rho_Y(\gamma(s'), \gamma(t')) < t' - s'$$

for some $s', t' \in [0, L]$, $s' < t'$. Then, applying the triangle inequality and then (5), we infer

$$\rho_Y(x, y) \le \rho_Y(x, \gamma(s')) + \rho_Y(\gamma(s'), \gamma(t')) + \rho_Y(\gamma(t'), y) < s' + (t' - s') + (L - t') = L,$$

which contradicts the initial equality $\rho_Y(x, y) = L$. The so-obtained contradiction completes the proof of identity (3). □

Remark. Identity (3) means that the curve γ of Theorem 3 is a geodesic in the metric ρ_Y, i.e., the length of its fragment between points $\gamma(s)$, $\gamma(t)$ calculated in ρ_Y is equal to $\rho_Y(\gamma(s), \gamma(t)) = t - s$. Nevertheless, if we compute the length of the above-mentioned fragment of the curve in the initial Riemannian metric then this length need not coincide with $t - s$; only the easily verifiable estimate $l(\gamma|_{[s,t]}) \le t - s$ holds (see (4)). In the general case, the equality $l(\gamma|_{[s,t]}) = t - s$ can only be guaranteed if $n = 2$ (if $n \ge 3$ then the corresponding counterexample is constructed by analogy with the counterexample in the proof of Theorem 2, see above). In particular, though, by Theorem 3, the metric ρ_Y is always intrinsic in the sense of the definitions in [1, §6], the space (Y, ρ_Y) may fail to be *a space with intrinsic metric* in the sense of [1].

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Sobolev Institute of Mathematics,
pr. Akad. Koptyuga 4, 630090, Novosibirsk, Russia
and
Novosibirsk State University,
ul. Pirogova 2, 630090, Novosibirsk, Russia
E-mail address: apkopylov@yahoo.com
E-mail address: korob@math.nsc.ru