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VARIATIONAL THEORY ON GRASSMANN FIBRATIONS: EXAMPLES

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. Simple examples of variational functionals on Grassmann fibrations are analysed on the basis of the Hilbert form. The Lagrange, Euler – Lagrange, and Noether classes, characterizing the functionals, their extremals and invariance properties are discussed. The relationship of equations for extremals and conservation law equations is established; in the examples the system of Euler – Lagrange equations turns out to be equivalent with the system of conservation law equations.

1. Introduction

This paper is aimed to explain new concepts and methods of the variational theory on Grassmann fibrations by simple examples. It partially covers invited lectures, given by both authors at the Joint Event *Colloquium on Differential Geometry and Finsler Extension of Relativity Theory*, August 2013, Debrecen, dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday. The results can be considered as an extension of our paper [6].

In Section 2 and Section 3 we briefly recall basic concepts of the theory of Grassmann fibrations and variational functionals on them. Then in Section 4 and Section 5 we discuss examples of variational functionals on the Grassmann fibrations $G^1\mathbb{R}^2$ and $G^1\mathbb{R}^3$, constructed from the Euclidean metric on $\mathbb{R}^2$ and its extension to $\mathbb{R}^3$. Our aim is to characterize the meaning of the definition of the first order Lepage form, the *Hilbert form*, its invariance properties and the consequences arising from the Noether theorem in this context ("conservation laws") for extremals of the underlying variational functionals. In

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particular, differences between basic underlying concepts, the *Lagrange form*, *Euler – Lagrange form*, etc. on fibred manifolds on one side, and the *Lagrange class*, *Euler – Lagrange class*, etc. on *Grassmann fibrations* are described. The methods, based on the Hilbert form, do not require any parametrisations. The examples also provide us with the methods how the invariance properties can be used for the study of extremals: to this purpose one should first solve the Noether equations for the generators of invariance groups, and then to formulate and solve the "conservation law" equations. It turns out that in our examples the *conservation law equations* are completely equivalent with the Euler – Lagrange equations for extremals. The functions, representing the first integrals, or "conserved quantities", can naturally be interpreted as a part of the *adapted coordinates* to the extremal submanifolds.

We consider in our examples three variational functionals for 1-dimensional (non-parametrized) submanifolds of $\mathbb{R}^2$ and $\mathbb{R}^3$, defined on the Grassmann fibrations $G^1\mathbb{R}^2$ and $G^1\mathbb{R}^3$; the functionals are defined by means of the 1st-order Lepage forms (the *Hilbert forms*). The following topics are included:

– construction of the Hilbert form from a homogeneous Lagrangian,
– the Euler – Lagrange class and equations for extremals as *set solutions*,
– invariance transformations of the Lagrange class, classification,
– "conserved quantities" (first integrals) within the Grassmann fibration framework, conservation law equations and their solutions.

2. Grassmann fibrations

Throughout, we consider curves in a smooth manifold Y of dimension $m+1$, where $m \geq 0$ is an integer. By an *r-velocity* at a point $y \in Y$, we mean an r-jet $J^r_0\zeta$ with source at the origin $0 \in \mathbb{R}$, and target at $y = \zeta(0)$. A 1-velocity $J^1_0\zeta$ is just a *tangent vector* of Y at y. r-velocities form a manifold denoted by T^rY. If (V, ψ), $\psi = (y^K)$, is a chart on Y, then the associated chart on T^rY is denoted by (V^r, ψ^r), $\psi^r = (y^K_{(0)}, y^K_{(1)}, y^K_{(2)}, \dots, y^K_{(r)})$.

If $I \subset \mathbb{R}$ is an open interval and $\gamma\colon I \to Y$ a curve in Y, the *r-jet prolongation* of γ is the curve $T^r\gamma\colon I \to T^rY$, where $T^r\gamma(t) = J^r_0(\gamma \circ \mathrm{tr}_{-t})$, and tr_{-t} is the translation, sending the origin $0 \in \mathbb{R}$ to the point $t \in \mathbb{R}$. The open set of *regular* velocities in T^rY (the velocities represented by *immersions*) is denoted by $\mathrm{Imm}\, T^rY$.

The *r-th differential group* of the real line $\mathbb{R}$, denoted L^r, is the manifold of invertible r-jets with source and target at $0 \in \mathbb{R}$; the composition of jets $L^r \times L^r \ni (J^r_0\alpha, J^r_0\beta) \to J^r_0(\alpha \circ \beta) \in L^r$ defines on L^r a *Lie group* structure. The manifold $\mathrm{Imm}\, T^rY$ is endowed with the *canonical right action* of the differential group L^r, $\mathrm{Imm}\, T^rY \times L^r \ni (J^r_0\zeta, J^r_0\alpha) \to J^r_0(\zeta \circ \alpha) \in \mathrm{Imm}\, T^rY$.

This action can be described by means of *adapted charts*. If (V, ψ), $\psi = (y^K)$, is a chart on Y, then the associated chart (V^r, ψ^r) induces a chart on $\mathrm{Imm}\, T^rY$ by restricting the coordinate functions ψ^r to $V^r \cap \mathrm{Imm}\, T^rY$. Since regular velocities are represented by immersions, for every element $J^r_0\zeta \in \mathrm{Imm}\, T^rY$

at least one coordinate $y^K_{(1)}(J^r_0\zeta)$ does not vanish. Setting for every L, $1 \leq L \leq m+1$, $V^{r(L)} = \{J^r_0\zeta \in V^r \,|\, y^L_{(1)}(J^r_0\zeta) \neq 0\}$, we get an open and L^r-invariant subset of V^r. Restriction of ψ^r to $V^{r(L)}$ yields a chart $(V^{r(L)}, \psi^{r(L)})$ on $\operatorname{Imm} T^rY$.

To describe the adapted charts we need a specific summation convention. A q-tuple $(I_1, I_2, \dots, I_q)$ is said to be a *q-partition* of a set $I = \{i_1, i_2, \dots, i_l\}$ of integers, if I_j, $1 \leq j \leq q$, are disjoint subsets of I, and $\bigcup I_j = I$. $|I_j|$ is the number of elements of the set I_j. By the symbol

$$\sum_{(I_1,I_2,\dots,I_q)}$$

we understand summation through all q-partitions of the set I, such that $i_1 = i_2 = \ldots = i_l = 1$. If an index L from the sequence $\{1, 2, \dots, m+1\}$ has been chosen, we let the Greek indices $\sigma, \nu, \dots$ run through the complementary sequence $\{1, 2, \dots, L-1, L+1, \dots, m+1\}$.

Lemma 1. *Let (V, ψ), $\psi = (y^K)$, be a chart on Y. Let L be a fixed index, $1 \leq L \leq m+1$, and let $r \geq 1$.*

(a) *There exist unique functions $w^\sigma, w^\sigma_{(1)}, w^\sigma_{(2)}, \dots, w^\sigma_{(r)}$, defined on $V^{r(L)}$, satisfying*

$$y^\sigma = w^\sigma, \quad y^\sigma_{(l)} = \sum_{q=1}^{l} \sum_{(I_1,I_2,\dots,I_q)} y^L_{|I_1|} y^L_{|I_2|} \dots y^L_{|I_q|} w^\sigma_{(q)}.$$

The functions $w^\sigma_{(1)}, w^\sigma_{(2)}, \dots, w^\sigma_{(r)}$ are given by

$$w^\sigma_{(l)}(J^r_0\zeta) = D^l(y^\sigma\zeta \circ (y^L\zeta)^{-1})(y^L\zeta(0)),$$

and are L^r-invariant.

(b) *The pair $(V^{r(L)}, \psi^{r(L)})$, where*

$$\psi^{r(L)} = (w^L, w^L_{(1)}, w^L_{(2)}, \dots, w^L_{(r)}, w^\sigma, w^\sigma_{(1)}, w^\sigma_{(2)}, \dots, w^\sigma_{(r)}),$$

and $w^L = y^L$, $w^L_{(l)} = y^L_{(l)}$, $1 \leq l \leq r$, is a chart on $\operatorname{Imm} T^rY$.

(c) *The L^r-orbits are on $V^{r(L)}$ defined by the equations*

$$w^\sigma_{(l)} = c^\sigma_l, \quad c^\sigma_l \in \mathbb{R}.$$

We call the pair $(V^{r(L)}, \psi^{r(L)})$ the *L-subordinate chart* (associated with (V, ψ)) on $\operatorname{Imm} T^rY$.

The right action of L^r on $\operatorname{Imm} T^rY$ defines the orbit space

$$G^rY = \operatorname{Imm} T^rY / L^r.$$

Its elements (*contact elements*, or *classes* of regular velocities $J^r_0\zeta$) are denoted by $[J^r_0\zeta]$.

Lemma 2. *The canonical right action of L^r defines on $\mathrm{Imm}\, T^rY$ the structure of a right principal L^r-bundle with base the orbit space G^rY. G^rY is endowed with a unique smooth structure such that the quotient projection is a submersion.*

The set G^rY endowed with the structures defined by Lemma 2 is called the *Grassmann fibration* (of 1-dimensional submanifolds) over Y (cf. Grigore and Krupka [1]). The canonical projection of G^rY onto Y is denoted by $\rho^{r,0}$.

The associated smooth structure of G^rY is described as follows. We denote $V_G^{r(L)} = \pi^r(V^{r(L)})$, $\psi_G^{r(L)} = (\tilde{w}^L, \tilde{w}^\sigma, \tilde{w}^\sigma_{(1)}, \tilde{w}^\sigma_{(2)}, \dots, \tilde{w}^\sigma_{(r)})$, where the coordinate functions on $V_G^{r(L)}$ are defined by $w^L = \tilde{w}^L \circ \pi^r$ and $w^\sigma_{(l)} = \tilde{w}^\sigma_{(l)} \circ \pi^r$, $\pi^r \colon \mathrm{Imm}\, T^rY \to G^rY$ denotes the quotient projection, and $0 \le l \le r$. The pair $(V_G^{r(L)}, \psi_G^{r(L)})$ is a chart on G^rY, called the *L-subordinate chart.* We usually omit the tilde symbol if no misunderstanding may arise.

Any immersion $f \colon X \to Y$ of a 1-dimensional manifold X, can be prolonged to an immersion $G^rf \colon X \to G^rY$ into G^rY. If X is an open interval in $\mathbb{R}$, then $G^r\zeta(t) = [T^r\zeta(t)]$, and this formula extends to an arbitrary manifold by means of charts on X.

Let $W \subset Y$ be an open set, $W^r = (\rho^{r,0})^{-1}(W) \subset G^rY$, and let $\alpha \colon W \to Y$ be a diffeomorphism. We define a diffeomorphism $G^r\alpha \colon W^r \to G^rY$ by $G^r\alpha([J^r_0\zeta]) = [J^r_0(\alpha \circ \zeta)]$; $G^r\alpha$ is called the *r-th Grassmann prolongation* of α. This definition applies to vector fields on W. If α_t is the local one-parameter group of a vector field Ξ, then the formula

$$G^r\Xi([J^r_0\zeta]) = \left(\frac{d}{dt} G^r\alpha_t([J^r_0\zeta])\right)_0$$

defines a vector field $G^r\Xi$ on W^r, the *r-th Grassmann prolongation* of Ξ.

Remark 1. For further use we express Lemma 1 for regular 1-velocities. If (V, ψ), $\psi = (y^K)$, is a chart on Y and (V^1, ψ^1), $\psi^1 = (y^K, \dot{y}^K)$, is the associated chart, then the L-subordinate chart on $\mathrm{Imm}\, T^1Y$, $(V^{1(L)}, \psi^{1(L)})$, $\psi^{1(L)} = (w^L, \dot{w}^L, w^\sigma, w^\sigma_{(1)})$, is defined by the coordinate transformations

$$y^L = w^L, \quad \dot{y}^L = \dot{w}^L, \quad y^\sigma = w^\sigma, \quad \dot{y}^\sigma = w^\sigma_{(1)}\dot{w}^L,$$

and

$$w^L = y^L, \quad \dot{w}^L = \dot{y}^L, \quad w^\sigma = y^\sigma, \quad w^\sigma_{(1)} = \frac{\dot{y}^\sigma}{\dot{y}^L}. \tag{1}$$

Equations of the L^1-orbits are

$$w^\sigma = c^\sigma, \quad w^\sigma_{(1)} = c^\sigma_1, \quad c^\sigma, c^\sigma_1 \in \mathbb{R}.$$

3. Variational functionals on Grassmann fibrations

Let η be a 1-form on G^rY. To any 1-dimensional manifold X, any immersion $f\colon X \to Y$, and any *compact* subset $\Omega \subset X$ we assign the *variational integral*

$$\eta_\Omega(f) = \int_\Omega G^r f^{\,*}\eta.$$

The mapping $f \to \eta_\Omega(f)$ is the *variational functional*, associated with η.

To investigate the structure of variational integrals we need the concept of a contact form. For any open set $W \subset Y$, denote by $\Omega_0^r W$ the ring of differentiable functions, by $\Omega_k^r W$ the $\Omega_0^r W$-module of differential k-forms, and by $\Omega^r W$ the exterior algebra on W^r. We say that a 1-form $\eta \in \Omega_1^r W$ is *contact*, if $G^r\zeta^{\,*}\eta = 0$ for all immersions $\zeta\colon I \to W$, where I is an open interval in $\mathbb{R}$. For any fixed L, the forms dw^L, $\omega_{(l)}^\sigma$, $dw_{(r)}^\sigma$, where the Greek index is complementary to L and $\omega_{(l)}^\sigma = dw_{(l)}^\sigma - w_{(l+1)}^\sigma dw^L$, constitute the *contact basis* of 1-forms on $V_G^{r(L)}$. A k-form $\eta \in \Omega_k^r W$ is said to be *contact*, if it is locally generated by the contact 1-forms $\omega_{(l)}^\sigma$. Contact forms define an ideal in $\Omega^r W$, called the *contact ideal*, and denoted by $\Theta^r W$.

We restrict ourselves without loss of generality to the case when η is a *Lepage form*. The meaning of this condition for the first variation formula is explained e.g. in the chapters of the handbook Krupka and Saunders [4], and in Krupka [3]. Formally, we say that η is a *Lepage form*, if $i_Z d\eta$ belongs to the contact ideal $\Theta^r W$ for every $\rho^{r,0}$-*vertical* vector field Z on W^r.

Let η be a form on G^rY, expressible in a chart $(V_G^{r(L)}, \psi_G^{r(L)})$ by

$$\eta = \mathcal{L}_\eta^L dw^L + \sum_{0\le l\le r-1} B_\sigma^l \omega_l^\sigma. \tag{2}$$

The term $\mathcal{L}_\eta^L dw^L$, complementary to the contact part, is said to be the *local Lagrangian*, and its component $\mathcal{L}_\eta^L$ the *local Lagrange function* associated to η (and $(V_G^{r(L)}, \psi_G^{r(L)})$). The class, defined globally,

$$[\eta] = \mathcal{L}_\eta^L dw^L \mod \Theta^r W,$$

is called the *Lagrange class*, associated with η.

The following theorem characterizes Lepage forms by means of charts.

Theorem 1. *Let (V,ψ), $\psi = (y^K)$, be a chart on Y such that $V \subset W$, and let $\eta \in \Omega_1^r W$ be a 1-form on W^r locally expressed by* (2). *Then η is a Lepage form if and only if*

$$(\rho^{2r-1,r})^*\eta = \mathcal{L}_\eta^L dw^L + \sum_{0\le l\le r-1}\left(\sum_{0\le k\le r-l-1} (-1)^k \frac{d^k}{d(w^L)^k}\frac{\partial \mathcal{L}_\eta^L}{\partial w_{(k+l+1)}^\sigma}\right)\omega_{(l)}^\sigma. \tag{3}$$

Remark 2. If $r = 1$, the Lepage form defined by (3) is given by

$$\eta = \mathcal{L}_\eta^L dw^L + \frac{\partial \mathcal{L}_\eta^L}{\partial w^\sigma_{(1)}} \omega^\sigma, \tag{4}$$

and coincides with the well-known *Hilbert form.* To show the projectability of the Hilbert form, associated with the Lagrange function $\mathcal{L}\colon \operatorname{Imm} T^1 Y \to \mathbb{R}$, it is sufficient to find its expression in the *adapted coordinates* on the Grassmann fibration $G^1 Y$. Indeed, in the adapted chart $(w^L, w^\sigma, \dot{w}^L, w^\sigma_{(1)})$, defined on an open subset of $\operatorname{Imm} T^1 Y$ where $\dot{w}^L \neq 0$ by (1), we get

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{y}^K} dy^K &= \frac{\partial \mathcal{L}}{\partial \dot{w}^L} dw^L - \frac{\partial \mathcal{L}}{\partial w^\sigma_{(1)}} \frac{w^\sigma_{(1)}}{\dot{w}^L} dw^L + \frac{\partial \mathcal{L}}{\partial w^\sigma_{(1)}} \frac{1}{\dot{w}^L} dw^\sigma \\ &= \frac{\partial \mathcal{L}}{\partial \dot{w}^L} dw^L + \frac{\partial \mathcal{L}}{\partial w^\sigma_{(1)}} \frac{1}{\dot{w}^L} \omega^\sigma = \mathcal{L}_\eta^L dw^L + \frac{\partial \mathcal{L}_\eta^L}{\partial w^\sigma_{(1)}} \omega^\sigma = \eta,\end{aligned}$$

where $\mathcal{L}_\eta^L = (1/\dot{w}^L)\mathcal{L}$ is defined on $G^1 Y$ and $\mathcal{L}$ is a positive-homogeneous function on $\operatorname{Imm} T^1 \mathbb{R}^2$.

If $r = 2$ we get

$$\eta = \mathcal{L}_\eta^L dw^L + \left(\frac{\partial \mathcal{L}_\eta^L}{\partial w^\sigma_{(1)}} - \frac{d}{dw^L} \frac{\partial \mathcal{L}_\eta^L}{\partial w^\sigma_{(2)}} \right) \omega^\sigma + \frac{\partial \mathcal{L}_\eta^L}{\partial w^\sigma_{(2)}} \omega^\sigma_{(1)},$$

which is an analogue of the Lepage equivalent of a 2nd order Lagrangian on a fibred manifold (cf. Krupka [3]).

The following theorem characterizes the meaning of Lepage forms for the variational theory. $\Theta^r W \wedge \Theta^r W$ denotes the exterior power of $\Theta^r W$.

Theorem 2. *Let $\eta \in \Omega_1^r W$ be a Lepage form. Then*

$$d\eta = E_\sigma(\mathcal{L}_\eta^L) \omega^\sigma \wedge dw^L \mod (\Theta^{2r} W \wedge \Theta^{2r} W),$$

where

$$E_\sigma(\mathcal{L}_\eta^L) = \sum_{k=0}^{r} (-1)^k \frac{d^k}{d(w^L)^k} \left(\frac{\partial \mathcal{L}_\eta^L}{\partial w^\sigma_{(k)}} \right).$$

We say that an immersion $f\colon X \to W$ is an *extremal* of the variational functional η_Ω, if for every vector field Ξ such that $\operatorname{supp} \Xi \cap f(X) \subset f(\Omega)$,

$$(\partial_{G^r \Xi} \eta)_\Omega (f) = 0.$$

f is called an *extremal*, if it is an extremal for every piece $\Omega \subset X$. In this definition we consider vector fields, which vanish on the boundary $\partial\Omega$ of the piece Ω *along* f. The class

$$E_\eta = E_\sigma(\mathcal{L}_\eta^L) \omega^\sigma \wedge dw^L \mod \Theta^{2r} W$$

is called the *Euler – Lagrange class* associated to η.

Theorem 3. *Let $\eta \in \Omega_1^r W$ be a Lepage form, let $f\colon X \to W$ be an immersion. The following conditions are equivalent:*

(a) *f is an extremal of η_Ω.*

(b) *For every chart (V, ψ) on Y there exists a subordinate chart $(V_G^{r(L)}, \psi_G^{r(L)})$ on $G^r Y$ such that in this chart the following system of partial differential equations for f is satisfied,*

$$\sum_{k=0}^{r} (-1)^k \frac{d^k}{d(w^L)^k} \left(\frac{\partial \mathcal{L}_\eta^L}{\partial w_{(k)}^\sigma} \right) \circ G^{2r} f = 0, \quad 1 \leq \sigma \leq m. \tag{5}$$

Equations (5) are the *Euler – Lagrange equations* associated with η.

Now we study variational functionals, invariant under Lie groups of diffeomorphisms. For generalities on invariant variational structures on fibred manifolds we refer to Krupka and Saunders [4].

We say that a 1-form $\eta \in \Omega_1^r W$ is *invariant* with respect to a diffeomorphism $\alpha\colon W \to Y$, if

$$G^r \alpha^* \eta = \eta \mod \Theta^r W. \tag{6}$$

If condition (6) is satisfied, we also say that α is an *invariance transformation* of η. This condition means that $G^r\alpha$ preserves *classes* of forms under the equivalence relation "η_1 *is equivalent with* η_2 *if* $\eta_1 - \eta_2 \in \Theta^r W$". We say that a vector field Ξ on W^r is the *generator* of invariance transformations of η, if its 1-parameter group consists of invariance transformations of η.

The following is an extension of the classical Noether equation for the Grassmann fibrations.

Theorem 4. *Let $\eta \in \Omega_1^r W$ be a 1-form and Ξ be a vector field on Y. The following two conditions are equivalent:*

(a) *Ξ is the generator of invariance transformations of η.*

(b) *The Lie derivative of η with respect to $G^r\Xi$ belongs to the contact ideal $\Theta^r W$,*

$$\partial_{G^r\Xi}\eta \in \Theta^r W.$$

The following theorem is a restatement of the first variation formula for *invariant* Lepage forms.

Theorem 5. *Let $\eta \in \Omega_1^r W$ be a Lepage form, and let Ξ be the generator of invariance transformations of η. Then*

$$i_{G^{2r}\Xi} E_\eta + d\rho^{2r,r\,*} i_{G^r\Xi}\eta = 0 \mod \Theta^{2r} W.$$

We are now in a position to extend the classical Emmy Noether's analysis of invariant variational functionals to Grassmann fibrations. Let $\phi\colon W^r \to \mathbb{R}$ be a function, and let $f\colon X \to Y$ be an immersion. Suppose that X is connected. We shall say that ϕ is *constant along* f, if

$$d(G^r f^* \phi) = 0.$$

We also say that ϕ is a *level set function* for the immersion f.

Theorem 6. *Let $\eta \in \Omega_1^r W$ be a Lepage form, and let an immersion $f \colon X \to Y$ be an extremal. Then for every generator Ξ of invariance transformations of η, $i_{G^r\Xi}\eta$ is a level-set function for the immersion $G^r f$,*

$$d(G^r f^* i_{G^r\Xi}\eta) = 0. \tag{7}$$

Remark 3. According to Theorem 6, formula (7) includes a construction of *adapted charts* to the submanifolds $G^r f(X) \subset G^r Y$; analogues of the "*conserved quantities*" from geometric mechanics are the *coordinate functions*, defining the submanifolds.

4. Example: Arc-length functional on the Grassmann fibration $G^1\mathbb{R}^2$

Consider the manifold $\mathbb{R}^2$, with the canonical coordinates (x, y) and the associated coordinates $(x, y, \dot{x}, \dot{y})$ on $\operatorname{Imm} T^1\mathbb{R}^2$. Formula

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

defines a Lagrange function on $\operatorname{Imm} T^1\mathbb{R}^2$; $\mathcal{L}$ satisfies, for any $\tau \in \mathbb{R}$, the condition $\mathcal{L}(x, y, \tau\dot{x}, \tau\dot{y}) = |\tau|\mathcal{L}(x, y, \dot{x}, \dot{y})$, and is positive-homogeneous.

(a) Extremals as set solutions The Hilbert form associated with $\mathcal{L}$ is

$$\eta = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}dx + \dot{y}dy). \tag{8}$$

To find the Euler – Lagrange equations for extremals, we choose a vector field

$$\Xi = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \Xi^1 \frac{\partial}{\partial \dot{x}} + \Xi^2 \frac{\partial}{\partial \dot{y}},$$

on $\operatorname{Imm} T^1\mathbb{R}^2$ and find an expression for the 1-form $i_\Xi d\eta$. First we have

$$\begin{aligned} d\eta = &-\frac{\dot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}} d\dot{x} \wedge dy + \frac{\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}} d\dot{x} \wedge dx \\ &- \frac{\dot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}} d\dot{y} \wedge dx + \frac{\dot{x}^2}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}} d\dot{y} \wedge dy. \end{aligned}$$

Contracting this form by Ξ we have

$$i_\Xi d\eta = \frac{\dot{x}\Xi^2 - \dot{y}\Xi^1}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}dy - \dot{y}dx) + \frac{\dot{y}\xi^1 - \dot{x}\xi^2}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}d\dot{y} - \dot{y}d\dot{x}). \tag{9}$$

Omitting the term containing the contact form $\dot{x}dy - \dot{y}dx$ we get for the *Euler – Lagrange class*

$$i_\Xi d\eta \approx \frac{\dot{y}\xi^1 - \dot{x}\xi^2}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}d\dot{y} - \dot{y}d\dot{x}). \tag{10}$$

Consequently, extremals are exactly the curves satisfying the equation

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = 0. \tag{11}$$

For curves, that can be parametrized by the coordinate x, that is, the curves of the form $x \to (x, y(x))$, $\dot{x} = 1$, $\ddot{x} = 0$, and equation (11) reduces to $\ddot{y} = 0$. Analogously, for curves expressible as $y \to (y, x(y))$, (11) reduces to $\ddot{x} = 0$. Consequently, solving these equations and writing them in a unique way we get all *extremals* described as the *set solutions*

$$\{(x, y) \in \mathbb{R}^2 \,|\, Px + Qy + R = 0,\ P, Q, R \in \mathbb{R}\}.$$

Remark 4. Equation (11) is invariant with respect to reparametrisations, and its solutions are the *set solutions* (cf. Urban and Krupka [5]). Indeed, if t is a parameter, and $\kappa = \kappa(t)$ a reparametrisation, then

$$\frac{dx}{dt} = \frac{dx}{d\kappa}\frac{d\kappa}{dt}, \quad \frac{d^2x}{dt^2} = \frac{d^2x}{d\kappa^2}\left(\frac{d\kappa}{dt}\right)^2 + \frac{dx}{d\kappa}\frac{d^2\kappa}{dt^2}$$

hence

$$\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2} = \left(\frac{d\kappa}{dt}\right)^3\left(\frac{dx}{d\kappa}\frac{d^2y}{d\kappa^2} - \frac{dy}{d\kappa}\frac{d^2x}{d\kappa^2}\right).$$

Remark 5. Equation (11), where the dot means differentiation with respect to a parameter t, can be resolved immediately. Indeed, if we have a solution, then writing

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} = -\phi, \tag{12}$$

we get a function $\phi = \phi(t)$, and (12) splits in two conditions $\ddot{x} + \phi(t)\dot{x} = 0$ and $\ddot{y} + \phi(t)\dot{y} = 0$. Setting

$$\mu(t) = \mathrm{e}^{-\int_0^t \phi(\tau)d\tau},$$

we have

$$\frac{d}{dt}\frac{\dot{x}}{\mu(t)} = \frac{1}{\mu(t)}(\ddot{x} + \dot{x}\phi(t)) = 0$$

hence $\dot{x} = A\mu(t)$, and

$$x(t) = A\int_0^t \mu(\tau)d\tau + B, \quad A, B \in \mathbb{R}.$$

Analogously

$$y(t) = C\int_0^t \mu(\tau)d\tau + D, \quad C, D \in \mathbb{R}.$$

Changing the parametrisation by the formula

$$\kappa(t) = \int_0^t \mu(\tau)d\tau,$$

we get the parametric equations of a *straight line*

$$x = A\kappa + B, \quad y = C\kappa + D.$$

(b) Invariance and the Noether currents In accordance with the general theory, the Hilbert form (8) is *invariant* with respect to a vector field Ξ, if the Lie derivative $\partial_\Xi \eta$ satisfies $\partial_\Xi \eta = 0 \mod \Theta^1\mathbb{R}^2$. We suppose that Ξ is the first *Grassmann prolongation* of a vector field ξ on Y, that is, $\Xi = G^1\xi$, where

$$\xi = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y}.$$

The Lie derivative is given by $\partial_\Xi \eta = i_\Xi d\eta + d i_\Xi \eta$, where $i_\Xi d\eta$ is given by (9). The second term, the *Noether current* associated with ξ, is

$$i_\Xi \eta = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} (\xi^1 \dot{x} + \xi^2 \dot{y}),$$

hence

$$\begin{aligned} d i_\Xi \eta = {} & \frac{\dot{y}(-\dot{x}\xi^2 + \dot{y}\xi^1)}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}} d\dot{x} + \frac{\dot{x}(-\dot{y}\xi^1 + \dot{x}\xi^2)}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}} d\dot{y} \\ & + \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{\partial \xi^1}{\partial x}\dot{x} + \frac{\partial \xi^2}{\partial x}\dot{y} \right) dx + \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{\partial \xi^1}{\partial y}\dot{x} + \frac{\partial \xi^2}{\partial y}\dot{y} \right) dy. \end{aligned} \tag{13}$$

Then

$$\begin{aligned} \partial_\Xi \eta = {} & \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \left(-\frac{\dot{x}\Xi^2 - \dot{y}\Xi^1}{\dot{x}^2 + \dot{y}^2}\dot{y} + \frac{\partial \xi^1}{\partial x}\dot{x} + \frac{\partial \xi^2}{\partial x}\dot{y} \right) dx \\ & + \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{\dot{x}\Xi^2 - \dot{y}\Xi^1}{\dot{x}^2 + \dot{y}^2}\dot{x} + \frac{\partial \xi^1}{\partial y}\dot{x} + \frac{\partial \xi^2}{\partial y}\dot{y} \right) dy. \end{aligned}$$

Consequently, passing to classes, the invariance equation reads

$$\frac{\partial \xi^1}{\partial x}\dot{x}^2 + \left(\frac{\partial \xi^2}{\partial x} + \frac{\partial \xi^1}{\partial y} \right) \dot{x}\dot{y} + \frac{\partial \xi^2}{\partial y}\dot{y}^2 = 0.$$

Supposing that the functions ξ^1 and ξ^2 do not depend on $\dot{x}$ and $\dot{y}$,

$$\frac{\partial \xi^1}{\partial x} = 0, \quad \frac{\partial \xi^2}{\partial y} = 0, \quad \frac{\partial \xi^2}{\partial x} + \frac{\partial \xi^1}{\partial y} = 0.$$

The solution is

$$\xi^1 = Ay + B, \quad \xi^2 = -Ax + C, \quad A, B, C \in \mathbb{R},$$

and implies that the Lie algebra of invariance vector fields is 3-dimensional,

$$\xi = (Ay + B)\frac{\partial}{\partial x} + (-Ax + C)\frac{\partial}{\partial y}.$$

The Noether current associated with ξ is a function given by

$$i_\Xi \eta = \frac{A}{\sqrt{\dot{x}^2 + \dot{y}^2}}(y\dot{x} - x\dot{y}) + \frac{B}{\sqrt{\dot{x}^2 + \dot{y}^2}}\dot{x} + \frac{C}{\sqrt{\dot{x}^2 + \dot{y}^2}}\dot{y}.$$

This determines the *conservation law equation*

$$\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}(A(y\dot{x} - x\dot{y}) + B\dot{x} + C\dot{y}) = \mathrm{const} \tag{14}$$

with the right-hand side an arbitrary constant. Since A, B, C are arbitrary, the conservation law equations are

$$\frac{y\dot{x} - x\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = S, \qquad \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = P, \qquad \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = Q,$$

where P, Q, $S \in \mathbb{R}$. These conditions imply that

$$\frac{y\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \frac{x\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = Py - Qx = S \tag{15}$$

so every solution is a straight line. This proves, that every solution of the conservation law equations is an extremal.

(c) An extension to $G^1\mathbb{R}^3$ Consider the manifold $\mathbb{R}^3$, with the canonical coordinates (x, y, z), the associated coordinates $(x, y, z, \dot{x}, \dot{y}, \dot{z})$, and a Lagrange function on $\operatorname{Imm} T^1\mathbb{R}^3$, defined by

$$\mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \sqrt{\dot{x}^2 + \dot{y}^2} + \dot{z}.$$

$\mathcal{L}$ is positive homogeneous. The Hilbert form associated with $\mathcal{L}$ is

$$\eta = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}dx + \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}dy + dz,$$

and comparison with (10) and (11) implies that extremal equation is

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = 0.$$

Solving this equation we get the *extremals* as the *set solutions*

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, Px + Qy + R = 0\},$$

where P, Q and R are *arbitrary functions* of the variable z.

We shall now study invariance properties of the Hilbert form. First we find an expression for the Lie derivative $\partial_\Xi \eta = i_\Xi d\eta + di_\Xi \eta$ with respect to the Grassmann prolongation $\Xi = G^1\xi$ of a vector field ξ defined on $\mathbb{R}^3$,

$$\xi = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial z}. \tag{16}$$

The first term in the Lie derivative $\partial_\Xi \eta$ coincides with (9),

$$i_\Xi d\eta = \frac{\dot{x}\Xi^2 - \dot{y}\Xi^1}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}dy - \dot{y}dx) + \frac{\dot{y}\xi^1 - \dot{x}\xi^2}{(\dot{x}^2 + \dot{y}^2)\sqrt{\dot{x}^2 + \dot{y}^2}}(\dot{x}d\dot{y} - \dot{y}d\dot{x}). \tag{17}$$

The second term $di_\Xi\eta$ is defined by the current associated with Ξ,

$$i_\Xi\eta = \frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}(\xi^1\dot{x}+\xi^2\dot{y})+\xi^3.$$

Using (13),

$$(18)\quad \begin{aligned} di_\Xi\eta = {} & \frac{(-\dot{x}\xi^2+\dot{y}\xi^1)(\dot{y}d\dot{x}-\dot{x}d\dot{y})}{(\dot{x}^2+\dot{y}^2)\sqrt{\dot{x}^2+\dot{y}^2}} \\ & + \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial x}\dot{x}+\frac{\partial\xi^2}{\partial x}\dot{y}\right)+\frac{\partial\xi^3}{\partial x}\right)dx \\ & + \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial y}\dot{x}+\frac{\partial\xi^2}{\partial y}\dot{y}\right)+\frac{\partial\xi^3}{\partial y}\right)dy \\ & + \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial z}\dot{x}+\frac{\partial\xi^2}{\partial z}\dot{y}\right)+\frac{\partial\xi^3}{\partial z}\right)dz \end{aligned}$$

Thus, summarizing, formulas (17) and (18) yield

$$\begin{aligned} \partial_\Xi\eta = {} & \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial x}\dot{x}+\frac{\partial\xi^2}{\partial x}\dot{y}-\frac{\dot{x}\Xi^2-\dot{y}\Xi^1}{\dot{x}^2+\dot{y}^2}\dot{y}\right)+\frac{\partial\xi^3}{\partial x}\right)dx \\ & + \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial y}\dot{x}+\frac{\partial\xi^2}{\partial y}\dot{y}+\frac{\dot{x}\Xi^2-\dot{y}\Xi^1}{\dot{x}^2+\dot{y}^2}\dot{x}\right)+\frac{\partial\xi^3}{\partial y}\right)dy \\ & + \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial z}\dot{x}+\frac{\partial\xi^2}{\partial z}\dot{y}\right)+\frac{\partial\xi^3}{\partial z}\right)dz. \end{aligned}$$

Supposing that the class of $\partial_\Xi\eta$ is *invariant* with respect to Ξ, and omitting the term with the contact form $\dot{x}dy-\dot{y}dx$, we get the *invariance condition*

$$\begin{aligned} & \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial x}\dot{x}+\frac{\partial\xi^2}{\partial x}\dot{y}\right)+\frac{\partial\xi^3}{\partial x}\right)\dot{x}+\left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial y}\dot{x}+\frac{\partial\xi^2}{\partial y}\dot{y}\right)+\frac{\partial\xi^3}{\partial y}\right)\dot{y} \\ & + \left(\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}\left(\frac{\partial\xi^1}{\partial z}\dot{x}+\frac{\partial\xi^2}{\partial z}\dot{y}\right)+\frac{\partial\xi^3}{\partial z}\right)\dot{z}=0. \end{aligned}$$

The components ξ^1, ξ^2 , ξ^3 of the vector field ξ (16), leaving invariant the Hilbert form, can be determined from this condition by an elementary calculation. We get

$$\xi = (Ax+B)\frac{\partial}{\partial x}+(-Ay+C)\frac{\partial}{\partial y}+D\frac{\partial}{\partial z},$$

where A, B, C are functions of z, and $D\in\mathbb{R}$.

The Noether current, associated with ξ, is given by

$$i_\Xi\eta = \frac{A}{\sqrt{\dot{x}^2+\dot{y}^2}}(y\dot{x}-x\dot{y}) + \frac{B}{\sqrt{\dot{x}^2+\dot{y}^2}}\dot{x} + \frac{C}{\sqrt{\dot{x}^2+\dot{y}^2}}\dot{y} + D.$$

The corresponding *conservation law equation* for 1-dimensional submanifolds of $\mathbb{R}^2$ reads

$$\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2}}(A(y\dot{x}-x\dot{y}) + B\dot{x} + C\dot{y}) + D = \text{const.}$$

Since $D \in \mathbb{R}$, this equation reduces, for any fixed value of the coordinate z, to equation (14). Comparing this result to (15) we see that the solutions are the sets of the form

$$Py - Qx = S,$$

where P, Q and S are arbitrary functions of the coordinate z (straight lines on the plane defined by the condition $z = \text{const}$).

5. Example: Extremals on the sphere S_r^2 with free radius

Consider the manifold $\mathbb{R}^3$, with the *spherical coordinates* (ϑ, φ, r), where for instance $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, and $r > 0$, defined by the transformation equations $x = r\cos\varphi\sin\vartheta$, $x = r\sin\varphi\sin\vartheta$, $z = r\cos\vartheta$, where (x, y, z) are the canonical coordinates on $\mathbb{R}^3$. The spherical coordinates form an *adapted chart* to the 2-dimensional sphere S_r^2 in $\mathbb{R}^3$ of radius r and centre $0 \in \mathbb{R}^3$. The associated coordinates on the manifold of regular velocities $\operatorname{Imm} T^1\mathbb{R}^3$ are denoted $(\vartheta, \varphi, r, \dot{\vartheta}, \dot{\varphi}, \dot{r})$, the ϑ-*subordinate* coordinates $(\vartheta, \varphi, r, \dot{\vartheta}, \varphi_{(1)}, r_{(1)})$ on the domain $V^{1(\vartheta)} = \{P \in \operatorname{Imm} T^1\mathbb{R}^3 \mid \dot{\vartheta}(P) \neq 0\}$ are defined by the formulas $\varphi_{(1)} = \dot{\varphi}/\dot{\vartheta}$, $r_{(1)} = \dot{r}/\dot{\vartheta}$. The *associated* chart on the Grassmann fibration $G^1\mathbb{R}^3 = \operatorname{Imm} T^1\mathbb{R}^3/L^1$ is formed by the functions $(\vartheta, \varphi, r, \varphi_{(1)}, r_{(1)})$ on the set $V_G^{1(\vartheta)} = \pi^1(V^{1(\vartheta)})$, where $\pi^1\colon \operatorname{Imm} T^1\mathbb{R}^3 \to G^1\mathbb{R}^3$ is the quotient projection.

Let λ be a Lagrangian on $V_G^{1(\vartheta)} \subset G^1\mathbb{R}^3$, defined by

$$\lambda = \mathcal{L}d\vartheta,$$

where the Lagrange function $\mathcal{L}$ is given by

$$\mathcal{L}(\vartheta, \varphi, r, \varphi_{(1)}, r_{(1)}) = \sqrt{1 + \varphi_{(1)}^2 \sin^2\vartheta} + r_{(1)}. \tag{19}$$

Note that (19) originates in the metric Lagrangian on the unit sphere S^2 in $\mathbb{R}^3$.

(a) Extremals as set solutions The Euler – Lagrange expressions read

$$\begin{aligned} E_1(\mathcal{L}) &= \frac{\partial\mathcal{L}}{\partial\varphi} - \frac{d}{d\vartheta}\frac{\partial\mathcal{L}}{\partial\varphi_{(1)}} = -\frac{d}{d\vartheta}\frac{\varphi_{(1)}\sin^2\vartheta}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}, \\ E_1(\mathcal{L}) &= \frac{\partial\mathcal{L}}{\partial r} - \frac{d}{d\vartheta}\frac{\partial\mathcal{L}}{\partial r_{(1)}} = 0. \end{aligned} \tag{20}$$

We search for solutions $\zeta\colon X \to \mathbb{R}^3$ of the associated Euler–Lagrange equations that are *immersions*, defined on a 1-dimensional manifold X such that $G^1\zeta(X) \subset V_G^{1(\vartheta)}$, where $G^1\zeta$ is the *Grassmann prolongation* of ζ. The Euler–Lagrange equations (20) reduce to one equation

$$\frac{\varphi_{(1)} \sin^2 \vartheta}{\sqrt{1 + \varphi_{(1)}^2 \sin^2 \vartheta}} = c \tag{21}$$

for some $c \in \mathbb{R}$, $-1 < c < 1$. Solutions of (21) can be parametrized by the coordinate ϑ, i.e. the curves $\vartheta \to (\vartheta, \varphi(\vartheta), r(\vartheta))$ in $\mathbb{R}^3$, and satisfy the condition

$$\varphi = \arccos\left(\frac{c}{\sqrt{1-c^2}} \frac{\cos\vartheta}{\sin\vartheta}\right) + c_0,$$

where $c_0 \in \mathbb{R}$ is an integration constant. Hence we get all *extremals* described as *set-solutions*, where the radius coordinate r is an *arbitrary function* of the polar angle ϑ. It is well-known that for $r = r(\vartheta) = \text{const}$ the solutions of (21), the geodesics on a sphere of radius r, are the *great circles* (cf. Jost and Li-Jost [2]).

(b) Invariance and first integrals The Lepage form associated with the Lagrange function (19) is given by

$$\eta = \frac{1}{\sqrt{1 + \varphi_{(1)}^2 \sin^2 \vartheta}} d\vartheta + \frac{\varphi_{(1)} \sin^2 \vartheta}{\sqrt{1 + \varphi_{(1)}^2 \sin^2 \vartheta}} d\varphi + dr, \tag{22}$$

We note that (22) coincides with the standard Hilbert form associated with the Lagrange function (19) on the manifold of regular velocities $\operatorname{Imm} T^1\mathbb{R}^3$.

Let Ξ be a vector field on $\mathbb{R}^3$, and $G^1\Xi$ its first-order Grassmann prolongation, expressed by means of the ϑ-subordinate chart on $G^1\mathbb{R}^3$,

$$\Xi = \Xi^\vartheta \frac{\partial}{\partial\vartheta} + \Xi^\varphi \frac{\partial}{\partial\varphi} + \Xi^r \frac{\partial}{\partial r},$$
$$G^1\Xi = \Xi^\vartheta \frac{\partial}{\partial\vartheta} + \Xi^\varphi \frac{\partial}{\partial\varphi} + \Xi^r \frac{\partial}{\partial r} + \Xi^\varphi_{(1)} \frac{\partial}{\partial\varphi_{(1)}} + \Xi^r_{(1)} \frac{\partial}{\partial r_{(1)}}.$$

We compute the Lie derivative of the Lepage form η. From expression (22),

$$\begin{aligned}
\partial_{G^1\Xi}\eta &= i_{G^1\Xi} d\eta + d i_{G^1\Xi}\eta \\
&= \left(\frac{\partial\mathcal{L}}{\partial\vartheta}\Xi^\vartheta + \left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\varphi_{(1)}}\varphi_{(1)} - \frac{\partial\mathcal{L}}{\partial r_{(1)}} r_{(1)}\right)\frac{d\Xi^\vartheta}{d\vartheta}\right. \\
&\quad \left. + \frac{\partial\mathcal{L}}{\partial\varphi_{(1)}}\frac{d\Xi^\varphi}{d\vartheta} + \frac{\partial\mathcal{L}}{\partial r_{(1)}}\frac{d\Xi^r}{d\vartheta}\right) d\vartheta + \left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\varphi_{(1)}}\varphi_{(1)} - \frac{\partial\mathcal{L}}{\partial r_{(1)}} r_{(1)}\right)\frac{\partial\Xi^\vartheta}{\partial\varphi}\omega^\varphi
\end{aligned}$$

$$
\begin{aligned}
&+\left(\mathcal{L}-\frac{\partial \mathcal{L}}{\partial \varphi_{(1)}}\varphi_{(1)}-\frac{\partial \mathcal{L}}{\partial r_{(1)}}r_{(1)}\right)\frac{\partial \Xi^{\vartheta}}{\partial r}\omega^{r}\\
&+\frac{\partial \mathcal{L}}{\partial \varphi_{(1)}}\frac{\partial \Xi^{\varphi}}{\partial \varphi}\omega^{\varphi}+\frac{\partial \mathcal{L}}{\partial \varphi_{(1)}}\frac{\partial \Xi^{\varphi}}{\partial r}\omega^{r}+\frac{\partial \mathcal{L}}{\partial r_{(1)}}\frac{\partial \Xi^{r}}{\partial \varphi}\omega^{\varphi}+\frac{\partial \mathcal{L}}{\partial r_{(1)}}\frac{\partial \Xi^{r}}{\partial r}\omega^{r}\\
&+\frac{\partial^2 \mathcal{L}}{\partial \varphi_{(1)}^2}\frac{d\Xi^{\varphi}}{d\vartheta}\omega^{\varphi}-\frac{\partial^2 \mathcal{L}}{\partial \varphi_{(1)}^2}\varphi_{(1)}\frac{d\Xi^{\vartheta}}{d\vartheta}\omega^{\varphi}+\frac{\partial^2 \mathcal{L}}{\partial \varphi_{(1)}\partial \vartheta}\Xi^{\vartheta}\omega^{\varphi},
\end{aligned}
$$

where $\omega^{\varphi}=d\varphi-\varphi_{(1)}d\vartheta$, $\omega^{r}=dr-r_{(1)}d\vartheta$. The *Noether's equation* for Ξ, $\partial_{G^1\Xi}\eta=0 \mod \Theta^1\mathbb{R}^3$, reads

$$
\frac{1}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}\left(\varphi_{(1)}^2\sin\vartheta\cos\vartheta\,\Xi^{\vartheta}+\frac{d\Xi^{\vartheta}}{d\vartheta}+\varphi_{(1)}\sin^2\vartheta\frac{d\Xi^{\varphi}}{d\vartheta}\right)+\frac{d\Xi^{r}}{d\vartheta}=0. \tag{23}
$$

We find all solutions of this equation. (23) can equivalently be written as two equations

$$
\frac{1}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}\frac{\partial \Xi^{\vartheta}}{\partial r}+\frac{\varphi_{(1)}\sin^2\vartheta}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}\frac{\partial \Xi^{\varphi}}{\partial r}+\frac{\partial \Xi^{r}}{\partial r}=0, \tag{24}
$$

and

$$
\begin{aligned}
&\frac{\varphi_{(1)}^2\sin\vartheta\cos\vartheta}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}\Xi^{\vartheta}+\frac{1}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}\left(\frac{\partial \Xi^{\vartheta}}{\partial \vartheta}+\frac{\partial \Xi^{\vartheta}}{\partial \varphi}\varphi_{(1)}\right)\\
&\quad+\frac{\varphi_{(1)}\sin^2\vartheta}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}}\left(\frac{\partial \Xi^{\varphi}}{\partial \vartheta}+\frac{\partial \Xi^{\varphi}}{\partial \varphi}\varphi_{(1)}\right)+\frac{\partial \Xi^{r}}{\partial \vartheta}+\frac{\partial \Xi^{r}}{\partial \varphi}\varphi_{(1)}=0.
\end{aligned} \tag{25}
$$

Differentiating (24) with respect to $\varphi_{(1)}$ we get

$$
\frac{\sin^2\vartheta}{\sqrt{(1+\varphi_{(1)}^2\sin^2\vartheta)^3}}\left(\frac{\partial \Xi^{\varphi}}{\partial r}-\varphi_{(1)}\frac{\partial \Xi^{\vartheta}}{\partial r}\right)=0,
$$

hence and from (24) we conclude that

$$
\frac{\partial \Xi^{\vartheta}}{\partial r}=0,\quad \frac{\partial \Xi^{\varphi}}{\partial r}=0,\quad \frac{\partial \Xi^{r}}{\partial r}=0. \tag{26}
$$

Differentiating now (25) with respect to $\varphi_{(1)}$ and setting $\varphi_{(1)}=0$, we obtain

$$
\frac{\partial \Xi^{r}}{\partial \vartheta}=0,\quad \frac{\partial \Xi^{r}}{\partial \varphi}=0,\quad \frac{\partial \Xi^{\vartheta}}{\partial \vartheta}=0. \tag{27}
$$

Equation (25) now transforms into

$$
\left(\frac{\partial \Xi^{\vartheta}}{\partial \varphi}+\sin^2\vartheta\frac{\partial \Xi^{\varphi}}{\partial \vartheta}\right)\varphi_{(1)}+\sin\vartheta\left(\cos\vartheta\,\Xi^{\vartheta}+\sin\vartheta\frac{\partial \Xi^{\varphi}}{\partial \varphi}\right)\varphi_{(1)}^2=0,
$$

which implies the following two conditions on Ξ^ϑ and Ξ^φ,

$$\frac{\partial \Xi^\vartheta}{\partial \varphi} + \sin^2 \vartheta \frac{\partial \Xi^\varphi}{\partial \vartheta} = 0, \tag{28}$$

and

$$\cos \vartheta \Xi^\vartheta + \sin \vartheta \frac{\partial \Xi^\varphi}{\partial \varphi} = 0. \tag{29}$$

Since Ξ^ϑ does not depend on ϑ (27), differentiating (28) with respect to ϑ we get

$$2 \cos \vartheta \frac{\partial \Xi^\varphi}{\partial \vartheta} + \sin \vartheta \frac{\partial^2 \Xi^\varphi}{\partial \vartheta^2} = 0. \tag{30}$$

However, equation (30) can be integrated and we get its solution of the form

$$\Xi^\varphi = C_0 \frac{\cos \vartheta}{\sin \vartheta} + C, \tag{31}$$

where C_0 and C are functions of φ. Indeed, we substitute $\xi = \partial \Xi^\varphi / \partial \vartheta$ in (30) and obtain the equation

$$2\xi \cos \vartheta + \sin \vartheta \frac{\partial \xi}{\partial \vartheta} = 0, \tag{32}$$

which has a solution $\xi = -C_0 (1/ \sin^2 \vartheta)$, where $C_0 = C_0(\varphi)$. Integrating this substitution for ξ we obtain (31). From (31) and (32) we get

$$\Xi^\vartheta = \int C_0 d\varphi + C_1, \tag{33}$$

where $C_1 \in \mathbb{R}$. Now we determine the integration constants C, C_0 and C_1 using condition (29). Applying expressions for Ξ^φ (31) and for Ξ^ϑ (33), into (29), we have

$$\left(\int C_0 d\varphi + C_1 + \frac{dC_0}{d\varphi} \right) \cos \vartheta + \frac{dC}{d\varphi} \sin \vartheta = 0,$$

which implies that

$$\frac{dC}{d\varphi} = 0,$$

and

$$\int C_0 d\varphi + C_1 + \frac{dC_0}{d\varphi}. \tag{34}$$

Thus, C does not depend on φ hence it is a constant. Differentiating (34) with respect to φ we get a linear second-order ordinary equation for C_0,

$$\frac{d^2 C_0}{d\varphi^2} + C_0 = 0, \tag{35}$$

the *harmonic oscillation equation* with angular frequency equal to 1. The general solution of (35) reads

$$C_0 = A\cos\varphi + B\sin\varphi, \tag{36}$$

where $A, B \in \mathbb{R}$. Substituting now (36) into the expressions for ξ^φ and Ξ^ϑ (31), (33), we conclude that

$$\Xi^\varphi = (A\cos\varphi + B\sin\varphi)\frac{\cos\vartheta}{\sin\vartheta} + C, \tag{37}$$

and

$$\Xi^\vartheta = A\sin\varphi - B\cos\varphi + C_1. \tag{38}$$

Finally, using (37) and (38), we get from (29) that $C_1 = 0$.

Summarizing, the general solution of the Noether equation (23) for vector field Ξ is of the form

$$\begin{aligned}\Xi &= (A\sin\varphi - B\cos\varphi)\frac{\partial}{\partial\vartheta} + \left((A\cos\varphi + B\sin\varphi)\frac{\cos\vartheta}{\sin\vartheta} + C\right)\frac{\partial}{\partial\varphi} + D\frac{\partial}{\partial r}\\ &= A\Xi_1 + B\Xi_1' + C\Xi_2 + D\Xi_3,\end{aligned}$$

where

$$\begin{aligned}\Xi_1 &= \sin\varphi\frac{\partial}{\partial\vartheta} + \cos\varphi\frac{\cos\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi}, \quad \Xi_2 = \frac{\partial}{\partial\varphi}, \quad \Xi_3 = \frac{\partial}{\partial r},\\ \Xi_1' &= \sin\varphi\frac{\cos\vartheta}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\vartheta},\end{aligned}$$

and $A, B, C, D \in \mathbb{R}$. Since $\Xi_1' = [\Xi_1, \Xi_2]$, we have 3-dimensional Lie algebra of generators of invariance transformations of the given Lepage form, generated by vector fields Ξ_1, Ξ_2 and Ξ_3.

Contracting the Lepage form η by the generators of invariance transformations Ξ_1, Ξ_2 and Ξ_3, we obtain the first integrals (Noether currents),

$$i_{\Xi_1}\eta = \frac{\sin\varphi + \varphi_{(1)}\sin\vartheta\cos\varphi\cos\vartheta}{\sqrt{1 + \varphi_{(1)}^2\sin^2\vartheta}}, \quad i_{\Xi_2}\eta = \frac{\varphi_{(1)}\sin^2\vartheta}{\sqrt{1 + \varphi_{(1)}^2\sin^2\vartheta}}, \quad i_{\Xi_3}\eta = 1. \tag{39}$$

The functions (39) are the *level-set functions* for extremals of the variational principle, given by the Lepage form η. If an extremal is a great circle on a sphere of radius $r = r(\vartheta) = \text{const}$, which lies in a plane given by equation $Pr\sin\vartheta\cos\varphi + Qr\sin\vartheta\sin\varphi + Rr\cos\vartheta = 0$, then along this extremal we obtain

$$i_{\Xi_1}\eta = \frac{\operatorname{sgn}(R)P}{\sqrt{P^2 + Q^2 + R^2}}, \quad i_{\Xi_2}\eta = -\frac{\operatorname{sgn}(R)R}{\sqrt{P^2 + Q^2 + R^2}}, \quad i_{\Xi_3}\eta = 1.$$

On the other hand, every immersion ζ, which is constant along the level-set functions (39), is an extremal for η. Indeed, if

$$\frac{\sin\varphi + \varphi_{(1)}\sin\vartheta\cos\varphi\cos\vartheta}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}} = c_1, \qquad \frac{\varphi_{(1)}\sin^2\vartheta}{\sqrt{1+\varphi_{(1)}^2\sin^2\vartheta}} = c_2, \tag{40}$$

along ζ for some $c_1, c_2 \in \mathbb{R}$, $-1 < c_2 < 1$, then the second equation of (40) coincides with the Euler − Lagrange equation (21) hence ζ is an extremal. Moreover, the first equation of (40) specifies the plane in $\mathbb{R}^3$, containing the extremal. Consequently, in this example the Noether currents do *not* determine the extremal uniquely.

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