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MULTIPLY WARPED PRODUCT ON QUASI-EINSTEIN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we have studied warped products and multiply warped product on quasi-Einstein manifold with semi-symmetric non-metric connection. Then we have applied our results to generalized Robertson-Walker space times with a semi-symmetric non-metric connection.

1. Introduction

Let (M^n, g), $(n > 2)$ be a Riemannian manifold and $U_S = \{x \in M : S \neq \frac{r}{n} g$ at $x\}$, then the manifold (M^n, g) is said to be quasi-Einstein manifold [4, 6] if on $U_S \subset M$, we have

$$S - \alpha g = \beta A \otimes A, \tag{1.1}$$

where A is a 1-form on U_S and α and β some functions on U_S. It is clear that the 1-form A as well as the function β are nonzero at every point on U_S.The scalars α, β are known as the associated scalars of the manifold. Also, the 1-form A is called the associated 1-form of the manifold defined by $g(X, \rho) = A(X)$ for any vector field X, ρ being a unit vector field, called the generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and $f > 0$ is a differential function on B. Consider the product manifold $B \times F$ with its projections $\pi\colon B \times F \to B$ and $\sigma\colon B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any vector field X on M. Thus we have $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M and F the fiber. The function f is called the warping function of the warped product [9].

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The concept of warped products was first introduced by Bishop and O'Neil [3] to construct examples of Riemannian manifold with negative curvature.

Now, we can generalize warped products to multiply warped products. A multiply warped product is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \ldots \oplus b_m^2 g_{F_m}$, where each $i \in \{1, 2, \ldots, m\}$, $b_i \colon B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian manifold. In particular, when $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold. We call M as the multiply generalized Robertson-Walker space-time.

A multiply twisted product (M, g) is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \ldots \oplus b_m^2 g_{F_m}$, where each $i \in \{1, 2, \ldots, m\}$, $b_i \colon B \times F_i \to (0, \infty)$ is smooth.

In 1924, Friedmann and Schouten was introduced the notion of a semi-symmetric linear connection on a differential manifold [5]. The idea of metric connection with torsion on Riemannian manifold has given by Hayden (1932) in [7]. In 1970, Yano [15] was introduced a systematic study of semi-symmetric metric connection on Riemannian manifold. Later K. S. Amur and S. S. Pujar [1], C. S. Bagewadi [2], Sharafuddin and Hussian (1976) [11], S. Sular, C. Özgür [12], M. Tripathi [13] have also studied semi-symmetric metric connection on Riemannian manifold. In [10], S. Sular and C. Özgür has studied warped product on semi-symmetric non-metric connection. Y. Wang has considered multiply warped product with a semi-symmetric non-metric connection, then applied the results to generalized Robertson-Walker space-time in [14].

In this paper, we have considered quasi-Einstein warped product manifolds endowed with semi-symmetric metric non-connection. First we have obtained the necessary and sufficient conditions of quasi-Einstein warped product manifold with semi-symmetric non-metric connection. Next we have established that under certain conditions Robertson-Walker space times would be converted to quasi-Einstein manifold with the above connection. Later we have shown that $(\bar{n}-1)$-dimensional base is isometric to a $(\bar{n}-1)$-dimensional sphere of a particular radius with respect to semi-symmetric non-metric connection. In the last section we have studied special multiply warped product with semi symmetric non-metric connection.

2. Preliminaries

Let (M^n, g) be a Riemannian manifold with the Levi-civita connection ∇. A linear connection $\tilde{\nabla}$ on (M^n, g) is said to be semi-symmetric if its torsion tensor T can be written as

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$

satisfies the condition

$$T(X, Y) = \pi(Y)X - \pi(X)Y,$$

where π is an $1-$ form on M^n with the associated vector field P defined by $\pi(X) = g(X, P)$, for all vector fields $X \in \chi(M^n)$.

A connection $\tilde{\nabla}$ is called semi-symmetric non-metric connection if $\tilde{\nabla} g \neq 0$.

The relation between semi-symmetric non-metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of M^n and it is given by [14]

$$\tilde{\nabla}_X Y = \nabla_X Y + \pi(Y)X, \tag{2.1}$$

where $g(X, P) = \pi(X)$.

Further, a relation between the curvature tensors R and $\tilde{R}$ of type (1,3) of the connections ∇ and $\tilde{\nabla}$ respectively is given by [14],

$$\begin{aligned} \tilde{R}(X,Y)Z = R(X,Y)Z + g(Z, \nabla_X P)Y - g(Z, \nabla_Y P)X \\ + \pi(Z)[\pi(Y)X - \pi(X)Y], \end{aligned} \tag{2.2}$$

for any vector field X, Y, Z on M^n.

3. Generalized Robertson-Walker Space-times with a Semi-Symmetric Non-Metric Connection

In this section we have considered quasi-Einstein warped product manifolds with respect to semi-symmetric non-metric connection. Now, we have proved the following theorem.

Theorem 3.1. *Let (M, g) be a warped product $I \times_f F$ where I is an open interval in $\mathbb{R}$, $\dim I = 1$ and $\dim F = \bar{n} - 1$, $(\bar{n} \geq 3)$. Then (M, g) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection iff F is quasi-Einstein manifold for $P \in \chi(B)$ with respect to the Levi-Civita connection or the warping function f is a constant on I for $P \in \chi(F)$.*

Proof. Assume that $P \in \chi(B)$ and let g_I be the metric on I. Taking $f = e^{\frac{q}{2}}$ and by using the proposition use of [10] we get

$$\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\frac{\bar{n}-1}{4}[2q'' + (q')^2 - 4]g_I(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}), \tag{3.1}$$

$$\tilde{S}(\frac{\partial}{\partial t}, V) = 0, \tag{3.2}$$

$$\tilde{S}(V, W) = S^F(V, W) + e^q[\frac{\bar{n}-1}{2}q' + \frac{q''}{2} - \frac{\bar{n}-3}{4}(q')^2]g_F(V, W), \tag{3.3}$$

for any vector field V, W on F.

Since, M is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection, we have

$$\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \alpha g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) + \beta\eta(\frac{\partial}{\partial t})\eta(\frac{\partial}{\partial t}),$$

and

$$\tilde{S}(V, W) = \alpha g(V, W) + \beta\eta(V)\eta(W),$$

Then the last equations reduce to

$$\tilde{S}(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = \alpha g_I(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) + \beta\eta(\frac{\partial}{\partial t})\eta(\frac{\partial}{\partial t}), \tag{3.4}$$

and

$$\tilde{S}(V,W) = \alpha e^q g_F(V,W) + \beta\eta(V)\eta(W). \tag{3.5}$$

Decomposing the vector field U uniquely into its components U_I and U_F on I and F, respectively, then we have $U = U_I + U_F$. Since $dim I = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = \upsilon\frac{\partial}{\partial t} + U_F$, where υ is a function on M. Then we can write

$$\eta(\frac{\partial}{\partial t}) = g(U,\frac{\partial}{\partial t}) = \upsilon. \tag{3.6}$$

Using the equations (3.6), the equations (3.4), (3.5) reduce to

$$\tilde{S}(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) = \alpha + \beta\upsilon^2, \tag{3.7}$$

and

$$\tilde{S}(V,W) = \alpha e^q g_F(V,W) + \beta\eta(V)\eta(W). \tag{3.8}$$

Comparing the right hand sides of (3.1) and (3.7) we get,

$$\alpha + \beta\upsilon^2 = -\frac{\bar{n}-1}{4}[2q'' + (q')^2 - 4]. \tag{3.9}$$

Similarly comparing the right hand sides of (3.3) and (3.8) we obtain

$$S^F(V,W) = e^q[\alpha + \frac{\bar{n}-3}{4}(q')^2 - \frac{(\bar{n}-1)}{2}q' - \frac{q''}{2}]g_F(V,W) + \beta\eta(V)\eta(W), \tag{3.10}$$

which gives that F is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(B)$.

Now taking $P \in \chi(F)$ and by use of [10] we get,

$$\tilde{S}(\frac{\partial}{\partial t}, V) = (\bar{n}-1)\frac{q'}{2}\pi(V)g_I(\frac{\partial}{\partial t},\frac{\partial}{\partial t}) \tag{3.11}$$

and

$$\tilde{S}(V,\frac{\partial}{\partial t}) = (1-\bar{n})\frac{q'}{2}\pi(V)g_I(\frac{\partial}{\partial t},\frac{\partial}{\partial t}), \tag{3.12}$$

for any vector field $V \in \chi(F)$.

Since M is a quasi-Einstein manifold, we have

$$\tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V,\frac{\partial}{\partial t}) = \alpha g(V,\frac{\partial}{\partial t}) + \beta\eta(V)\eta(\frac{\partial}{\partial t}). \tag{3.13}$$

Now $g(V,\frac{\partial}{\partial t}) = 0$ as $\frac{\partial}{\partial t} \in \chi(B)$ and $V \in \chi(F)$.

Hence from the last equation we get

$$\tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V,\frac{\partial}{\partial t}) = \beta\eta(V)\eta(\frac{\partial}{\partial t}). \tag{3.14}$$

Therefore we have

$$\eta(V)\eta(\frac{\partial}{\partial t}) = (\bar{n}-1)\frac{q'}{2}\pi(V)g_I(\frac{\partial}{\partial t},\frac{\partial}{\partial t}), \tag{3.15}$$

$$\eta(V)\eta(\frac{\partial}{\partial t}) = (1-\bar{n})\frac{q'}{2}\pi(V)g_I(\frac{\partial}{\partial t},\frac{\partial}{\partial t}). \tag{3.16}$$

Comparing from (3.15), (3.16) we get

$$q' = 0.$$

Hence, q is constant. Therefore f is constant. □

Now, we consider the warped product $M = B \times_f I$ with $\dim B = \bar{n} - 1$, $\dim I = 1$ ($\bar{n} \geq 3$). Under this assumption we have obtained the following theorem.

Theorem 3.2. *Let (M, g) be a warped product $B \times_f I$, where $dimI = 1$ and $dimB = \bar{n} - 1$ ($\bar{n} \geq 3$).*

i) *If (M, g) is a quasi-Einstein manifold with scalars α, β respect to the semi-symmetric non-metric connection, $P \in \chi(B)$ is parallel on B with respect to the Levi-Civita connection on B and f is a constant on B, then $\alpha = 0$.*
ii) *If (M, g) is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection for $P \in \chi(F)$, then f is a constant on B.*
iii) *If f is a constant on B and B is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(F)$, then M is an quasi-Einstein manifold with respect to the semi-symmetric non-metric connection.*

Proof. Assume that (M, g) is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection. Then we write

$$\tilde{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{3.17}$$

Decomposing the vector field U uniquely into its components U_B and U_I on B and I, respectively, then we have

$$U = U_B + U_I. \tag{3.18}$$

Since $\dim I = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = \upsilon\frac{\partial}{\partial t} + U_B$, where υ is a function on M. From (3.17), (3.18) and from the proposition of [10], we have,

$$\tilde{S}^B(X,Y) = \alpha g_B(X,Y) + \beta g_B(X,U_B)g_B(Y,U_B) - \frac{H^f(X,Y)}{f} - g(Y,\nabla_X P) + \pi(X)\pi(Y). \tag{3.19}$$

By contraction over X and Y and we get

$$\tilde{r}^B = \alpha(\bar{n}-1) + \beta g_B(U_B,U_B) - \frac{\Delta f}{f} + \pi(P) - \sum_{i=1}^{\bar{n}-1} g(e_i,\nabla_{e_i}P). \tag{3.20}$$

Also from (3.17) we have

$$\tilde{r}^M = \alpha\bar{n} + \beta g_B(U_B, U_B), \tag{3.21}$$

So, by the use of (3.21) in (3.20) we get

$$\tilde{r}^B = \tilde{r}^M - \alpha - \frac{\Delta f}{f} + \pi(P) - \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) \tag{3.22}$$

Also from the proposition of [10] we get

$$\tilde{r}^M = \tilde{r}^B - 2\frac{\Delta f}{f} - \pi(P) + \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) + (\bar{n}-1)\frac{Pf}{f}.$$

Therefore, from above two relation we get

$$\alpha + \frac{\Delta f}{f} - \pi(P) + \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) = -2\frac{\Delta f}{f} - \pi(P) + \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P) + (\bar{n}-1)\frac{Pf}{f}.$$

Since $P \in \chi(B)$ is parallel and f is a constant on B, then we get $\alpha = 0$.

ii) Let $P \in \chi(F)$. By the use of the proposition of [10] we get,

$$\tilde{S}(X, V) = (\bar{n}-1)\pi(V)\frac{Xf}{f}$$

and

$$\tilde{S}(V, X) = (1-\bar{n})\pi(V)\frac{Xf}{f},$$

for any vector field $X \in \chi(B)$ and $V \in \chi(F)$. Since $F = I$, then taking $V = P$ we have

$$\tilde{S}(X, P) = (\bar{n}-1)\pi(P)\frac{Xf}{f}, \tag{3.23}$$

and

$$\tilde{S}(P, X) = (1-\bar{n})\pi(P)\frac{Xf}{f}. \tag{3.24}$$

Since M is a quasi-Einstein manifold, we have

$$\tilde{S}(X, P) = \tilde{S}(P, X) = \alpha g(P, X) + \beta\eta(P)\eta(X).$$

Again we have $g(P, X) = 0$ for $X \in \chi(B)$ and $P \in \chi(F)$.

Hence, we have $Xf = 0$. This implies that f is constant.

iii) Assume that B is a quasi-Einstein manifold with respect to Levi-Civita connection. Then we have

$$S^B(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{3.25}$$

for any vector field X, Y tangent to B.

$$\tilde{S}^M(X, Y) = S^B(X, Y) + \frac{H^f(X, Y)}{f},$$

for any vector field $P \in \chi(F)$. Since f is a constant, then $H^f(X,Y) = 0$ for all $X, Y \in \chi(B)$.

The above equation reduces to

$$\tilde{S}^M(X,Y) = S^B(X,Y). \tag{3.26}$$

By the use of (3.25) in (3.26) we get

$$\tilde{S}^M(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{3.27}$$

which shows that M is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection. □

Next, we consider generalized Robertson-Walker space time with a semi-symmetric non-metric connection. Now we prove the following theorem.

Theorem 3.3. *Let (M, g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, \dim F = l$. Then (M, g) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ with constant associated scalars α and β, if and only if the following conditions are satisfied.*

i) *(F, g_F) is quasi-Einstein manifold with scalar α_F, β_F.*
ii) *$l(1 - \frac{f''}{f}) = \alpha - \upsilon^2\beta$,*
iii) *$\alpha_F + (1-l)f'^2 - \alpha f^2 + f''f + lf'f = 0$ and $\beta = \beta_F$.*

Proof. By the proposition of [10] we have

$$\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = l(\frac{f''}{f} - 1),$$

$$\tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V, \frac{\partial}{\partial t}) = 0,$$

$$\tilde{S}(V,W) = S^F(V,W) + g_F(V,W)\{ff'' - (l-1)f'^2 + lff'\}.$$

Then by the quasi-Einstein condition, we get the theorem 3.3. □

From the theorem 3.3. Putting $\dim F = 1$ we get the following corollary.

Corollary 3.1. *Let (M, g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, dimF = 1$. Then (M, g) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection if and only if*

$$f'' + (\alpha - \upsilon^2\beta - 1)f = 0.$$

By the corollary 3.1. and elementary methods for ordinary differential equations we get

Theorem 3.4. *Let (M, g) be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}, dimF = 1$. Then (M, g) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection if and only if*

i) *when $\alpha - \upsilon^2\beta < 1$, $f(t) = c_1 e^{(\sqrt{1-(\alpha-\upsilon^2\beta)})t} + c_2 e^{-(\sqrt{1-(\alpha-\upsilon^2\beta)})t}$,*
ii) *when $\alpha - \upsilon^2\beta = 1$, $f(t) = c_1 + c_2 t$,*

iii) *when* $\alpha - \upsilon^2\beta > 1$, *we have that* $f(t) = c_1 \cos((\sqrt{\alpha - \upsilon^2\beta - 1})t) + c_2 \sin((\sqrt{\alpha - \upsilon^2\beta - 1})t)$.

Next the following theorem shows when base of quasi-Einstein warped product manifold is isometric to a sphere of a particular radius.

Theorem 3.5. *Let* (M, g) *be a warped product* $B \times_f I$ *connected with* $(\bar{n} - 1)-$*dimensional Riemannian manifold* B *where* $\bar{n} \geq 3$ *and one-dimensional Riemannian manifold* I. *If* (M, g) *is a quasi-Einstein manifold with constant associated scalars* α *and* β, $U \in \chi(M)$ *with respect to semi-symmetric non-metric connection,* $P \in \chi(B)$ *and the Hessian of* f *is proportional to metric tensor* g_B, *then* (B, g_B) *is a* $(\bar{n} - 1)-$*dimensional sphere of radius* $\rho = \frac{\bar{n}-1}{\sqrt{\tilde{r}^B + \alpha}}$.

Proof. Let M be a warped product manifold. Then from the proposition of [10] we have

$$\tilde{S}^M(X, Y) = \tilde{S}^B(X, Y) + \left[\frac{H^f(X, Y)}{f} + g(\nabla_X P, Y) - \pi(X)\pi(Y)\right], \tag{3.28}$$

for any vector field X, Y on B. Since M is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection, we have

$$\tilde{S}^M(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y). \tag{3.29}$$

Decomposing the vector field U uniquely into its components U_B and U_I on B and I, respectively, then we have

$$U = U_B + U_I. \tag{3.30}$$

Since $\dim I = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = \upsilon\frac{\partial}{\partial t} + U_B$, where υ is a function on M. Putting the value of (3.29), (3.30) in (3.28) we get

$$\begin{aligned}\tilde{S}^B(X, Y) = {} & \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) \\ & - \left[\frac{H^f(X, Y)}{f} + g(\nabla_X P, Y) - \pi(X)\pi(Y)\right].\end{aligned} \tag{3.31}$$

By contraction over X and Y we get,

$$\tilde{r}^B = \tilde{r}^M - \alpha - \frac{\Delta f}{f} + \pi(P) - \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla_{e_i} P). \tag{3.32}$$

Again from the proposition of [10] we get

$$\frac{\tilde{r}^M}{\bar{n}} = (\bar{n} - 1)\frac{Pf}{f} - \frac{\Delta f}{f}. \tag{3.33}$$

From the last two equations we get

$$(\tilde{r}^B + \alpha)f = \bar{n}(\bar{n} - 1)Pf - (\bar{n} + 1)\Delta f + f\pi(P) - \sum_{i=1}^{\bar{n}-1} fg(e_i, \nabla_{e_i} P). \tag{3.34}$$

Hence we get

$$(3.35)\qquad \frac{(\tilde{r}^B+\alpha)f}{\bar{n}(\bar{n}-1)} = Pf - \frac{(\bar{n}+1)\Delta f}{\bar{n}(\bar{n}-1)} + \frac{f\pi(P)}{\bar{n}(\bar{n}-1)} - \sum_{i=1}^{\bar{n}-1} \frac{fg(e_i, \nabla_{e_i}P)}{\bar{n}(\bar{n}-1)}.$$

Since, the Hessian of f is proportional to metric tensor g_B, then we have

$$(3.36)\quad H^f(X,Y) = \frac{\bar{n}}{\bar{n}-1}\left[-Pf + \frac{(\bar{n}+1)\Delta f}{\bar{n}(\bar{n}-1)} - \frac{f\pi(P)}{\bar{n}(\bar{n}-1)} + \sum_{i=1}^{\bar{n}-1}\frac{fg(e_i,\nabla_{e_i}P)}{n(n-1)}\right] g_B(X,Y).$$

Hence from the equations (3.35), (3.36) we get

$$(3.37)\qquad H^f(X,Y) + \frac{\tilde{r}^B+\alpha}{(\bar{n}-1)^2} f g_B(X,Y) = 0.$$

So, B is isometric to the $(\bar{n}-1)-$dimensional sphere of radius $\frac{\bar{n}-1}{\sqrt{\tilde{r}^B+\alpha}}$ [8]. Thus the theorem is proved. □

4. Special Multiply Warped Product Manifolds with Semi-Symmetric Non-Metric Connection

Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and I is an open interval in $\mathbb{R}$ and $b_i \in C^\infty(I)$.

Now, we prove the following theorem for multiply generalized Robertson-Walker space time.

Theorem 4.1. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ and $P = \frac{\partial}{\partial t}$. Then (M, g) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ with constant associated scalars α and β, if and only if the following conditions are satisfied.*

i) *(F_i, g_{F_i}) is quasi-Einstein manifold with scalar $\alpha_{F_i}, \beta_{F_i}$, $i \in \{1, 2, \ldots, m\}$,*

ii) $\displaystyle\sum_{i=1}^{m} l_i(1 - \frac{b_i''}{b_i}) = \alpha - \upsilon^2\beta,$

iii) $\displaystyle\alpha b_i^2 - \alpha_{F_i} + b_i b_i'' + (l_i - 1){b_i'}^2 + b_i b_i' \sum_{j\neq i} l_j \frac{b_j'}{b_j'} - b_i^2 \sum_{j=1}^{m} l_j \frac{b_j'}{b_j'} = 0$ *and* $\beta = \beta_{F_i}$.

Proof. By the proposition of [14] we have

$$(4.1)\qquad \tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\sum_{i=1}^{m} l_i(1 - \frac{b_i''}{b_i}),$$

$$(4.2)\qquad \tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V, \frac{\partial}{\partial t}) = 0,$$

(4.3) $\tilde{S}(V,W) =$

$$S^{F_i}(V,W) + g_{F_i}(V,W)\{-b_i b_i'' - (l_i - 1){b_i'}^2 - b_i b_i' \sum_{j\neq i} l_j \frac{b_j'}{b_j} + b_i^2 \sum_{j=1}^{m} l_j \frac{b_j'}{b_j'}.$$

Since, M is a quasi-Einstein manifold. So,

$$\tilde{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y).$$

Now,

$$\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \alpha g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) + \beta\eta(\frac{\partial}{\partial t})\eta(\frac{\partial}{\partial t}).$$

Decomposing the vector field U uniquely into its components U_I and U_F on I and F, respectively, then we have $U = U_I + U_F$. Since $dimI = 1$, we can take $U_I = \upsilon\frac{\partial}{\partial t}$ which gives $U = \upsilon\frac{\partial}{\partial t} + U_F$, where υ is a function on M. Then we can write

$$\eta(\frac{\partial}{\partial t}) = g(U, \frac{\partial}{\partial t}) = \upsilon. \tag{4.4}$$

Hence, we get

$$\sum_{i=1}^{m} l_i(1 - \frac{b_i''}{b_i}) = \alpha - \upsilon^2\beta.$$

Again, $\tilde{S}(V,W) = \alpha g(V,W) + \beta\eta(V)\eta(W)$.

From by the proposition of [14] and the equation (4.3) we get (F_i, g_{F_i}) is quasi-Einstein manifold.

Also, after some calculation we can show that

$$\alpha b_i^2 - \alpha_{F_i} + b_i b_i'' + (l_i - 1){b_i'}^2 + b_i b_i' \sum_{j\neq i} l_j \frac{b_j'}{b_j'} - b_i^2 \sum_{j=1}^{m} l_j \frac{b_j'}{b_j'} = 0$$

and $\beta = \beta_{F_i}$. □

Next, we have obtained the following theorem with some condition of fibre and warping function with semi-symmetric non-metric connection.

Theorem 4.2. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $-dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ with $P \in \chi(F_r)$ and $g_{F_r}(P,P) = 1$ and $\bar{n} \geq 3$. Then (M,g) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection $\breve{\nabla}$ with constant associated scalars α and β, if and only if the following conditions are satisfied.*

i) *(F_i, g_{F_i}) $(i \neq r)$ is quasi-Einstein manifold with scalar $\alpha_{F_i}, \beta_{F_i}$, $i \in \{1, 2, \ldots, m\}$.*

ii) $\displaystyle\sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = -\alpha + \upsilon^2\beta.$

iii) $\displaystyle\alpha_{F_i} - b_i b_i'' - (l_i - 1){b_i'}^2 - b_i b_i' \sum_{j\neq i} l_j \frac{b_j'}{b_j'} - \alpha b_i^2 = 0$ *and* $\beta = \beta_{F_i}$.

iv)

$$S^{F_i}(V,W) - g_{F_i}(V,W)[b_i b_i'' + (l_i-1){b_i'}^2 + \alpha b_i^2 + b_i b_i' \sum_{j\neq i} l_j \frac{b_j'}{b_j'}] =$$
$$(\bar{n}-1)[\pi(V)\pi(W) - \frac{g(W,\nabla_V P) + g(V,\nabla_W P)}{2}],$$

for $V,W \in \Gamma(TF_r), r = i.$

Proof. By the proposition of [14] and $g_{F_r}(P,P) = 1$, we have that b_r is constant. So, we have

$$\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = -\alpha + \upsilon^2\beta.$$

By variables separation, we have

$$\tilde{S}(V,W) = S^{F_i}(V,W) + b_i^2 g_{F_i}(V,W)[-\frac{b_i''}{b_i} - (l_i-1)\frac{{b_i}'^2}{b_i^2} - \sum_{j\neq i} l_j \frac{b_i' b_j'}{b_i b_j}]$$
$$+ (\bar{n}-1)[g(W,\nabla_V P) - \pi(V)\pi(W)].$$

When $i \neq r$, then $\pi(V) = \nabla_V P = \nabla_W P = 0$.

$$\tilde{S}(V,W) = S^{F_i}(V,W) + b_i^2 g_{F_i}(V,W)[-\frac{b_i''}{b_i} - (l_i-1)\frac{{b_i}'^2}{b_i^2} - \sum_{j\neq i} l_j \frac{b_i' b_j'}{b_i b_j}]$$
$$= \alpha b_i^2 g_F(V,W) + \beta\eta(V)\eta(W).$$

By variables separation, we have (F_i, g_{F_i}) $(i \neq r)$ is quasi-Einstein manifold with scalar $\alpha_{F_i}, \beta_{F_i}$, $i \in \{1,2,\ldots,m\}$.

When $i = r$, we get

$$S^{F_i}(V,W) - g_{F_i}(V,W)[b_i b_i'' + (l_i-1){b_i'}^2 + \alpha b_i^2 + b_i b_i' \sum_{j\neq i} l_j \frac{b_j'}{b_j'}] =$$
$$(\bar{n}-1)[\pi(V)\pi(W) - \frac{g(W,\nabla_V P) + g(V,\nabla_W P)}{2}]. \quad \square$$

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