Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
33 (2017), 207–213
www.emis.de/journals
ISSN 1786-0091

ON A HYPERBOLIC KAEHLERIAN SPACE

B. B. CHATURVEDI AND B. K. GUPTA

Abstract. The object of the present paper is to study some curvature properties in a hyperbolic Kaehlerian manifold equipped with quarter-symmetric metric connection.

1. Introduction

Hyperbolic Kaehlerian manifold has been studied by different differential geometer through different approaches. Nevena Pušić [5] studied hyperbolic Kaehlerian space equipped with quarter-symmetric metric connection. In 1985, G. Ganchev and A. Borisov [3] discussed the isotropic sections and curvature properties of hyperbolic Kaehlerian manifolds. Nevena Pušić [6] discussed HB-parallel hyperbolic Kaehlerian spaces. In 2013, Arif Salimov and S. Turanli [1] has been discussed some curvature properties of anti-Kaehler-codazzi manifold. Recently, hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection has been studied by B.B. Chaturvedi and B.K. Gupta [2] in 2015. In the consequences of these studies, in this paper we have studied some curvature properties of a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection.

If F_i^h satisfies the relation

$$F_j^i\, F_i^h = \delta_j^h, \tag{1.1}$$

$$F_{i\,j} = -F_{j\,i}, \quad (F_{i\,j} = g_{j\,k}\, F_i^k), \tag{1.2}$$

and

$$F_{i,j}^h = 0, \tag{1.3}$$

then the manifold is called hyperbolic Kaehlerian (space) manifold.

Where F_i^h is a tensor field of type (1,1) and $F_{i,j}^h$ is a covariant derivative of F_i^h with respect to Riemannian connection.

2010 *Mathematics Subject Classification.* 32Q60; 32Q15.
Key words and phrases. Riemannian manifold, quarter-symmetric metric connection, hyperbolic Kaehlerian manifold, Nijenhuis tensor.

In 1975 S. Global [7] defined

Definition 1.1. A linear connection ∇ on a n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection if the torsion tensor T, defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.4}$$

of the connection ∇, satisfies

$$T(X,Y) = \eta(X)\phi Y - \eta(Y)\phi X, \tag{1.5}$$

where η is a 1-form and ϕ is a tensor field of type (1,1) .

A quarter-symmetric connection ∇ is said to be a quarter-symmetric metric connection if the covariant derivative of metric g vanishes otherwise it is called a quarter-symmetric non-metric connection.

Yano and Imai [4] considered a quarter-symmetric metric connection ∇ and Riemannian connection D with coefficients $\Gamma^h_{i\,j}$ and $\{^{\,h}_{i\,j}\}$ respectively. According to them if the torsion tensor T of the connection ∇ on (M^n, g), $(n > 2)$, satisfies

$$T^i_{j\,k} = p_j A^i_k - p_k\, A^i_j, \tag{1.6}$$

then the relation between the coefficients of quarter-symmetric metric connection ∇ and Riemannian connection D is given by

$$\Gamma^i_{j\,k} = \{^{\,i}_{j\,k}\} - p_k\, U^i_j + p_j\, V^i_k - p^i\, V_{j\,k}, \tag{1.7}$$

where

$$U_{i\,j} = \frac{1}{2}(A_{i\,j} - A_{j\,i}), \quad V_{i\,j} = \frac{1}{2}(A_{i\,j} + A_{j\,i}), \tag{1.8}$$

where $\nabla g = 0$ and p_i are the components of 1-form . Also A^i_j denotes the components of a tensor of type (1,1). $U_{i\,j}$ and $V_{i\,j}$ are covariant skew symmetric and symmetric tensors respectively.

Equation (1.8) implies

$$A_{i\,j} = U_{i\,j} + V_{i\,j}. \tag{1.9}$$

Nevena Pušić [6] found a relation between $\Gamma^h_{i\,j}$ and $\{^{\,h}_{i\,j}\}$ by putting $V_{i\,j} = g_{i\,j}$ and $U_{i\,j} = F_{i\,j}$ in (1.7), given by

$$\Gamma^i_{j\,k} = \{^{\,i}_{j\,k}\} - p_k\, F^i_j + p_j\, \delta^i_k - p^i\, g_{j\,k}. \tag{1.10}$$

where $\omega^h = \omega_t g^{t\,h}$, ω^h is a contravariant components of generating vector w_h.

Also, Nevena Pušić [6] found a relation between curvature tensor with respect to a quarter-symmetric metric connection ∇ and a Riemannian connection D given by

$$\begin{aligned}\overline{R}_{i\,j\,k\,h} =& R_{i\,j\,k\,h} - g_{i\,h} p_{k\,j} + g_{i\,k}\, p_{h\,j} - g_{j\,k}\, p_{h\,i} + g_{h\,j}\, p_{k\,i} \\ &+ p_j\, p_h\, F_{i\,k} + p_i\, p_k\, F_{j\,h} - p_j\, p_k\, F_{i\,h} - p_i\, p_h\, F_{j\,k},\end{aligned} \tag{1.11}$$

where

$$p_{jk} = \nabla_j p_k - p_j p_k + p_k q_j + \frac{1}{2} p_s p^s g_{jk}. \tag{1.12}$$

Taking covariant derivative of F_i^h with respect to quarter-symmetric metric connection ∇ and Riemannian connection D, we have

$$\nabla_k F_i^h = \partial_k F_i^h + F_i^r \Gamma_{rk}^h - F_r^h \Gamma_{ik}^r, \tag{1.13}$$

and

$$D_k F_i^h = \partial_k F_i^h + F_i^r \{ {}^{\,h}_{rk} \} - F_r^h \{ {}^{\,r}_{ik} \}. \tag{1.14}$$

Subtracting (1.14) from (1.13), we have

$$\nabla_k F_i^h - D_k F_i^h = F_i^r (\Gamma_{rk}^h - \{ {}^{\,h}_{rk} \}) - F_r^h (\Gamma_{ik}^r - \{ {}^{\,r}_{ik} \}). \tag{1.15}$$

Using (1.10) in (1.15), we have

$$\nabla_k F_i^h - D_k F_i^h = F_i^r (-p_k F_r^h + p_r \delta_k^h - p^h g_{rk}) - F_r^h (-p_k F_i^r + p_i \delta_k^r - p^r g_{ik}). \tag{1.16}$$

Using (1.1) and (1.2) in (1.16), we have

$$\nabla_k F_i^h = D_k F_i^h. \tag{1.17}$$

Again taking covariant derivative of (1.13) with respect to quarter-symmetric metric connection, we get

$$\begin{aligned} \nabla_j \nabla_k F_i^h =& \partial_j \partial_k F_i^h - \partial_r F_i^h \Gamma_{jk}^r - \partial_k F_r^h \Gamma_{ij}^r \\ & + \partial_k F_i^r \Gamma_{rj}^h + (\partial_j F_i^r + F_i^m \Gamma_{mj}^r - F_m^r \Gamma_{ij}^m) \Gamma_{rk}^h \\ & + F_i^r \nabla_j \Gamma_{rk}^h - (\partial_j F_r^h + F_r^m \Gamma_{mj}^h - F_m^h \Gamma_{rj}^m) \Gamma_{ik}^r - F_r^h \nabla_j \Gamma_{ik}^r. \end{aligned} \tag{1.18}$$

Interchanging j and k in equation (1.18), we get

$$\begin{aligned} \nabla_k \nabla_j F_i^h =& \partial_k \partial_j F_i^h - \partial_r F_i^h \Gamma_{jk}^r - \partial_j F_r^h \Gamma_{ik}^r \\ & + \partial_j F_i^r \Gamma_{rk}^h + (\partial_k F_i^r + F_i^m \Gamma_{mk}^r - F_m^r \Gamma_{ik}^m) \Gamma_{rj}^h \\ & + F_i^r \nabla_k \Gamma_{rj}^h - (\partial_k F_r^h + F_r^m \Gamma_{mk}^h - F_m^h \Gamma_{rk}^m) \Gamma_{ij}^r - F_r^h \nabla_k \Gamma_{ij}^r. \end{aligned} \tag{1.19}$$

Subtracting (1.18) from (1.19), we get

$$\begin{aligned} \nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h =& F_i^m (\Gamma_{mk}^r \Gamma_{rj}^h - \Gamma_{mj}^r \Gamma_{rk}^h + \nabla_k \Gamma_{mj}^h - \nabla_j \Gamma_{mk}^h) \\ & - F_r^h (\Gamma_{mk}^r \Gamma_{ij}^m - \Gamma_{mj}^r \Gamma_{ik}^m + \nabla_j \Gamma_{ik}^r - \nabla_k \Gamma_{ij}^r). \end{aligned} \tag{1.20}$$

Equation (1.20) implies

$$\nabla_k \nabla_j F_i^h - \nabla_j \nabla_k F_i^h = \overline{R}_{kjm}^{\,h} F_i^m - \overline{R}_{kji}^{\,r} F_r^h. \tag{1.21}$$

2. Twin anti-Hermitian metric

A skew symmetric tensor ω defined by

$$\omega(Y,Z) = g(FY,Z), \tag{2.1}$$

is said to be a killing-yano tensor if

$$(D_X\omega)(Y,Z) + (D_Y\omega)(X,Z) = 0. \tag{2.2}$$

The twin anti-Hermitian metric G defined by

$$G(Y,Z) = g(FY,Z), \tag{2.3}$$

where $G(Y,Z) = G(Z,Y)$.

Since G is symmetric but 2-form ω is not symmetric so the killing-yano equation (2.2) has no immediate meaning. Therefore, we can change the killing-yano equation by Codazzi equation

$$(D_X\, G)(Y,Z) - (D_Y\, G)(X,Z) = 0. \tag{2.4}$$

Equation (2.4) equivalent to

$$(D_X\, F)Y - (D_Y\, F)X = 0. \tag{2.5}$$

3. Curvature properties

We know that for Riemannian connection D

$$D_k D_j\, F_i^h - D_j D_k\, F_i^h = R^h_{k\,j\,m}\, F_i^m - R^r_{k\,j\,i}\, F_r^h. \tag{3.1}$$

Now subtracting (1.21) from (3.1), we get

(3.2)
$$(D_k D_j\, F_i^h - \nabla_k \nabla_j\, F_i^h) - (D_j D_k\, F_i^h - \nabla_j \nabla_k\, F_i^h) = R^h_{k\,j\,m}\, F_i^m - R^r_{k\,j\,i}\, F_r^h - \overline{R}^h_{k\,j\,m}\, F_i^m + \overline{R}^r_{k\,j\,i}\, F_r^h.$$

After contracting (3.2) by h and k and using (1.3) and (1.17), we get

$$(D_h D_j\, F_i^h - \nabla_h \nabla_j\, F_i^h) = S_{j\,m}\, F_i^m - R^r_{h\,j\,i}\, F_r^h - \overline{S}_{j\,m}\, F_i^m + \overline{R}^r_{h\,j\,i}\, F_r^h. \tag{3.3}$$

Equation (3.3) can be written as

$$(D_h D_j\, F_i^h - \nabla_h \nabla_j\, F_i^h) = S_{j\,m}\, F_i^m - R_{h\,j\,i\,l}\, g^{r\,l}\, F_r^h - \overline{S}_{j\,m}\, F_i^m + \overline{R}_{h\,j\,i\,l}\, g^{r\,l}\, F_r^h. \tag{3.4}$$

Using $g^{r\,l}\, F_r^h = G^{h\,l}$ in (3.4), we get

$$(D_h D_j\, F_i^h - \nabla_h \nabla_j\, F_i^h) = S_{j\,m}\, F_i^m - R_{h\,j\,i\,l}\, G^{h\,l} - \overline{S}_{j\,m}\, F_i^m + \overline{R}_{h\,j\,i\,l}\, G^{h\,l}. \tag{3.5}$$

In 2013, Arif Salimov and S. Turanli [6] defined

$$H_{j\,i} = R_{h\,j\,i\,l}\, G^{h\,l}. \tag{3.6}$$

Now we are taking

$$\overline{H}_{j\,i} = \overline{R}_{h\,j\,i\,l}\, G^{h\,l}, \tag{3.7}$$

using (3.6) and (3.7) in (3.5), we have

$$(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - H_{ji} - \overline{S}_{jm} F_i^m + \overline{H}_{ji}, \tag{3.8}$$

where S_{jm} and $\overline{S}_{jm}$ are Ricci tensors with respect to Riemannian connection and quarter-symmetric metric connection respectively and G^{hl} is twin anti-Hermitian metric.

We know that the curvature tensor of type (0,4) with respect to Riemannian connection D satisfies the following relations

$$(i) \quad R_{(hj)il} = 0 \quad \text{and} \quad (ii) \quad R_{hj(il)} = 0. \tag{3.9}$$

Now, equation (3.6) can be written as

$$H_{ji} = \frac{1}{2}(R_{hjil} + R_{ljih})\, G^{lh}. \tag{3.10}$$

Interchanging i and j in (3.10), we get

$$H_{ij} = \frac{1}{2}(R_{hijl} + R_{lijh})\, G^{lh}. \tag{3.11}$$

Subtracting (3.11) from (3.10), we have

$$H_{ji} - H_{ij} = \frac{1}{2}(R_{hjil} + R_{ljih} - R_{hijl} - R_{lijh})\, G^{lh} = 0. \tag{3.12}$$

Equation (3.12) implies that

$$H_{ji} = H_{ij}. \tag{3.13}$$

If we take

$$\begin{aligned} &(i) \quad p_{hi} F_i^h = p_{ih} F_j^h, \\ &(ii) \quad p_j F_{hi} = p_i F_{jh} \\ &(iii) \quad p_i^r g_{hj} = p_j^r g_{hj}, \end{aligned} \tag{3.14}$$

then from (1.11) and (3.7) we can say that $\overline{R}_{hjil}$ will be symmetric in first and last indices.

Therefore we can write

$$\overline{H}_{ji} = \frac{1}{2}(\overline{R}_{hjil} + \overline{R}_{ljih})\, G^{lh}. \tag{3.15}$$

Interchanging i and j in (3.15), we get

$$\overline{H}_{ij} = \frac{1}{2}(\overline{R}_{hijl} + \overline{R}_{lijh})\, G^{lh}. \tag{3.16}$$

Subtracting (3.16) from (3.15), we get

$$\overline{H}_{ji} - \overline{H}_{ij} = \frac{1}{2}(\overline{R}_{hjil} + \overline{R}_{ljih} - \overline{R}_{hijl} - \overline{R}_{lijh})\, G^{lh} = 0, \tag{3.17}$$

equation (3.17) implies that

$$\overline{H}_{ji} = \overline{H}_{ij}. \tag{3.18}$$

Thus we conclude

Theorem 3.1. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection H_{ij} is symmetric with respect to quarter-symmetric metric connection ∇ if equation (3.14) holds.*

Now equation (3.8) can be written as

(3.19)
$$D_h(D_j F_i^h - D_i F_j^h) - \nabla_h(\nabla_j F_i^h - \nabla_i F_j^h) = (S_{jm} F_i^m - H_{ji}) - (S_{im} F_j^m - H_{ij}) + (\overline{S}_{im} F_j^m - \overline{H}_{ij}) - (\overline{S}_{jm} F_i^m - \overline{H}_{ji}),$$

using (1.17) and (2.5) in equation (3.19), we have

$$S_{jm} F_i^m - S_{im} F_j^m = \overline{S}_{jm} F_i^m - \overline{S}_{im} F_j^m. \tag{3.20}$$

Thus we conclude

Theorem 3.2. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection if the Ricci tensor is pure with respect to Riemannian connection then it is also pure with respect quarter-symmetric metric connection if the equation (3.14) holds.*

In 2013, Arif Salimov and S. Turanli [6] defined

$$S^*_{ji} = -H_{jr} F_i^r = -R_{hjrl} G^{lh} F_i^r, \tag{3.21}$$

where S^*_{jr} is *Ricci tensor with respect to Riemannian connection.

Now we are taking

$$\overline{S^*}_{ji} = -\overline{H}_{jr} F_i^r = -\overline{R}_{hjrl} G^{lh} F_i^r, \tag{3.22}$$

where $\overline{S^*}_{ji}$ is *Ricci tensor with respect to quarter-symmetric metric connection.

With the help of equation (1.1), equation (3.21) and (3.22) can be written as

$$S^*_{jr} F_i^r = -H_{ji} \quad \text{and} \quad \overline{S^*}_{jr} F_i^r = -\overline{H}_{ji}, \tag{3.23}$$

using equation (3.23) in (3.8), we have

$$(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = (S_{jr} F_i^r + S^*_{jr} F_i^r) - (\overline{S}_{jm} F_i^m + \overline{S}^*_{jm} F_i^m), \tag{3.24}$$

from (3.24), if

$$D_h D_j F_i^h = \nabla_h \nabla_j F_i^h, \tag{3.25}$$

then, we have

$$S_{jr} F_i^r - \overline{S}_{jm} F_i^m = S^*_{jr} F_i^r - \overline{S}^*_{jm} F_i^m, \tag{3.26}$$

which implies $S_{jr} F_i^r = \overline{S}_{jm} F_i^m$, if only if $\overline{S}^*_{jm} F_i^m = S^*_{jr} F_i^r$.

Thus we conclude:

Theorem 3.3. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection the Ricci tensor with respect to Riemannian connection will be equal to Ricci tensor with respect to quarter-symmetric metrc connection if only if* **Ricci tensor with respect to Riemannian connection be equal to* **Ricci tensor with respect to quarter symmetric metric connection if equations (3.14) and (3.25) hold.*

4. Conclusions

In this paper we have found that in a hyperbolic Kaehlerian manifold the Ricci tensor is pure with respect to quarter-symmetric metric connection if only if it is pure with respect to Riemannian connection with some conditions.

5. Acknowledgments

The second author express his thanks to (UGC) New Delhi, India for providing Junior Research Fellowship (JRF).

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Received October 13, 2016.

Department of Pure & Applied Mathematics,
Guru Ghasidas Vishwavidyalaya Bilaspur (C.G.),
India
E-mail address: brajbhushan25@gmail.com
E-mail address: brijeshggv75@gmail.com