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# QUASI-CONFORMALLY FLAT AND PROJECTIVELY FLAT TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study quasi-conformally flat and projectively flat contact manifolds. We have also studied Quasi-conformally flat and projectively flat trans-Sasakian manifolds. We obtained condition for trans-Sasakian manifold to be quasi-conformally flat and projectively flat. The value of scalar curvature has been obtained in quasi-conformally flat and projectively flat trans-Sasakian manifolds.

## 1. Introduction

Oubina [11] introduced a manifold which generalizes both $\alpha$-Sasakian and $\beta$-Kenmotsu manifolds such manifold was called a trans-Sasakian manifold of type $(\alpha, \beta)$. Sasakian, Kenmotsu, and cosympletic manifold are particular cases of trans-Sasakian manifolds. Trans-Sasakian manifolds of type $(0,0), (\alpha, 0)$ and $(0, \beta)$ are called cosympletic [2], $\alpha$-Sasakian ([3], [14]) and $\beta$-Kenmotsu ([3], [8]) respectively. Concept of nearly trans-Sasakian manifold was introduced by C. Gherghe [7]. Marrero [9] constructed a three dimensional trans-Sasakian manifold. Prasad and Srivastava [12] obtained certain results on trans-Sasakian manifolds. Jeong- Sik kim et al. [7] studied a generalized Ricci-recurrent trans-Sasakian manifolds. A quasi-conformal curvature tensor $\check{C}$ was defined by Yano and Sawaki [17] as follows:

$$\begin{aligned} \check{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1.1}$$

where $a$ and $b$ are constants and $R, S, Q$ and $r$ are the Riemannian curvature-tensor, the Ricci - tensor, the Ricci operator and the scalar curvature of the manifold respectively. A $(2n+1)-$dimensional Riemannian manifold $(M, g)$

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is quasi-Conformally flat if $\check{C} = 0$. Various properties of the quasi-conformal curvature tensor on contact metric manifolds have been studied by several geometers [1, 4, 5, 13] etc. If $a = 1$ and $b = -\frac{1}{2n-1}$, then $\check{C}$ becomes the conformal curvature tensor $C$ which is given by

(1.2) $C(X,Y)Z$

$$= R(X,Y)Z - \frac{1}{2n-1}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}.$$

The Weyl projective curvature tensor $P$ of type $(1,3)$ on a $(2n+1)-$dimensional Riemannian manifold $(M,g)$ is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.3}$$

for any $X, Y, Z \in TM$. The manifold $(M,g)$ is said to be projectively flat if $P$ vanishes identically on $M$. In this paper we study quasi-conformally flat and projectively flat trans-Sasakian manifolds.

## 2. Preliminaries

Let $M$ be a $(2n+1)$- dimensional almost contact metric manifold [1] with almost contact metric structure $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is a compatible Riemannian metric on $M$ such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \tag{2.3}$$

for all $X, Y \in TM$.

An almost contact metric manifold is said to be contact manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y), \tag{2.4}$$

$\Phi(X,Y)$ is being called fundamental $2-$form of $M$.

An almost contact metric manifold $M$ is called trans-Sasakian manifold if

$$(\nabla_X \phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta g\{(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.5}$$

where $\nabla$ is Levi-Civita connection of Riemannian metric $g$ and $\alpha$ and $\beta$ are smooth functions on $M$. From equation (2.5) and equations (2.1), (2.2) and (2.3), we get

$$(\nabla_X \phi)\xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \tag{2.6}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

In a $(2n+1)$- dimensional trans-Sasakian manifold, we have ([6], [3])

$$R(X,Y)\xi = \left(\alpha^2-\beta^2\right)[\eta(Y)X-\eta(X)Y] \\ +2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] \\ +(Y\alpha)\phi X-(X\alpha)\phi Y \\ +(Y\beta)\phi^2X-(X\beta)\phi^2Y, \tag{2.8}$$

$$R(\xi,Y)X = \left(\alpha^2-\beta^2\right)[g(X,Y)\xi-\eta(X)Y] \\ +2\alpha\beta[g(\phi X,Y)\xi-\eta(X)\phi Y] \\ +(X\alpha)\phi Y+g(\phi X,Y)(\text{grad}\alpha) \\ +(X\beta)[Y-\eta(Y)\xi]-g(\phi X,\phi Y)(\text{grad}\beta), \tag{2.9}$$

$$\eta(R(\xi,Y)X) \\ = g(R(\xi,Y)X,\xi) = \left(\alpha^2-\beta^2-\xi\beta\right)[g(X,Y)-\eta(X)\eta(Y)] \tag{2.10}$$

and

$$2\alpha\beta+\xi\alpha=0. \tag{2.11}$$

In a trans-Sasakian manifold, we also have [6]

$$S(X,\xi) = \left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)\eta(X)-(2n-1)X\beta-(\phi X)\alpha \tag{2.12}$$

and

$$Q\xi = \left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)\xi-(2n-1)\,\text{grad}\beta+\phi(\text{grad}\alpha). \tag{2.13}$$

An almost contact metric manifold $M$ is said to be $\eta-$Einstein if its Ricci-tensor $S$ is of the form

$$S(X,Y) = ag(X,Y)+b\eta(X)\eta(Y), \tag{2.14}$$

where $a$ and $b$ are smooth functions on $M$. An $\eta-$Einstein manifold becomes Einstein manifold if $b=0$, i.e

$$S(X,Y) = ag(X,Y). \tag{2.15}$$

If $\{e_1, e_2, ..., e_{2n}, e_{2n+1}=\xi\}$ be a local ortho-normal basis of tangent space in a $(2n+1)-$ dimensional almost contact manifold $M$, then we have

$$\sum_{i=1}^{2n+1} g(e_i,e_i) = (2n+1). \tag{2.16}$$

$$\sum_{i=1}^{2n+1} g(e_i,Y)S(X,e_i) = \sum_{i=1}^{2n+1} R(e_i,Y,X,e_i) = S(X,Y). \tag{2.17}$$

## 3. QUASI-CONFORMALLY FLAT AND PROJECTIVELY FLAT MANIFOLDS

Let $M$ be a $(2n+1)$- dimensional quasi-conformally flat manifold, then from equation (1.1) we have

$$R(X,Y)Z = \frac{b}{a}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX] + \frac{r}{(2n+1)\,a}(\frac{a}{2n} + 2b)[g(Y,Z)X - g(X,Z)Y]. \tag{3.1}$$

Let $\{e_1, e_2, ... e_{2n}, e_{2n+1} = \xi\}$ be a local ortho-normal basis of tangent space. Putting $Y = Z = e_i$, in equation (3.1), we get

$$S(X,W) = \frac{r}{2n+1} g(X,W) \qquad if\ a + (2n-1)\,b \neq 0. \tag{3.2}$$

Hence from equation (3.2), we can state the following theorem:

**Theorem 3.1.** *A quasi-conformally flat manifold is an Einstein manifold if* $a + (2n-1)\,b \neq 0$*. If* $a + (2n-1)\,b = 0$*, then* $\check{C}(X,Y)Z = aC(X,Y)Z$*, or* $\check{C}(X,Y)Z = -(2n-1)\,bC(X,Y)Z$*.*

This leads to:

**Corollary 3.2.** *A quasi-conformally flat manifold is conformally flat if* $a + (2n-1)\,b = 0$ *and* $a \neq 0$ *(equivalently if* $a + (2n-1)\,b = 0$ *and* $b \neq 0$*).*

**Corollary 3.3.** *If* $a + (2n-1)\,b = 0$*, then quasi-conformally curvature becomes constant multiple of conformal curvature tensor.*

If the manifold is projectively flat then from equation(1.3), we have

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{3.3}$$

Putting $Y = Z = e_i$ in equation (3.3), we get

$$S(X,W) = \frac{1}{2n}\left[rg(X,W) - S(X,W)\right], \tag{3.4}$$

$$S(X,W) = \frac{r}{2n+1} g(X,W). \tag{3.5}$$

Hence a projectively flat manifold is an Einstein manifold.

## 4. Quasi-conformally flat trans-Sasakian manifolds

If a trans-Sasakian manifold is quasi-conformally flat then from equation (1.1) we have

$$
\begin{aligned}
(4.1)\qquad R(X,Y,Z,W) = {} & \frac{b}{a}[S(X,Z)g(Y,W) \\
& -S(Y,Z)g(X,W) + g(X,Z)S(Y,W) \\
& -g(Y,Z)S(X,W)] + \frac{r}{(2n+1)a}(\frac{a}{2n}+2b)[g(Y,Z)g(X,W) \\
& -g(X,Z)g(Y,W)].
\end{aligned}
$$

On putting $Z=\xi$ in equation (4.1) and using equations (2.3) and (2.12), we get

$$
\begin{aligned}
(4.2)\qquad g(R(X,Y)\xi,W) = {} & \frac{b}{a}[\{\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)\eta(X) \\
& -(2n-1)X\beta-(\phi X)\alpha\}g(Y,W) \\
& -\{\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)\eta(Y) \\
& -(2n-1)Y\beta-(\phi Y)\alpha\}g(X,W) \\
& +\eta(X)S(Y,W)-\eta(Y)S(X,W)] \\
& +\frac{r}{(2n+1)a}(\frac{a}{2n}+2b)\{\eta(Y)g(X,W) \\
& -\eta(X)g(Y,W)\}
\end{aligned}
$$

Again putting $X=\xi$ in equation (4.2) and using equations $(2.1)$, $(2.3)$, $(2.10)$, $(2.11)$ and (2.12), we get

$$
\begin{aligned}
(4.3)\quad S(Y,W) = {} & [\frac{r}{(2n+1)a}(\frac{a}{2n}+2b)-\frac{a}{b}(2n\left(\alpha^2-\beta^2\right)-\xi\beta)]g(Y,W) \\
& +[2\left(2n\left(\alpha^2-\beta^2\right)-\xi\beta\right)+\frac{a}{b}\left(\alpha^2-\beta^2-\xi\beta\right) \\
& -\frac{r}{(2n+1)b}(\frac{a}{2n}+2b)]\eta(Y)\eta(W) \\
& -\{(2n-1)Y\beta-(\phi Y)\alpha\}\eta(W) \\
& -\{((2n-1)W\beta-(\phi W)\alpha)\}\eta(Y).
\end{aligned}
$$

From equations (3.2) and (4.3), we get

$$(4.4)\qquad r = 2n(2n+1)\left(\alpha^2-\beta^2-\xi\beta\right),$$

and

$$(4.5)\qquad (2n-1)\,\mathrm{grad}\,\beta-\phi(\mathrm{grad}\,\alpha) = (2n-1)(\xi\beta)\xi.$$

This leads to:

**Theorem 4.1.** *A trans-Sasakian manifold can not be quasi-conformally flat unless* $(2n-1)\,\mathrm{grad}\,\beta-\phi(\mathrm{grad}\,\alpha)-(2n-1)(\xi\beta)\xi$ *is zero.*

From the equations (4.4) and (4.5), we have the following:

**Corollary 4.2.** *If a trans-Sasakian manifold is quasi-conformally flat then scalar curvature tensor* $r = 2n\,(2n+1)\,(\alpha^2 - \beta^2 - \xi\beta)$*, where* $\alpha$ *and* $\beta$ *are related by* $(2n-1)\,\mathrm{grad}\,\beta - \phi\,(grad\alpha) = (2n-1)\,(\xi\beta)\,\xi$.

## 5. Projectively flat trans-Sasakian manifolds

If a trans-Sasakian manifold is projectively flat then from equation (1.3), we have

$$R(X,Y,Z,W) = g\,(R(X,Y)\,Z,W) = \frac{1}{2n}[S(Y,Z)g\,(X,W) - S(X,Z)g\,(Y,W)]. \tag{5.1}$$

On putting $W = \xi$ in equation (5.1) and using equations (2.3) and (2.12), we get

$$\eta\,((R(X,Y)\,Z) = \frac{1}{2n}[S(Y,Z)\eta\,(X) - S(X,Z)\eta\,(Y)]. \tag{5.2}$$

Again putting $X = \xi$ in equation (5.2) and using equations $(2.1)\,,(2.3)\,,(2.10)\,,(2.11)$ and (2.12), we get

$$S\,(Y,Z) = 2n\left(\alpha^2 - \beta^2 - \xi\beta\right)g\,(Y,Z) + (2n-1)\,(\xi\beta)\,\eta\,(Y)\,\eta\,(Z) + ((2n-1)\,(Z\beta) + (\phi Z)\,\alpha)\,\eta\,(Y)\,.$$

From equation (3.5) and (5.3), we get

$$r = 2n\,(2n+1)\left(\alpha^2 - \beta^2 - \xi\beta\right), \tag{5.3}$$

and

$$(2n-1)\,(d\beta - \xi\,(\beta)\,\eta) + d\alpha o\phi = 0. \tag{5.4}$$

This leads to:

**Theorem 5.1.** *A trans-Sasakian manifold can not be projectively flat unless* $(2n-1)\,(d\beta - (\xi\beta)\,\eta) + d\alpha o\phi$ *is zero.*

From the equations(5.4) and (5.5), we have the following:

**Corollary 5.2.** *If a trans-Sasakian manifold is projectively flat than scalar curvature tensor* $r = 2n\,(2n+1)\,(\alpha^2 - \beta^2 - \xi\beta)$*, where* $\alpha$ *and* $\beta$ *are related* $(2n-1)\,(d\beta - \xi\,(\beta)\,\eta) + d\alpha 0\phi = 0$.

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