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# DERIVATIONS OF CONVOLUTION ALGEBRAS ON FINITE PERMUTATION SEMIGROUPS

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ABSTRACT. If $n \in \mathbb{N}$ let $S_n$ be the lexicographically ordered discrete semigroup of permutations of $\{1, \ldots, n\}$. Our matter is to seek about the structure and behauviour of derivations of the convolution algebra $l^1(S_n)$. This problem has its own interest even in the finite case and emerges from studies of several kinds of amenability on Banach algebras supported on infinite discrete groups or semigroups.

## 1. INTRODUCTION

The problem of derivations on convolution algebras has a long-standing interest and even today it is a matter of research (among a huge literature the reader can see [2], [11]). Given a semigroup $S$ and $u, v \in S$ we shall write

$$\left[uv^{-1}\right] = \{x \in S : xv = u\} \text{ and } \left[v^{-1}u\right] = \{x \in S : vx = u\}.$$

It is known that if $S$ is discrete and contains an infinite pairwise disjoint sequence of sets $X(u_n) = u_n S \cap [u_n u_n^{-1}]$ then $l^1(S, w)$ is not amenable for any weight function $w$, that is there is always a Banach $l^1(S, w)$-bimodule $\mathcal{H}$ and a non-inner derivation $D_w : l^1(S, w) \to \mathcal{H}^*$ (cf. [5], Theorem 1.). In particular, let $S_{\mathbb{N}}$ be the discrete semigroup of functions of the positive integers into itself and let $\mathbb{P}$ be the set of prime positive integers. Given a fix $p \in \mathbb{P}$ and $n \in \mathbb{N}$ we write $u_p(n) = p^{n(p)}$ if $n = p^{n(p)}m$, with $(m : p) = 1$. We can represent $n = \Pi_{q \in \mathbb{P}} q^{\upsilon_q(n)}$, where $\upsilon_q(n) = \max\{s \in \mathbb{N}_0 : q^s \mid n\}$ for each $q \in \mathbb{P}$. Thus, if $\eta(n) \triangleq p^{\upsilon_p(n)}$ then $\eta u_p = u_p$ and since $u_p$ is idempotent then $u_p \eta \in X(u_p)$. Further, since $u_p \eta u_p = u_p$ then $(u_p \eta)(p^s) = p^s$ for all $s \in \mathbb{N}_0$. Hence it is readily seen that $\{X(u_p)\}$ is an infinite disjoint sequence of non empty subsets of $S_{\mathbb{N}}$ and $l^1(S_{\mathbb{N}}, w)$ is never amenable.

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Now, remember that a semigroup $S$ is called *inverse semigroup* if for all $v \in S$ there exist a unique element $v^{-1} \in S$ so that $vv^{-1}v = v$ and $v^{-1}vv^{-1} = v^{-1}$. Hence, if $S$ is an inverse semigroup with an infinite set $E_S$ of idempotents $l^1(S,w)$ is never amenable (cf. [5], Corollary 1.). Further, combining this result with the investigation in [4] the algebra $l^1(S,w)$ on an inverse semigroup $S$ is amenable if and only if $E_S$ is finite and every subgroup of $S$ is amenable. Certainly, for all $n \in \mathbb{N}$, by finiteness, each group contained in $S_n$ is amenable but not necessarily an inverse semigroup. For instance, let us consider $(1,1,2) \in S_3$, i.e., the function $(1,1,2) : 1 \to 1,\ 2 \to 1,\ 3 \to 2$. Then $(1,1,2)$ has no unique inverse, for

$$(1,1,2)(1,3,1)(1,1,2) = (1,1,2),\ (1,1,2)(1,3,3)(1,1,2) = (1,1,2).$$
$$(1,3,1)(1,1,2)(1,3,1) = (1,3,1),\ (1,3,3)(1,1,2)(1,3,3) = (1,3,3).$$

It is worth mentioning that any derivation on the group algebra of a discrete group is inner (cf. [7], Theorem 4.). A closer look on the structure and properties of derivations on $l^1(S_n)$ will allow us to derive the results described in Subsection 1.2.

1.1. **Notations.** If $n \in \mathbb{N}$ and $g \in S_n$, let $\delta_g = \{\delta_{g,h}\}_{h \in X_n}$, where $\delta_{g,h}$ denotes the usual Kronecker symbol. Clearly $\{\delta_g\}_{g \in S_n}$ is a basis of $l^1(S_n)$ and given $D \in L(l^1(S_n))$ there is a unique subset $\{\lambda_g^h\}_{g,h \in S_n}$ of $\mathbb{C}$ such that $D(\delta_g) = \sum_{h \in S_n} \lambda_g^h \delta_h$ for $g \in S_n$. If $\{\delta^g\}_{g \in S_n}$ denotes the dual basis of $\{\delta_g\}_{g \in S_n}$ then $\lambda_g^h = \langle D(\delta_g), \delta^h \rangle$ for all $g, h \in S_n$. If $\mathfrak{m}, \mathfrak{p} \in l^1(S_n)$ then there is a unique set $\{F_h^g\}_{g,h \in S_n}$ of linear forms on $l^1(S_n)$ so that

$$\mathfrak{m} * \mathfrak{p} = \sum_{g \in S_n} \left[ \sum_{h \in S_n} \mathfrak{m}^h F_h^g(\mathfrak{p}) \right] \delta_g. \tag{1}$$

We shall also write $S_n$ as the disjoint union $S_n = \cup_{k=1}^n S_{n,k}$, where $S_{n,k}$ contains the elements $g \in S_n$ so that $\#\operatorname{Im}(g) = k$. The Jacobson radical of $l^1(S_n)$ is denoted by $J(l^1(S_n))$ and the set of derivations on $l^1(S_n)$ by $\mathcal{Z}(l^1(S_n))$.

1.2. **Our results.** In Section 2 we shall seek about derivations on $l^1(S_n)$. In Theorem 1, (2) we determine the precise conditions on the coefficients $\lambda_g^h$ in order that $D$ be a derivation; (3) will be related to the innerness matter while (4) will describe the beauviour of the transpose of derivations on the forms $F_h^g$. In Proposition 1 it is shown how $l^1(S_n)$ can be antihomomorphically embedded into a subalgebra $\mathfrak{F}$ of $\mathrm{M}_{n^n}(\mathbb{C})$. In Theorem 2 we shall see that $\mathcal{Z}(l^1(S_n))$ is linearly isomorphic to a quotient of the Lie algebra of matrices $\lambda \in \mathrm{M}_{n^n}(\mathbb{C})$ so that $ad_\lambda(\mathfrak{F}) \subseteq \mathfrak{F}$. Here upon in Proposition 2 we will give a complete description of $\mathcal{Z}(l^1(S_2))$. Among other properties, it will seen that any derivation maps onto the explicitly evaluated Jacobson radical. In [8] it is shown that any element in the image of a bounded derivation on a Banach algebra $\mathcal{U}$ so that $[[\mathcal{U},\mathcal{U}],\mathcal{U}] = (0)$ is quasi-nilpotent. In [10] it is proved that any *centralizing* derivation $D$ on a Banach algebra $\mathcal{U}$ maps into the radical. Although those conditions are sufficient by Proposition 2, they are not necessary, even in the finite dimensional context. From Proposition 2 we deduce that

$$\left[\left[\delta_{(2,1)}, \delta_{(2,2)}\right], \delta_{(1,1)}\right] = \mathfrak{q}_0 \neq 0.$$

Further, we know that any non zero derivation $D$ on $l^1(S_2)$ maps onto the radical and is given as $D(\mathfrak{m}) = \langle \mathfrak{m}, \lambda \rangle \mathfrak{q}_0$ for some suitable linear form $\lambda$. But

$$[D(\mathfrak{m}), \mathfrak{m}] = \langle \mathfrak{m}, \lambda \rangle \left(\mathfrak{m}^{(1,1)} + 2\mathfrak{m}^{(2,1)} + \mathfrak{m}^{(2,2)}\right) \mathfrak{q}_0,$$

i.e., $[D(\mathfrak{m}), \mathfrak{m}]$ is not centralizing since $\mathcal{Z}(l^1(S_2)) = \mathbb{C}\delta_{(1,2)}$.

## 2. Derivations on $l^1(S_n)$

The proof of the following theorem is straightforward:

**Theorem 1.**

(i) *$D$ is a derivation if and only if the following identity holds*

$$\lambda_{gh}^{l} = \sum_{k \in [lh^{-1}]} \lambda_g^k + \sum_{k \in [g^{-1}l]} \lambda_h^k \text{ if } l, g, h \in S_n. \tag{2}$$

(ii) *Let $D$ be an inner derivation, say $D = ad_{\mathfrak{m}}$ for some $\mathfrak{m} \in l^1(S_n)$. If $\mathfrak{m} = \sum_{k \in X_n} \mathfrak{m}^k \delta_k$ then*

$$\lambda_g^h = \sum_{k \in [hg^{-1}]} \mathfrak{m}^k - \sum_{k \in [g^{-1}h]} \mathfrak{m}^k. \tag{3}$$

(iii) *If $D$ is a derivation then*

$$D^*\left(F_h^g\right) = \sum_{l \in S_n} (\lambda_l^g F_h^l - \lambda_h^l F_l^g) \text{ if } g, h \in S_n. \tag{4}$$

**Corollary 1.** *(cf.* [1]*, Lemma 1.1. (ii)) If $D \in \mathcal{Z}(l^1(S_n))$ then $\varkappa_0(D(\mathfrak{m})) = 0$, where $\mathfrak{m} \in l^1(S_n)$,, $\varkappa_0$ is the augmentation functional, i.e.,*

$$\varkappa_0(D(\mathfrak{m})) = \langle D(\mathfrak{m}), 1 \rangle.$$

*Proof.* Given $l, h \in S_n$ by (2) we see that

$$\lambda_g^l = \sum_{k \in [lh^{-1}]} \lambda_g^k + \sum_{k \in [g^{-1}l]} \lambda_h^k. \tag{5}$$

As $S_n \circ S_{n,1} \cup S_{n,1} \circ S_n \subseteq S_{n,1}$ if $l \in S_n - S_{n,1}$ then $[g^{-1}l] = \varnothing$. By (5), if we choose $h \in S_{n,1}$ then $[lh^{-1}] = \varnothing$ and $\lambda_g^l = 0$. Now, if $g, h \in S_{n,1}$ we can write

$$\begin{aligned}\sum_{l \in S_{n,1}} \lambda_g^l \delta_l &= D(\delta_g) \\ &= D(\delta_{gh}) \\ &= D(\delta_g * \delta_h) \\ &= D(\delta_g) * \delta_h + \delta_g * D(\delta_h) \\ &= \sum_{l \in S_{n,1}} \lambda_g^l \delta_l + [\sum_{l \in S_{n,1}} \lambda_h^l] \delta_g,\end{aligned}$$

i.e., $\sum_{l \in S_{n,1}} \lambda_h^l = 0$. Now, given $h \in S_n$ we choose any $g \in S_{n,1}$. Now we get

$$\begin{aligned}D(\delta_g) &= D(\delta_g * \delta_h) \\ &= [\sum_{l \in S_{n,1}} \lambda_g^l \delta_l] * \delta_h + \delta_g * \sum_{l \in S_n} \lambda_h^l \delta_l \\ &= D(\delta_g) + [\sum_{l \in S_n} \lambda_h^l] \delta_g\end{aligned}$$

and so $\sum_{l \in S_n} \lambda_h^l = 0$. The general case now follows by simply spanning. □

**Proposition 1.**

(i) *If* $g, h \in S_n$ *then* $F_h^g = \delta^g \delta_h$ *and* $F_h^g = \sum_{i \in [h^{-1}g]} \delta^i$.

(ii) *There is a semigroup isomorphism* $\widehat{n} : S_n \to S_n$ *so that* $\widehat{n} = \widehat{n}^{-1}$ *and*

$$F_h^g(\delta_f) = F_{\widehat{n}(h)}^{\widehat{n}(g)}(\delta_{\widehat{n}(f)}) \tag{6}$$

*for all* $f, g, h \in S_n$.

(iii) *There is an anti-monomorphism* $F : l^1(S_n) \hookrightarrow \mathrm{M}_{n^n}(\mathbb{C})$.

*Proof.* (i): Let $\mathfrak{m} \in l^1(S_n)$, $l \in S_n$. By using (1) we have

$$\mathfrak{m} * \delta_l = \sum_{k \in S_n l} \left[ \sum_{f \in [kl^{-1}]} \mathfrak{m}^f \right] \delta_k = \sum_{k \in S_n} \left[ \sum_{f \in S_n} \mathfrak{m}^f F_f^k(\delta_l) \right] \delta_k.$$

If $k \notin S_n l$ then $\sum_{f \in S_n} \mathfrak{m}^f F_f^k(\delta_l) = 0$ and we deduce that $F_f^k(\delta_l) = 0$ for $f \in S_n$. If $k \in S_n l$ then

$$\sum_{f \in [kl^{-1}]} \mathfrak{m}^f = \sum_{f \in S_n} \mathfrak{m}^f F_f^k(\delta_l),$$

and since $\mathfrak{m}$ is arbitrary we see that $F_f^k(\delta_l) = 1$ if $fl = k$. Therefore,

$$\begin{aligned} F_h^g(\mathfrak{m}) &= \sum_{f \in S_n} \mathfrak{m}^f F_h^g(\delta_f) \\ &= \sum_{hf=g} \mathfrak{m}^f \\ &= \sum_{f \in [h^{-1}g]} \mathfrak{m}^f \\ &= \sum_{f \in S_n} \mathfrak{m}^f \langle \delta_{hf}, \delta^g \rangle \\ &= \langle \delta_h * \mathfrak{m}, \delta^g \rangle \\ &= \langle \mathfrak{m}, \delta^g \delta_h \rangle . \end{aligned}$$

(ii): Let $\widehat{n}(f)(i) = f(n - i + 1)$, $f \in S_n$ and $i \in \{1, ..., n\}$. It is readily seen that $\widehat{n}$ is a semigroup isomorphism of $S_n$ and $\widehat{n}^{-1} = \widehat{n}$. Moreover, if $\mathfrak{m} \in l^1(S_n)$, we can write

$$\begin{aligned} F_{\widehat{n}(h)}^{\widehat{n}(g)}(\mathfrak{m}) &= \sum_{f \in S_n} \mathfrak{m}^{\widehat{n}(f)} \left\langle \delta_{\widehat{n}(h)\widehat{n}(f)}, \delta^{\widehat{n}(g)} \right\rangle \\ &= \sum_{f \in S_n} \mathfrak{m}^{\widehat{n}(f)} \left\langle \delta_{\widehat{n}(hf)}, \delta^{\widehat{n}(g)} \right\rangle \\ &= \sum_{f \in S_n} \mathfrak{m}^{\widehat{n}(f)} \langle \delta_{hf}, \delta^g \rangle \\ &= F_h^g(\overset{\vee}{n}(\mathfrak{m})), \end{aligned} \tag{7}$$

with $\overset{\vee}{n}(\mathfrak{m}) = \sum_{f \in S_n} \mathfrak{m}^{\widehat{n}(f)} \delta_f$. If $f \in S_n$ we can see that $\overset{\vee}{n}(\delta_f) = \delta_{\widehat{n}(f)}$ and with this fact combined with (7) we get (6).

(iii): Let $F : l^1(S_n) \to \mathrm{M}_{n^n}(\mathbb{C})$ so that $F(\mathfrak{p}) = [F_h^g(\mathfrak{p})]_{g,h \in S_n}$ if $\mathfrak{p} \in l^1(S_n)$, where the upper and lower indexes denote rows and columns, respectively. Then $F$ is clearly linear. If $\mathfrak{m}, \mathfrak{p} \in l^1(S_n)$ and $g, h \in S_n$, by (1), we have $\delta_h * \mathfrak{m} = \sum_{k \in S_n} F_h^k(\mathfrak{m}) \delta_k$. Further,

$$\delta_h * \mathfrak{m} * \mathfrak{p} = \sum_{k \in S_n} F_h^k(\mathfrak{m}) \delta_k * \mathfrak{p} = \sum_{k \in S_n} F_h^k(\mathfrak{m}) \sum_{l \in S_n} F_k^l(\mathfrak{p}) \delta_l.$$

Consequently,

$$\begin{aligned} F_h^g(\mathfrak{m} * \mathfrak{p}) &= \langle \mathfrak{m} * \mathfrak{p}, \delta^g \delta_h \rangle \\ &= \langle \delta_h * \mathfrak{m} * \mathfrak{p}, \delta^g \rangle \\ &= \sum_{k \in S_n} F_h^k(\mathfrak{m}) F_k^g(\mathfrak{p}) \\ &= [F(\mathfrak{p}) F(\mathfrak{m})]_h^g . \end{aligned}$$

Since $F(\mathfrak{p}) = [\langle \delta_h * \mathfrak{p}, \delta_g \rangle]_{g,h \in S_n}$, the injectivity of $F$ is immediate. □

*Remark* 1. The anti-monomorphism $F$ of Proposition 1 provides a non trivial re-presentation of $l^1(S_n)$ into $\mathbb{C}^{n^n}$. Therefore, it suffices to consider

$$\varrho : l^1(S_n) \to L\left(\mathbb{C}^{n^n}\right), \varrho(\mathfrak{m})(x) = x \cdot F(\mathfrak{m}).$$

For matrix representations of finite dimensional convolution algebras the reader can see [6].

**Theorem 2.** *Let* $\mathfrak{F} = \mathrm{Im}(F)$ *and* $\mathcal{D}_n = \{\lambda \in \mathrm{M}_{n^n}(\mathbb{C}) : ad_\lambda(\mathfrak{F}) \subseteq \mathfrak{F}\}$. *There is a bijection between* $\mathcal{Z}(l^1(S_n))$ *and the quotient linear space* $\mathcal{D}_n/(\mathcal{D}_n \cap \mathfrak{F}^c)$.

*Proof.* By Theorem 1 (iii) and Proposition 1, if $D \in \mathcal{Z}(l^1(S_n))$ then there is $\lambda \in \mathrm{M}_{n^n}(\mathbb{C})$ so that $F(D(\mathfrak{p})) = ad_\lambda(F(\mathfrak{p}))$. Hence $\lambda \in \mathcal{D}_n$. As $ad_\lambda(\mathfrak{F}) = (0)$ if and only if $\lambda \in \mathfrak{F}^c$, we get an injection $\Psi : \mathcal{Z}(l^1(S_n)) \hookrightarrow \mathcal{D}_n/(\mathcal{D}_n \cap \mathfrak{F}^c)$. Now, if $\lambda \in \mathcal{D}_n$ let us write $D_\lambda = F^{-1} \circ ad_\lambda \circ F$ in $L(l^1(S_n))$. Then $D_\lambda$ is a derivation and the linear mapping $\lambda \to D_\lambda$ is zero on $\mathcal{D}_n \cap \mathfrak{F}^c$. Finally it is readily seen that the induced mapping $\mathcal{D}_n/(\mathcal{D}_n \cap \mathfrak{F}^c) \to \mathcal{Z}(l^1(S_n))$ equals $\Psi^{-1}$. □

**Proposition 2.**

(i) *The space* $\mathcal{Z}(l^1(S_2))$ *is two dimensional.*
(ii) *If* $D \in \mathcal{Z}(l^1(S_2)) - (0)$ *then* $\mathrm{Im}(D)$ *is a non-zero ideal and* $\mathrm{Im}(D)^{[2]} = (0)$.
(iii) *If* $D \in \mathcal{Z}(l^1(S_2))$ *then* $\mathrm{Im}(D) \subseteq J(l^1(S_2))$.
(iv) *Every non-zero derivation within* $l^1(S_2)$ *maps onto the radical.*

*Proof.* (i): By applying Theorem 1 (i) or (iii) it is seen that if $\mathfrak{m} \in l^1(S_2)$ and $D$ is a derivation on $l^1(S_2)$ then

$$D(\mathfrak{m}) = \left[\alpha \mathfrak{m}^{(1,1)} + (\alpha + \beta)\mathfrak{m}^{(2,1)} + \beta \mathfrak{m}^{(2,2)}\right]\mathfrak{q}_0, \ \alpha, \beta \in \mathbb{C}, \tag{8}$$

where $\mathfrak{q}_0 = \delta_{(1,1)} - \delta_{(2,2)}$. In this case we have

$$\begin{aligned}
&F_{(1,1)}^{(1,1)} = F_{(2,2)}^{(2,2)} : \mathfrak{m} \to \mathfrak{m}^{(1,1)} + \mathfrak{m}^{(1,2)} + \mathfrak{m}^{(2,1)} + \mathfrak{m}^{(2,2)},\\
&F_{(1,1)}^{(1,2)} = F_{(1,1)}^{(2,1)} = F_{(1,1)}^{(2,2)} = F_{(2,2)}^{(1,1)} = F_{(2,2)}^{(1,2)} = F_{(2,2)}^{(2,1)} = 0,\\
&F_{(1,2)}^{(1,1)} = F_{(2,1)}^{(2,2)} : \mathfrak{m} \to \mathfrak{m}^{(1,1)},\\
&F_{(1,2)}^{(1,2)} = F_{(2,1)}^{(2,1)} : \mathfrak{m} \to \mathfrak{m}^{(1,2)},\\
&F_{(2,1)}^{(1,2)} = F_{(1,2)}^{(2,1)} : \mathfrak{m} \to \mathfrak{m}^{(2,1)},\\
&F_{(2,1)}^{(1,1)} = F_{(1,2)}^{(2,2)} : \mathfrak{m} \to \mathfrak{m}^{(2,2)}.
\end{aligned}$$

(ii): Given $\mathfrak{m}, \mathfrak{p} \in l^1(S_2)$ it is sufficient to observe that

$$\begin{aligned}
&D(\mathfrak{m}) * \mathfrak{p} = D(\mathfrak{m})[\mathfrak{p}^{(1,1)} + \mathfrak{p}^{(1,2)} + \mathfrak{p}^{(2,1)} + \mathfrak{p}^{(2,2)}],\\
&\mathfrak{p} * D(\mathfrak{m}) = [\mathfrak{p}^{(1,2)} - \mathfrak{p}^{(2,1)}]D(\mathfrak{m}).
\end{aligned}$$

(iii): An element $\mathfrak{m} \in l^1(S_2)$ is singular if and only if $\left(\mathfrak{m}^{(1,2)}\right)^2 = \left(\mathfrak{m}^{(2,1)}\right)^2$ or $\mathfrak{m}^{(1,1)} + \mathfrak{m}^{(1,2)} + \mathfrak{m}^{(2,1)} + \mathfrak{m}^{(2,2)} = 0$. Besides $\mathfrak{q}_0 \in J(l^1(S_2))$ if and only if

$\delta_{(1,2)} - \mathfrak{p} * \mathfrak{q}_0$ is regular for any $\mathfrak{p} \in l^1(S_2)$ (cf. [3], p. 69). If for a fixed $\mathfrak{p}$ we write $\widehat{\mathfrak{p}} = \delta_{(1,2)} - \mathfrak{p} * \mathfrak{q}_0$ then $\widehat{\mathfrak{p}} = \delta_{(1,2)} - \left(\mathfrak{p}^{(1,2)} - \mathfrak{p}^{(2,1)}\right)\mathfrak{q}_0$. Hence $\widehat{\mathfrak{p}}^{(1,2)} = 1$, $\widehat{\mathfrak{p}}^{(2,1)} = 0$ and $\widehat{\mathfrak{p}}^{(1,1)} + \widehat{\mathfrak{p}}^{(1,2)} + \widehat{\mathfrak{p}}^{(2,1)} + \widehat{\mathfrak{p}}^{(2,2)} = 1$, i.e., $\widehat{\mathfrak{p}}$ becomes regular. Thus $\mathfrak{q}_0 \in J(l^1(S_2))$ and the claim follows by (8).

(iv): We observe that

$$J\left(l^1(S_2)\right) = \left\{\mathfrak{m} \in l^1(S_2) : \mathfrak{m}^{(1,1)} + \mathfrak{m}^{(2,2)} = \mathfrak{m}^{(1,2)} = \mathfrak{m}^{(2,1)} = 0\right\}$$

and by a dimensionality argument the claim follows. □

*Example* 3. The following is the list of non-zero complex homomorphisms on $l^1(S_3)$ induced by homomorphisms $h : (S_3, \circ) \to (\{-1,0,1\}, \cdot)$ :

$$(9) \qquad h_0(\mathfrak{m}) = \sum_{g \in S_3} \mathfrak{m}^g,$$

$$h_1(\mathfrak{m}) = \mathfrak{m}^{(1,2,3)} + \mathfrak{m}^{(1,3,2)} + \mathfrak{m}^{(2,1,3)} + \mathfrak{m}^{(2,3,1)} + \mathfrak{m}^{(3,1,2)} + \mathfrak{m}^{(3,2,1)},$$
$$h_2(\mathfrak{m}) = \mathfrak{m}^{(1,2,3)} - \mathfrak{m}^{(1,3,2)} + \mathfrak{m}^{(2,1,3)} - \mathfrak{m}^{(2,3,1)} - \mathfrak{m}^{(3,1,2)} - \mathfrak{m}^{(3,2,1)},$$
$$h_3(\mathfrak{m}) = \mathfrak{m}^{(1,2,3)} - \mathfrak{m}^{(1,3,2)} - \mathfrak{m}^{(2,1,3)} + \mathfrak{m}^{(2,3,1)} + \mathfrak{m}^{(3,1,2)} - \mathfrak{m}^{(3,2,1)},$$
$$h_4(\mathfrak{m}) = \mathfrak{m}^{(1,2,3)} + \mathfrak{m}^{(1,3,2)} - \mathfrak{m}^{(2,1,3)} - \mathfrak{m}^{(2,3,1)} - \mathfrak{m}^{(3,1,2)} + \mathfrak{m}^{(3,2,1)}.$$

If $D \in \mathcal{Z}^1(l^1(S_3))$ the matrix $M = \left[\left\langle D(\delta_g), \delta^h\right\rangle\right]_{g,h \in \mathrm{Inv}(S_3)}$ has the form

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & c_2 & -c_2 & -c_1 \\ c_3 & -c_1 - c_4 & c_5 & -c_2 - c_6 & c_2 - c_7 & c_1 - c_8 \\ c_8 & c_2 - c_3 & c_7 - c_2 & -c_5 & c_4 & c_6 \\ c_8 & -c_2 - c_3 & c_2 + c_6 & c_4 & -c_5 & c_7 \\ c_3 & c_1 - c_4 & -c_1 - c_8 & -c_7 & -c_6 & c_5 \end{bmatrix}.$$

By (9) the Jacobson radical of $l^1(S_3)$ is contained in the subspace $S$ of $l^1(S_3)$ defined as

$$S : \mathfrak{m}^{(1,2,3)} = m^{(2,1,3)} = \mathfrak{m}^{(1,3,2)} + \mathfrak{m}^{(3,2,1)} = \mathfrak{m}^{(2,3,1)} + \mathfrak{m}^{(3,1,2)} = 0.$$

Therefore, if $\mathrm{Im}(D) \subseteq J(l^1(S_3))$ the following identities must hold

$$c_1 = c_3 = c_4 = c_5 = c_8 = 0 \text{ and } c_2 = -c_6 = c_7.$$

## References


[1] S. Bowling, J. Duncan, *First order cohomology of Banach semigroup algebras*, Semigroup Forum, 56(1) (1998), 130-145.

[2] Y. Choi, M. J. Heath, *Characterizing derivations from the disk algebra to its dual*, Proc. Amer. Math. Soc., 139(3) (2011), 1073-1080.

[3] H. G. Dales, *Banach algebras and automatic continuity*, Lond. Math. Soc. Monogr., 24, Clarendon Press, New York, 2000.

[4] J. Duncan, I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Roy. Soc. Edinburgh Sect. A, 80(3-4) (1978), 309-321.

[5] J. Duncan, L. T. Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scand., 66(1) (1990), 141-146.

[6] E. Hewitt, H. S. Zuckerman, *Finite dimensional convolution algebras*, Acta Math., 93 (1955), 67-119.

[7] B. E. Johnson Cheng, J. R. Ringrose, *Derivations of operator algebras and discrete group algebras*, Bull. London Math. Soc., 1 (1969), 70-74.

[8] B-D. Kim, K-W. Jun, *The range of derivations on noncommutative Banach algebras*, Bull. Korean Math. Soc., 28(1) (1991), 65-68.

[9] M. Mathieu, G. J. Murphy, *Derivations mapping into the radical*, Arch. Math. (Basel), 57(5) (1991), 469-474.

[10] M. Mathieu, V. Runde, *Derivations mapping into the radical. II*, Bull. London Math. Soc., 24(5) (1992), 485-487.

[11] T. V. Pedersen, *Properties of derivations on some convolution algebras*, Cent. Eur. J. Math., 12(5) (2014), 742-751.

[12] I. E. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc., 61 (1947), 69-105.

[13] I. M. Singer, J. Wermer, *Derivations on commutative normed algebras*, Math. Ann., 129 (1955), 260-264.

[14] M. P. Thomas. *The image of a derivation is contained in the radical*, Ann. of Math. (2), 128(3) (1988), 435-460.

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