



EXPONENTIAL POMPEIU'S TYPE INEQUALITIES WITH APPLICATIONS TO OSTROWSKI'S INEQUALITY

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ABSTRACT. In this paper, some exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

1. INTRODUCTION

In 1946, Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Lemma 1 (Pompeiu, 1946 [6]). *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ between x_1 and x_2 such that*

$$(1.1) \quad \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

The following inequality is useful to derive some Ostrowski type inequalities.

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Corollary 1 (Pompeiu's Inequality). *With the assumptions of Lemma 1 and if $\|f - \ell f'\|_\infty = \sup_{t \in (a,b)} |f(t) - t f'(t)| < \infty$ where $\ell(t) = t$, $t \in [a, b]$, then*

$$(1.2) \quad |tf(x) - xf(t)| \leq \|f - \ell f'\|_\infty |x - t|$$

for any $t, x \in [a, b]$.

The inequality (1.2) was obtained by the author in [3].

In 1938, A. Ostrowski [4] proved the following result by the estimating the integral mean.

Theorem 1 (Ostrowski, 1938 [4]). *Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $|f'(t)| \leq M < \infty$ for all $t \in (a, b)$. Then for any $x \in [a, b]$, we have the inequality*

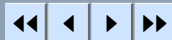
$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a).$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 2 (Dragomir, 2005 [3]). *Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$(1.4) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - \ell f'\|_\infty,$$

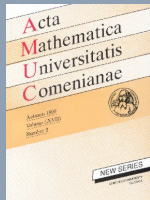


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where $\ell(t) = t$, $t \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [7], using a mean value theorem, E. C. Popa obtained a generalization of (1.4) as follows:

Theorem 3 (Popa, 2007 [7]). *Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Assume that $\alpha \notin [a, b]$. Then for any $x \in [a, b]$, we have the inequality*

$$(1.5) \quad \left| \left(\frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f - \ell_\alpha f'\|_\infty,$$

where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$.

In [5], J. Pečarić and S. Ungar proved a general estimate with the p -norm, $1 \leq p \leq \infty$ which for $p = \infty$, gives Dragomir's result.

Theorem 4 (Pečarić & Ungar, 2006 [5]). *Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the inequality*

$$(1.6) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq PU(x, p) \|f - \ell f'\|_p$$

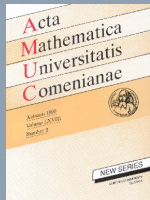


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for $x \in [a, b]$, where

$$PU(x, p) := (b - a)^{\frac{1}{p} - 1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right].$$

In the cases $(p, q) = (1, \infty)$, $(\infty, 1)$ and $(2, 2)$, the quantity $PU(x, p)$ has to be taken as the limit as $p \rightarrow 1, \infty$ and 2 , respectively.

For other inequalities in terms of the p -norm of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$ and $\alpha \notin [a, b]$, see [1] and [2].

In this paper, some exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.



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2. EXPONENTIAL INEQUALITIES

We can provide some similar results for complex-valued functions with the exponential instead of ℓ .



Lemma 2. Let $f: [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \neq 0$. Then for any $t, x \in [a, b]$, we have

$$(2.1) \quad \left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| \leq \begin{cases} \|f' - \alpha f\|_{\infty} & \text{if } f' - \alpha f \in L_{\infty}[a, b], \\ \times \left| \frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))} \right| \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p & \text{if } f' - \alpha f \in L_p[a, b] \\ \times \left| \frac{1}{\exp(tq \operatorname{Re}(\alpha))} - \frac{1}{\exp(xq \operatorname{Re}(\alpha))} \right|^{1/q} & p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - \alpha f\|_1 \frac{1}{\min\{\exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha))\}}, \end{cases}$$

or equivalently,

$$(2.2) \quad |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| \leq \begin{cases} \|f' - \alpha f\|_{\infty} & \text{if } f' - \alpha f \in L_{\infty}[a, b], \\ \times |\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))| \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p & \text{if } f' - \alpha f \in L_p[a, b] \\ \times |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} & p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - \alpha f\|_1 \max\{\exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha))\}. \end{cases}$$

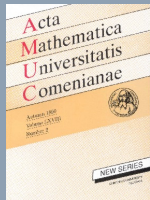


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Proof. If f is absolutely continuous, then $f/\exp(\alpha \cdot)$ is absolutely continuous on the interval $[a, b]$ and

$$\int_t^x \left(\frac{f(s)}{\exp(\alpha s)} \right)' ds = \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\begin{aligned} \int_t^x \left(\frac{f(s)}{\exp(\alpha s)} \right)' ds &= \int_t^x \frac{f'(s) \exp(\alpha s) - \alpha f(s) \exp(\alpha s)}{\exp(2\alpha s)} ds \\ &= \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds, \end{aligned}$$

then we get the following identity

$$(2.3) \quad \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} = \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds$$

for any $t, x \in [a, b]$ with $x \neq t$.

Taking the modulus in (2.3) we have

$$(2.4) \quad \begin{aligned} \left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| &= \left| \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds \right| \\ &\leq \left| \int_t^x \frac{|f'(s) - \alpha f(s)|}{|\exp(\alpha s)|} ds \right| := I \end{aligned}$$

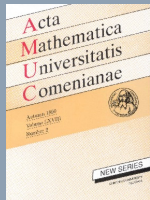


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and utilizing Hölder's integral inequality, we deduce

$$(2.5) \quad I \leq \begin{cases} \sup_{s \in [t, x] \setminus \{x, t\}} |f'(s) - \alpha f(s)| \left| \int_t^x \frac{1}{|\exp(\alpha s)|} ds \right|, \\ \left| \int_t^x |f'(s) - \alpha f(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|\exp(\alpha s)|^q} ds \right|^{1/q}, \\ \left| \int_t^x |f'(s) - \alpha f(s)| ds \right| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(\alpha s)|} \right\}, \end{cases}$$

$$\leq \begin{cases} \|f' - \alpha f\|_\infty \left| \int_t^x \frac{1}{|\exp(\alpha s)|} ds \right|, \\ \|f' - \alpha f\|_p \left| \int_t^x \frac{1}{|\exp(\alpha s)|^q} ds \right|^{1/q}, \\ \|f' - \alpha f\|_1 \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(\alpha s)|} \right\}. \end{cases}$$

Now, since $\alpha = \operatorname{Re}(\alpha) + i \operatorname{Im}(\alpha)$ and $s \in [a, b]$, then

$$\begin{aligned} |\exp(\alpha s)| &= \exp(s \operatorname{Re}(\alpha) + i s \operatorname{Im}(\alpha)) = |\exp(s \operatorname{Re}(\alpha)) \exp(i s \operatorname{Im}(\alpha))| \\ &= |\exp(s \operatorname{Re}(\alpha))| |\exp(i s \operatorname{Im}(\alpha))| = \exp(s \operatorname{Re}(\alpha)). \end{aligned}$$

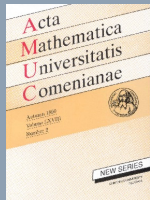


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We have

$$\begin{aligned}
 \int_t^x \frac{1}{|\exp(\alpha s)|} ds &= \int_t^x \frac{1}{\exp(s \operatorname{Re}(\alpha))} ds = \int_t^x \exp(-s \operatorname{Re}(\alpha)) ds \\
 &= -\operatorname{Re}(\alpha) \exp(-s \operatorname{Re}(\alpha)) \Big|_t^x \\
 &= -\operatorname{Re}(\alpha) \exp(-x \operatorname{Re}(\alpha)) + \operatorname{Re}(\alpha) \exp(-t \operatorname{Re}(\alpha)) \\
 &= \operatorname{Re}(\alpha) [\exp(-t \operatorname{Re}(\alpha)) - \exp(-x \operatorname{Re}(\alpha))] \\
 &= \operatorname{Re}(\alpha) \left[\frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))} \right].
 \end{aligned}$$

By (2.4) and (2.5), we get

$$\left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| \leq \|f' - \alpha f\|_\infty |\operatorname{Re}(\alpha)| \left| \frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))} \right|$$

and the first part of (2.1) is proved.

We have

$$\begin{aligned}
 \int_t^x \frac{1}{|\exp(\alpha s)|^q} ds &= \int_t^x \frac{1}{\exp(sq \operatorname{Re}(\alpha))} ds \\
 &= q \operatorname{Re}(\alpha) \left[\frac{1}{\exp(tq \operatorname{Re}(\alpha))} - \frac{1}{\exp(xq \operatorname{Re}(\alpha))} \right].
 \end{aligned}$$

By (2.4) and (2.5), we get the second part of (2.1).

We have

$$\sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(\alpha s)|} \right\} = \frac{1}{\min \{ \exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha)) \}}.$$

By (2.4) and ((2.5), we get the last part of (2.1).

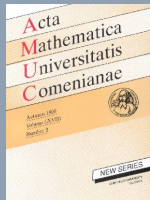


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The inequality (2.2) follows from (2.1) by multiplying by $|\exp(\alpha x) \exp(\alpha t)|$ and performing the required calculation. \square

The following particular case is of interest.

Corollary 2. *Let $f: [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$. Then for any $t, x \in [a, b]$, we have*

$$(2.6) \quad \left| \frac{f(x)}{\exp(x)} - \frac{f(t)}{\exp(t)} \right| \leq \begin{cases} \|f' - f\|_{\infty} \left| \frac{1}{\exp(t)} - \frac{1}{\exp(x)} \right| & \text{if } f' - f \in L_{\infty}[a, b], \\ q^{1/q} \|f' - f\|_p \left| \frac{1}{\exp(tq)} - \frac{1}{\exp(xq)} \right|^{1/q} & \text{if } f' - f \in L_p[a, b], \\ \|f' - f\|_1 \frac{1}{\min\{\exp(t), \exp(x)\}}, & p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

or equivalently,

$$(2.7) \quad \begin{aligned} & |\exp(t)f(x) - f(t)\exp(x)| \\ & \leq \begin{cases} \|f' - f\|_{\infty} |\exp(x) - \exp(t)| & \text{if } f' - f \in L_{\infty}[a, b], \\ q^{1/q} \|f' - f\|_p |\exp(xq) - \exp(tq)|^{1/q} & \text{if } f' - f \in L_p[a, b] \\ \|f' - f\|_1 \max\{\exp(t), \exp(x)\}. & p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \end{cases} \end{aligned}$$



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Remark 1. If $\operatorname{Re}(\alpha) = 0$, then the inequality (2.5) becomes

$$I \leq \begin{cases} \sup_{s \in [t, x] \setminus \{x, t\}} |f'(s) - i \operatorname{Im}(\alpha) f(s)| \left| \int_t^x \frac{1}{|\exp(i \operatorname{Im}(\alpha) s)|} ds \right|, \\ \left| \int_t^x |f'(s) - i \operatorname{Im}(\alpha) f(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|\exp(i \operatorname{Im}(\alpha) s)|^q} ds \right|^{1/q}, \\ \left| \int_t^x |f'(s) - i \operatorname{Im}(\alpha) f(s)| ds \right| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(i \operatorname{Im}(\alpha) s)|} \right\}, \end{cases}$$

$$\leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty \left| \int_t^x ds \right|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p \left| \int_t^x ds \right|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1, \end{cases} = \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty |x - t|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p |x - t|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1. \end{cases}$$

Therefore, we have

$$(2.8) \quad \left| \frac{f(x)}{\exp(i \operatorname{Im}(\alpha) x)} - \frac{f(t)}{\exp(i \operatorname{Im}(\alpha) t)} \right| \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty |x - t|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p |x - t|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1, \end{cases}$$

or equivalently,

$$(2.9) \quad \left| \exp(i \operatorname{Im}(\alpha) t) f(x) - f(t) \exp(i \operatorname{Im}(\alpha) x) \right| \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty |x - t|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p |x - t|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1 \end{cases}$$

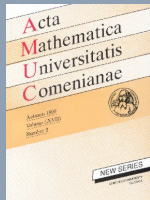


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for any $t, x \in [a, b]$.

In particular, we have

$$(2.10) \quad \left| \frac{f(x)}{\exp(ix)} - \frac{f(t)}{\exp(it)} \right| \leq \begin{cases} \|f' - if\|_{\infty} |x - t|, \\ \|f' - if\|_p |x - t|^{1/q}, \\ \|f' - if\|_1, \end{cases}$$

or equivalently,

$$(2.11) \quad |\exp(it) f(x) - f(t) \exp(ix)| \leq \begin{cases} \|f' - if\|_{\infty} |x - t|, \\ \|f' - if\|_p |x - t|^{1/q}, \\ \|f' - if\|_1 \end{cases}$$

for any $t, x \in [a, b]$.

3. INEQUALITIES OF OSTROWSKI TYPE

The following result holds.

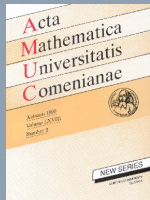


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Theorem 5. Let $f: [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$. Then for any $x \in [a, b]$ we have

$$(3.1) \quad \left| f(x) \frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x) \int_a^b f(t) dt \right| \leq \begin{cases} |\operatorname{Re}(\alpha)| \|f' - \alpha f\|_{\infty} B_1(a, b, x, \alpha) & \text{if } f' - \alpha f \in L_{\infty}[a, b], \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} (b-a)^{1/p} & \text{if } f' - \alpha f \in L_p[a, b] \\ \times \|f' - \alpha f\|_p |B_q(a, b, x, \alpha)|^{1/q} & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - \alpha f\|_1 B_{\infty}(a, b, x, \alpha), & \end{cases}$$

where

$$B_q(a, b, x, \alpha) := 2 \left[\exp(xq \operatorname{Re}(\alpha)) \left(x - \frac{a+b}{2} \right) + \frac{1}{q \operatorname{Re}(\alpha)} \left(\frac{\exp(bq \operatorname{Re}(\alpha)) + \exp(aq \operatorname{Re}(\alpha))}{2} - \exp(xq \operatorname{Re}(\alpha)) \right) \right]$$

for $q \geq 1$ and

$$B_{\infty}(a, b, x, \alpha) := \exp(x \operatorname{Re}(\alpha)) (x - a) + \frac{\exp(b \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}.$$

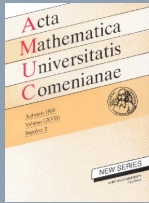


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Proof. Utilising the first inequality in (2.2), we have

$$\begin{aligned} (3.2) \quad & \left| f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \right| \\ & \leq \int_a^b |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| dt \\ & \leq |\operatorname{Re}(\alpha)| \|f' - \alpha f\|_\infty \int_a^b |\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))| dt \end{aligned}$$

for any $x \in [a, b]$.

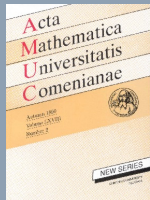


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Observe that since $\operatorname{Re}(\alpha) > 0$, then

$$\begin{aligned}
 & \int_a^b |\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))| dt \\
 &= \int_a^x (\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))) dt \\
 & \quad + \int_x^b (\exp(t \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))) dt \\
 &= \exp(x \operatorname{Re}(\alpha)) (x - a) - \frac{\exp(t \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)} \Big|_a^x \\
 & \quad + \frac{\exp(t \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)} \Big|_x^b - (b - x) \exp(x \operatorname{Re}(\alpha)) \\
 &= \exp(x \operatorname{Re}(\alpha)) (2x - a - b) - \frac{1}{\operatorname{Re}(\alpha)} (\exp(x \operatorname{Re}(\alpha)) - \exp(a \operatorname{Re}(\alpha))) \\
 & \quad + \frac{1}{\operatorname{Re}(\alpha)} (\exp(b \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))) \\
 &= \exp(x \operatorname{Re}(\alpha)) (2x - a - b) \\
 & \quad + \frac{1}{\operatorname{Re}(\alpha)} (\exp(b \operatorname{Re}(\alpha)) + \exp(a \operatorname{Re}(\alpha)) - 2 \exp(x \operatorname{Re}(\alpha))) \\
 &= 2 \left[\exp(x \operatorname{Re}(\alpha)) \left(x - \frac{a+b}{2} \right) \right. \\
 & \quad \left. + \frac{1}{\operatorname{Re}(\alpha)} \left(\frac{\exp(b \operatorname{Re}(\alpha)) + \exp(a \operatorname{Re}(\alpha))}{2} - \exp(x \operatorname{Re}(\alpha)) \right) \right]
 \end{aligned}$$

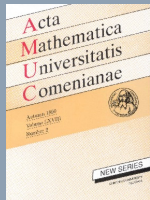


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for any $x \in [a, b]$.

Also

$$f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt = f(x) \frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x) \int_a^b f(t) dt$$

for any $x \in [a, b]$ and by (3.2), we get the first inequality in (3.1).

Using the second inequality in (2.2), we have

$$\begin{aligned} & \left| f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \right| \\ (3.3) \quad & \leq \int_a^b |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| dt \\ & \leq q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p \int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} dt \end{aligned}$$

for any $x \in [a, b]$.

By Hölder's integral inequality, we also have

$$\begin{aligned} & \int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} dt \\ & \leq \left(\int_a^b dt \right)^{1/p} \left[\int_a^b \left(|\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} \right)^q dt \right]^{1/q} \\ & = (b - a)^{1/p} \left[\int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))| dt \right]^{1/q}, \end{aligned}$$

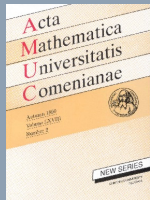


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for any $x \in [a, b]$.

Observe that as above we have

$$\int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))| dt = 2 \left[\exp(xq \operatorname{Re}(\alpha)) \left(x - \frac{a+b}{2} \right) + \frac{1}{q \operatorname{Re}(\alpha)} \left(\frac{\exp(bq \operatorname{Re}(\alpha)) + \exp(aq \operatorname{Re}(\alpha))}{2} - \exp(xq \operatorname{Re}(\alpha)) \right) \right] = B_q(a, b, x, \alpha)$$

for any $x \in [a, b]$ and by (3.3), we get the second part of (3.1).

Using the third inequality in (2.2), we have

$$\begin{aligned} \left| f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \right| &\leq \int_a^b |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| dt \\ &\leq \|f' - \alpha f\|_1 \int_a^b \max \{ \exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha)) \} dt \end{aligned} \tag{3.4}$$

for any $x \in [a, b]$.

Observe that

$$\begin{aligned} &\int_a^b \max \{ \exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha)) \} dt \\ &= \int_a^x \max \{ \exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha)) \} dt + \int_x^b \max \{ \exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha)) \} dt \\ &= \int_a^x \exp(x \operatorname{Re}(\alpha)) dt + \int_x^b \exp(t \operatorname{Re}(\alpha)) dt \\ &= \exp(x \operatorname{Re}(\alpha)) (x - a) + \frac{\exp(b \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)} \end{aligned}$$

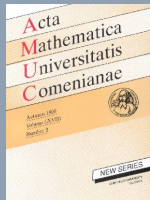


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and by (3.4), we get the third part of (3.1). \square

Remark 2. If $\operatorname{Re}(\alpha) < 0$, then a similar result may be stated. However, the details are left to the interested reader.

Corollary 3. *Let $f: [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$. Then for any $x \in [a, b]$, we have*

$$(3.5) \quad \left| f(x)[\exp(b) - \exp(a)] - \exp(x) \int_a^b f(t) dt \right| \leq \begin{cases} \|f' - f\|_{\infty} B_1(a, b, x) & \text{if } f' - f \in L_{\infty}[a, b], \\ q^{1/q} (b-a)^{1/p} \|f' - f\|_p \times |B_q(a, b, x)|^{1/q} & \text{if } f' - f \in L_p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_1 B_{\infty}(a, b, x), & \end{cases}$$



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where

$$B_q(a, b, x) := 2 \left[\left(x - \frac{a+b}{2} \right) \exp(xq) + \frac{1}{q} \left(\frac{\exp(bq) + \exp(aq)}{2} - \exp(xq) \right) \right]$$

for $q \geq 1$ and

$$B_\infty(a, b, x) := (x - a) \exp(x) + \exp(b) - \exp(x).$$

Remark 3. The midpoint case is as follows:

$$\left| f \left(\frac{a+b}{2} \right) [\exp(b) - \exp(a)] - \exp \left(\frac{a+b}{2} \right) \int_a^b f(t) dt \right|$$

$$(3.6) \quad \leq \begin{cases} \|f' - f\|_\infty B_1(a, b) & \text{if } f' - f \in L_\infty[a, b], \\ q^{1/q} (b-a)^{1/p} \|f' - f\|_p & \text{if } f' - f \in L_p[a, b], \\ \times |B_q(a, b)|^{1/q} & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_1 B_\infty(a, b), & \end{cases}$$

where

$$B_q(a, b, x) := \frac{2}{q} \left(\frac{\exp(bq) + \exp(aq)}{2} - \exp \left(\frac{a+b}{2} q \right) \right)$$

for $q \geq 1$ and

$$B_\infty(a, b) := \frac{b-a}{2} \exp \left(\frac{a+b}{2} \right) + \exp(b) - \exp \left(\frac{a+b}{2} \right).$$

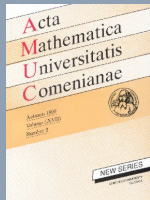


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The case $\operatorname{Re}(\alpha) = 0$ is different and may be stated as follows.

Theorem 6. *Let $f: [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) = 0$ and $\operatorname{Im}(\alpha) \neq 0$. Then for any $x \in [a, b]$, we have*

$$\begin{aligned}
 & \left| f(x) \frac{\exp(i \operatorname{Im}(\alpha) b) - \exp(i \operatorname{Im}(\alpha) a)}{i \operatorname{Im}(\alpha)} - \exp(i \operatorname{Im}(\alpha) x) \int_a^b f(t) dt \right| \\
 (3.7) \quad & \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_{\infty} & \text{if } f' - i \operatorname{Im}(\alpha) f \in L_{\infty}[a, b], \\
 \times \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2 & \\
 \frac{q}{q+1} \|f' - i \operatorname{Im}(\alpha) f\|_p & \text{if } f' - i \operatorname{Im}(\alpha) f \in L_p[a, b], \\
 \times \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}} & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
 \|f' - i \operatorname{Im}(\alpha) f\|_1 (b-a). & \end{cases}
 \end{aligned}$$

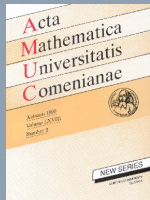


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Proof. Utilizing the inequality (2.9), we have

$$\begin{aligned}
 & \left| f(x) \frac{\exp(i \operatorname{Im}(\alpha) b) - \exp(i \operatorname{Im}(\alpha) a)}{i \operatorname{Im}(\alpha)} - \exp(i \operatorname{Im}(\alpha) x) \int_a^b f(t) dt \right| \\
 & \leq \int_a^b |\exp(i \operatorname{Im}(\alpha) t) f(x) - f(t) \exp(i \operatorname{Im}(\alpha) x)| dt \\
 (3.8) \quad & \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty \int_a^b |x - t| dt, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p \int_a^b |x - t|^{1/q} dt, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1 \int_a^b dt. \end{cases}
 \end{aligned}$$

Since

$$\int_a^b |x - t| dt = \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] (b - a)^2$$

and

$$\int_a^b |x - t|^{1/q} dt = \frac{q}{q + 1} \left[\left(\frac{b - x}{b - a} \right)^{\frac{q+1}{q}} + \left(\frac{x - a}{b - a} \right)^{\frac{q+1}{q}} \right] (b - a)^{\frac{q+1}{q}},$$

then from (3.8), we get the desired result (3.7). □

Corollary 4. *Let $f: [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$. Then for any $x \in [a, b]$, we have*



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$$(3.9) \quad \left| f(x) \frac{\exp(ib) - \exp(ia)}{i} - \exp(ix) \int_a^b f(t) dt \right| \leq \begin{cases} \|f' - if\|_\infty \times \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2 & \text{if } f' - if \in L_\infty[a, b], \\ \frac{q}{q+1} \|f' - if\|_p \times \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}} & \text{if } f' - if \in L_p[a, b], \\ \|f' - if\|_1 (b-a). & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{cases}$$

Remark 4. The midpoint case is as follows

$$(3.10) \quad \left| f\left(\frac{a+b}{2}\right) \frac{\exp(ib) - \exp(ia)}{i} - \exp\left(i\frac{a+b}{2}\right) \int_a^b f(t) dt \right| \leq \begin{cases} \frac{1}{4} \|f' - if\|_\infty (b-a)^2, & \text{if } f' - if \in L_\infty[a, b], \\ \frac{q}{(q+1)2^{1/q}} \|f' - if\|_p (b-a)^{\frac{q+1}{q}}, & \text{if } f' - if \in L_p[a, b]. \end{cases}$$

Similar inequalities may be stated if one uses (2.1) and integrates over t on $[a, b]$. The details are left to the interested reader.

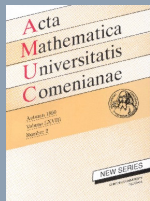


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