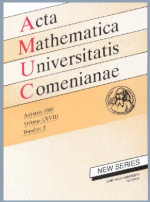


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# DIHEDRAL COVERS OF THE COMPLETE GRAPH $K_5$

M. GHASEMI

Abstract. A regular cover of a connected graph is called dihedral if its transformation group is dihedral. In this paper, the author classifies all dihedral coverings of the complete graph $K_5$ whose fibre-preserving automorphism subgroups act arc-transitively.

## 1. Introduction

Throughout this paper, we consider finite connected graphs without loops or multiple edges. For a graph $X$, every edge of $X$ gives rise to a pair of opposite arcs. By $\mathrm{V}(X)$, $\mathrm{E}(X)$, $\mathrm{A}(X)$ and $\mathrm{Aut}(X)$, we denote the vertex set, the edge set, the arc set and the automorphism group of the graph $X$, respectively. The *neighborhood* of a vertex $v \in \mathrm{V}(X)$ denoted by $N(v)$ is the set of vertices adjacent to $v$ in $X$. Let a group $G$ act on a set $\Omega$ and let $\alpha \in \Omega$. We denote by $G_\alpha$ the *stabilizer* of $\alpha$ in $G$, that is, the subgroup of $G$ fixing $\alpha$. The group $G$ is said to be *semiregular* if $G_\alpha = 1$ for each $\alpha \in \Omega$, and *regular* if $G$ is semiregular and transitive on $\Omega$. A graph $\widetilde{X}$ is called a *covering* of a graph $X$ with projection $p\colon \widetilde{X} \to X$ if there is a surjection $p\colon V(\widetilde{X}) \to V(X)$ such that $p|_{N(\tilde{v})}\colon N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in \mathrm{V}(X)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{X}$ is called the *covering graph* and $X$ is the *base graph*. A covering $\widetilde{X}$ of $X$ with a projection $p$ is said to be *regular* (or *$K$-covering*) if there is a semiregular subgroup $K$ of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that graph $X$ is isomorphic to the quotient graph $\widetilde{X}/K$, say by $h$, and the

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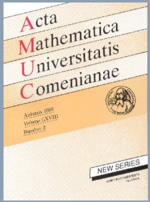



quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition $ph$ of $p$ and $h$ (for the purpose of this paper, all functions are composed from left to right). If $K$ is cyclic, elementary abelian or dihedral then $\widetilde{X}$ is called a *cyclic*, *elementary abelian* or *dihedral covering* of $X$, respectively. If $\widetilde{X}$ is connected, $K$ is the covering transformation group. The *fibre* of an edge or a vertex is its preimage under $p$. An automorphism of $\widetilde{X}$ is said to be *fibre-preserving* if it maps a fibre to a fibre while an element of the covering transformation group fixes each fibre setwise. All of fibre-preserving automorphisms form a group called the *fibre-preserving group*.

An *$s$-arc* in a graph $X$ is an ordered $(s+1)$-tuple $(v_0, v_1, \dots, v_s)$ of vertices of $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \le i \le s$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$; in other words, a directed walk of length $s$ which never includes a backtracking. A graph $X$ is said to be *$s$-arc-transitive* if $\mathrm{Aut}(X)$ is transitive on the set of $s$-arcs in $X$. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*. An $s$-arc-transitive graph is said to be *$s$-transitive* if it is not $(s+1)$-arc-transitive. In particular, a subgroup of the automorphism group of a graph $X$ is said to be *$s$-regular* if it acts regularly on the set of $s$-arcs of $X$. Also if the subgroup is the full automorphism group $\mathrm{Aut}(X)$ of $X$, then $X$ is said to be *$s$-regular*. Thus, if a graph $X$ is $s$-regular, then $\mathrm{Aut}(X)$ is transitive on the set of $s$-arcs and the only automorphism fixing an $s$-arc is the identity automorphism of $X$.

Regular coverings of a graph have received considerable attention. For example, for a graph $X$ which is the complete graph $K_4$, the complete bipartite graph $K_{3,3}$, hypercube $Q_3$ or Petersen graph $O_3$, the $s$-regular cyclic or elementary abelian coverings of $X$, whose fibre-preserving groups are arc-transitive, classified for each $1 \le s \le 5$ [**3, 4, 6, 7**]. As an application of these classifications, all $s$-regular cubic graphs of order $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, and $10p^2$ constructed for each $1 \le s \le 5$ and each prime $p$ [**3, 4, 6**]. In [**14**], it was shown that all cubic graphs admitting a solvable edge-transitive group of automorphisms arise as regular covers of one of the following basic graphs: the complete graph $K_4$, the dipole Dip3 with two vertices and three parallel edges, the complete bipartite graph $K_{3,3}$, the Pappus graph of order 18, and the Gray graph of order 54. Also all dihedral coverings of the complete graph $K_4$ and cubic symmetric graphs of order $2p$ were classified in [5, 8]. But apart from the octahedron graph [11], graphs of higher valencies have not received much attention. For more results see [1, 2, 13, 15]. In a series of reductions of this kind, the final, irreducible graph is often a complete graph. Thus studying $K_5$ is the obvious next choice in order to establish a base of examples for further investigation. All pairwise non-isomorphic connected arc-transitive $p$-elementary abelian covers of the complete graph $K_5$ are constructed in [10]. In this paper all dihedral coverings of the complete graph $K_5$ whose fibre-preserving automorphism subgroups act arc-transitively are determined. Also we give a family of 2-arc-transitive graphs.

Let $n$ be a non-negative integer. Let $\mathbb{Z}_n$ denote the cyclic group of order $n$ and $D_{2n}$ the dihedral group of order $2n$. Set

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

By $\{0, 1, 2, 3, 4\}$ denote the vertex set of $K_5$. For $n \geq 3$, the graph $DK(2n)$ is defined to have vertex set

$$V(DK(2n)) = \{0, 1, 2, 3, 4\} \times D_{2n}$$

and edge set

$$\begin{aligned} E(DK(2n)) = \{&(0,c)(3,c), (1,c)(3,c), (1,c)(4,c), (2,c)(4,c), (0,c)(1,bc), \\ &(0,c)(2,a^{-1}bc), (0,c)(4,ac), (1,c)(2,bc), (2,c)(3,ac), \\ &(3,c)(4,a^{-2}bc), (4,c)(0,a^{-1}c) \mid c \in D_{2n}\}. \end{aligned}$$

Note that the first four edges in the edge set $E(DK(2n))$ correspond with the tree edges in the spanning tree $T$ as depicted by the dashed lines in Fig. 1 and these four edges have the common $c$ as the second coordinates. In fact, the graph $DK(2n)$ is the covering graph derived from a

$T$-reduced voltage assignment $\phi\colon A(K_5) \to D_{2n}$ which assigns the six values $b$, $a^{-1}b$, $a$, $b$, $a^{-2}b$, $a^{-1}$ to the six cotree edges in $K_5$.

The following theorem is the main result of this paper.

**Theorem 1.1.** *Let $\widetilde{X}$ be a connected $D_{2n}$-covering ($n \geq 3$) of the complete graph $K_5$ whose fibre-preserving subgroup is arc-transitive. Then $\widetilde{X}$ is arc-transitive if and only if $\widetilde{X}$ is isomorphic to $DK(2n)$ for $n \geq 3$.*

## 2. Preliminaries related to coverings

Let $X$ be a graph and $K$ a finite group. By $a^{-1}$, we mean the reverse arc to an arc $a$. A *voltage assignment* (or *$K$-voltage assignment*) of $X$ is a function $\phi\colon A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of $\phi$ are called *voltages* and $K$ is the *voltage group*. The *graph* $X \times_\phi K$ derived from a voltage assignment $\phi\colon A(X) \to K$ has a vertex set $V(X) \times K$ and an edge set $E(X) \times K$, so that an edge $(e, g)$ of $X \times_\phi K$ joins a vertex $(u, g)$ to $(v, \phi(a)g)$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of $X$ with the first coordinate projection $p\colon X \times_\phi K \to X$ which is called the *natural projection*. By defining $(u, g^{'})^g := (u, g^{'}g)$ for any $g \in K$ and $(u, g^{'}) \in V(X \times_\phi K)$, $K$ becomes a subgroup of $\mathrm{Aut}(X \times_\phi K)$ which acts semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a $K$-*covering*. For each $u \in V(X)$ and $uv \in E(X)$, the vertex set $\{(u, g) \mid g \in K\}$ is the fibre of $u$ and the edge set $\{(u, g)(v, \phi(a)g) \mid g \in K\}$ is the fibre of $uv$, where $a = (u, v)$. Conversely, each regular covering $\widetilde{X}$ of $X$ with a covering transformation group $K$ can be derived from a $K$-voltage assignment. Given a spanning tree $T$ of the graph $X$, a voltage assignment $\phi$ is said to be *$T$-reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [9] showed that every regular covering $\widetilde{X}$ of a graph $X$ can be derived from a $T$-reduced voltage assignment $\phi$ with respect to an arbitrary fixed spanning tree $T$ of $X$. It is clear that if $\phi$ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group $K$.

Let $\widetilde{X}$ be a $K$-covering of $X$ with a projection $p$. If $\alpha\in \mathrm{Aut}(X)$ and $\tilde{\alpha}\in \mathrm{Aut}(\widetilde{X})$ satisfy $\tilde{\alpha}p = p\alpha$, we call $\tilde{\alpha}$ a *lift* of $\alpha$, and $\alpha$ the *projection* of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. In particular, if the covering graph $\widetilde{X}$ is connected, then the covering transformation group $K$ is the lift of the trivial group, that is $K=\{\tilde{\alpha}\in \mathrm{Aut}(\widetilde{X})\colon\ p = \tilde{\alpha}p\}$. Clearly, if $\tilde{\alpha}$ is a lift of $\alpha$, then $K\tilde{\alpha}$ consists of all the lifts of $\alpha$.

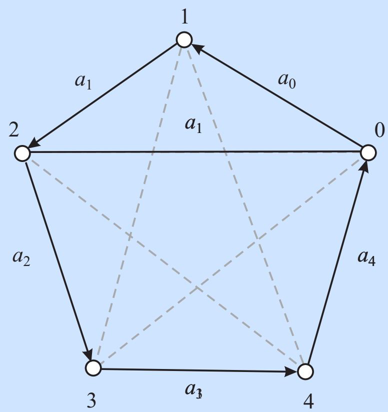
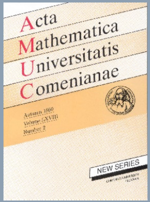



**Figure 1.** A choice of the six cotree arcs in $K_5$.



Let $X \times_\phi K \to X$ be a connected $K$-covering derived from a $T$-reduced voltage assignment $\phi$. The problem whether an automorphism $\alpha$ of $X$ lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha\in \mathrm{Aut}(X)$, we define a function $\bar{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group $K$ by

$$(\phi(C))^{\bar{\alpha}} = \phi(C^\alpha),$$

where $C$ ranges over all fundamental closed walks at $v$, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages on $C$ and $C^\alpha$, respectively. Note that if $K$ is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at $v$ can be substituted by the fundamental cycles generated by the cotree arcs of $X$.

The next proposition is a special case of [12, Theorem 3.5].

**Proposition 2.1.** *Let $X \times_\phi K \to X$ be a connected $K$-covering derived from a $T$-reduced voltage assignment $\phi$. Then, an automorphism $\alpha$ of $X$ lifts if and only if $\bar{\alpha}$ extends to an automorphism of $K$.*

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of $X$ with projections $p_1$ and $p_2$, respectively, are said to be *equivalent* if there exists a graph isomorphism $\tilde{\alpha}\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\tilde{\alpha} p_2 = p_1$. We quote the following proposition.

**Proposition 2.2** ([16])**.** *Two connected regular coverings $X \times_\phi K$ and $X \times_\psi K$, where $\phi$ and $\psi$ are $T$-reduced, are equivalent if and only if there exists an automorphism $\sigma\in$ Aut($K$) such that $\phi(u,v)^\sigma = \psi(u,v)$ for any cotree* $\mathrm{arc}(u,v)$ *of $X$.*

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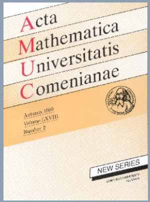
## 3. Proof of Theorem 1.1

Suppose that $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. If $n = 2$, then $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now since elementary abelian coverings of the complete graph $K_5$ were classified by Kuzman [10], we only consider $n \geq 3$.

By $K_5$, we denote the complete graph with vertex set $\{0, 1, 2, 3, 4\}$. Let $T$ be a spanning tree of $K_5$ as shown by dashed lines in Figure 2. Let $\phi$ be such a voltage assignment defined by $\phi = 1$ on $T$ and $\phi = a_0, a_1, a_2, a_3$, $a_4$, and $b_0$ on the cotree arcs $(0, 1)$, $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 0)$, and $(0, 2)$, respectively. Let $\rho = (01234)$, $\tau = (0132)$ and $\sigma = (024)$. Then $\rho$, $\tau$, and $\sigma$ are automorphisms of $K_5$.

By $i_1 i_2 \dots i_s$ denote a directed cycle which has vertices $i_1$, $i_2$, …, $i_s$ in a consecutive order. There are six fundamental cycles 130, 124, 1423, 134, 1403, and 13024 in $K_5$ which are generated by the six cotree arcs $(0, 1)$, $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 0)$ and $(0, 2)$, respectively. Each cycle is mapped to a cycle of the same length under the actions of $\rho$, $\tau$, $\sigma$. We list all these cycles and their voltages in Table 1 in which $C$ denotes a fundamental cycle of $K_5$ and $\phi(C)$ denotes the voltage of $C$.

Let $\widetilde{X} = K_5 \times_\phi D_{2n}$ be a covering graph of the graph $K_5$ satisfying the hypotheses in the theorem, where $\phi = 1$ on the spanning tree $T$ which is depicted by the dashed lines in Figure 2. Note that the vertices of $K_5$ are labeled by 0, 1, 2, 3, and 4. By the hypotheses, the fibre-preserving group, say $\widetilde{L}$, of the covering graph $K_5 \times_\phi D_{2n}$ acts arc-transitively on $K_5 \times_\phi D_{2n}$. Hence, the projection of $\widetilde{L}$, say $L$, is arc-transitive on the base graph $K_5$. Thus $L$ is isomorphic to $\mathrm{AGL}(1, 5) = \langle \rho, \tau \rangle$, $A_5 = \langle \rho, \sigma \rangle$, or $S_5 = \langle \rho, \sigma, \tau \rangle$. Consider the mapping $\bar{\rho}$ from the set $\{a_0, a_1, a_2, a_3, a_4, b_0\}$ of the voltages of the six fundamental cycles of $K_5$ to the group $D_{2n}$, defined by $(\phi(C))^{\bar{\rho}} = \phi(C^\rho)$, where $C$ ranges over the six fundamental cycles. From Table 1, one can see that $a_0^{\bar{\rho}} = a_1$, $a_1^{\bar{\rho}} = a_2 b_0$, $a_2^{\bar{\rho}} = b_0^{-1} a_3$, $a_3^{\bar{\rho}} = a_4 b_0$, $a_4^{\bar{\rho}} = b_0^{-1} a_0$ and $b_0^{\bar{\rho}} = b_0$. Similarly, we can define $\bar{\sigma}$ and $\bar{\tau}$.

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| $C$ | $\phi(C)$ | $C^\rho$ | $\phi(C^\rho)$ |
|---|---|---|---|
| 130 | $a_0$ | 241 | $a_1$ |
| 124 | $a_1$ | 230 | $a_2b_0$ |
| 1423 | $a_2$ | 2034 | $b_0^{-1}a_3$ |
| 134 | $a_3$ | 240 | $a_4b_0$ |
| 1403 | $a_4$ | 2014 | $b_0^{-1}a_0$ |
| 13024 | $b_0$ | 24130 | $b_0$ |
| $C^\sigma$ | $\phi(C^\sigma)$ | $C^\tau$ | $\phi(C^\tau)$ |
| 132 | $a_2^{-1}a_1^{-1}$ | 321 | $a_2^{-1}a_1^{-1}$ |
| 140 | $a_4a_0$ | 304 | $a_4^{-1}a_3^{-1}$ |
| 1043 | $a_0^{-1}a_4^{-1}a_3^{-1}$ | 3402 | $a_3a_4b_0a_2$ |
| 130 | $a_0$ | 324 | $a_2^{-1}a_3^{-1}$ |
| 1023 | $a_0^{-1}b_0a_2$ | 3412 | $a_3a_1a_2$ |
| 13240 | $a_2^{-1}a_4a_0$ | 32104 | $a_2^{-1}a_1^{-1}a_0^{-1}a_4^{-1}a_3^{-1}$ |

**Table 1.** Fundamental cycles and their images with corresponding voltages.

Here we make the following general assumption.

(I) Let $\widetilde{X}$ be a connected $D_{2n}$-covering ($n \geq 3$) of the complete graph $K_5$ whose fibre-preserving subgroup is arc-transitive.

For the three following lemmas we suppose that $n$ is an odd number.

**Lemma 3.1.** *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* $\rho$ *and* $\sigma$*, say* $L$*, lifts. Under the assumption* (I)*,* $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(6)$.

*Proof.* Since $\rho, \sigma \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\sigma}$ can be extended to automorphisms of $D_{2n}$. We denote by $\rho^*$ and $\sigma^*$ these extended automorphisms, respectively. In this case $o(a_0) = o(a_1) = o(a_3)$. Now we consider the following two subcases:

*Subcase I*. $o(a_0) = o(a_1) = o(a_3) = 2$.

By considering $a_1^{\sigma^*} = a_4 a_0$, we have $o(a_4 a_0) = 2$. It follows that $o(a_4) \neq 2$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(b_0^{-1} a_0) \neq 2$. So $o(b_0^{-1}) = 2$, and hence $o(a_2) \neq 2$, by $a_2^{\rho^*} = b_0^{-1} a_3$. Now we may assume that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n-1$ and $0 < r, s \leq n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2 we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$ and $0 < r, s \leq n-1$. Also since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, r, s\}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get the same results. First suppose that $(i, n) = 1$. Since $\sigma\colon a \mapsto a^i, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^r$, $a_4 = a^s$, and $b_0 = a^j b$, where $0 \leq i, j \leq n-1$ and $0 < r, s \leq n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = ab$, $a_1^{\rho^*} = (ab)^{\rho^*} = a^{\rho^*} b^{\rho^*} = a^{r+j} b$. Thus $a^{\rho^*} = a^{r+j-1}$. By considering the image of $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^j b$ under $\rho^*$, we conclude that $a^{r(r+j-1)} = a^{j-i}$, $a^{s(r+j-1)} = a^j$ and $a^{j(r+j-1)} ab = a^j b$. Also $a_0^{\sigma^*} = b^{\sigma^*} = a^{-r+1} b$ and $a_1^{\sigma^*} = (ab)^{\sigma^*} = a^{\sigma^*} b^{\sigma^*} = a^s b$. Thus $a^{\sigma^*} = a^{s+r-1}$.

Now by considering the image of $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^j b$ under $\sigma^*$, we conclude that $a^{r(r+s-1)} = a^{s-i}$, $a^{s(r+s-1)} = a^{-j+r}$ and $a^{j(s+r-1)} a^{-r+1} b = a^{s-r} b$.

Therefore, we have the following:

(1) $r(r+j-1) = j - i$, (2) $s(r+j-1) = j$,
(3) $j(r+j-1) = j - 1$, (4) $j(s+r-1) = s - 1$,
(5) $r(s+r-1) = s - i$ (6) $s(s+r-1) = -j + r$.

By (1) and (3), $rj(r+j-1) = j^2 - ij$ and $rj(r+j-1) = rj - r$. Thus $j^2 - ji = rj - r$. Also by (4) and (5), $rj(s+r-1) = sr - r$ and $rj(s+r-1) = sj - ij$. Thus $sj - ij = sr - r$. So $j^2 - rj = sj - sr$, and hence $(j-r)(j-s) = 0$. Also by (2) and (3), $sj(r+j-1) = j^2$ and $sj(r+j-1) = sj - s$. Thus $j^2 = sj - s$. By $(j-r)(j-s) = 0$, we have $j = r$ or $j = s$. If $j = r$, then $s^2 + sr - s = 0$, by (6). Thus $s = 0$ or $s = -r + 1$. If $s = 0$, then $j = 0$ by (2). Thus $r = 0$, a contradiction. If $s = -r + 1$, then $s = 1$ by $j(s + r - 1) = s - 1$. So $r = 0$, a contradiction. If $j = s$, then by $j^2 = sj - s$, we have $s = 0$, a contradiction.

Now suppose that $(r, n) = 1$. Since $\sigma\colon a \mapsto a^r, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a$, $a_4 = a^r$ and $b_0 = a^k b$, where $0 \le i, j, k \le n - 1$ and $0 < r \le n - 1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = a^i b$, $a_2^{\rho^*} = (a)^{\rho^*} = a^{k-j}$. By considering the image of $a_1 = a^i b$, $a_3 = a^j b$, $a_4 = a^r$ and $b_0 = a^k b$ under $\rho^*$, we conclude that $a^{i(k-j)} a^i b = a^{k+1} b$, $a^{j(k-j)} a^i b = a^{r+k} b$, $a^{r(k-j)} = a^k$ and $a^{k(k-j)} a^i b = a^k b$. Also $a_0^{\sigma^*} = b^{\sigma^*} = a^{i-1} b$ and $a_2^{\sigma^*} = (a)^{\sigma^*} = a^{r-j}$. Now by considering the image of $a_1 = a^i b$, $a_3 = a^j b$, $a_4 = a^r$ and $b_0 = a^k b$ under $\sigma^*$, we conclude that $a^{i(r-j)} a^{i-1} b = a^r b$, $a^{j(r-j)} a^{i-1} b = b$, $a^{r(r-j)} = a^{-k+1}$ and $a^{k(r-j)} a^{i-1} b = a^{r-1} b$.

Therefore, we have the following:

| | | | |
|---|---|---|---|
| (1) | $i(k - j) + i = k + 1,$ | (2) | $j(k - j) + i = r + k,$ |
| (3) | $r(k - j) = k,$ | (4) | $k(k - j) + i = k,$ |
| (5) | $i(r - j) + i - 1 = r,$ | (6) | $j(r - j) + i - 1 = 0,$ |
| (7) | $r(r - j) = -k + 1,$ | (8) | $k(r - j) = r - i.$ |

By (2) and (3), $rj(k-j) = r^2 + rk - ir$ and $rj(k-j) = kj$. Thus $r^2 + rk - ir = kj$. Also by (7) and (8), $rk(r-j) = -k^2 + k$ and $rk(r-j) = r^2 - ir$. Thus $-k^2 + k = r^2 - ir$. So $kj - rk = -k^2 + k$, and hence $k(j - r + k - 1) = 0$. Thus $k = 0$ or $j = r - k + 1$. If $k = 0$, then $i = 0$ by (4). Thus by $-k^2 + k = r^2 - ir$, we have $r = 0$, a contradiction. If $j = r - k + 1$, then $(k - 1)(r + 1) = 0$ by (7). Hence $k = 1$ or $r = -1$. If $k = 1$, then $j = r$. Now by (6), $i = 1$, and so by (8), we have $r = 1$. So by (5), $1 = 0$, a contradiction. If $r = -1$, then $j = -k$. Also by (5), $i(r - j + 1) = 0$, and so $i = 0$ or $r = j - 1$. If $i = 0$, then by (1), $k = -1$. Thus $j = 1$, and so by (3), $2 = -1$. Therefore, $n = 3$ and

$$a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1} b.$$

From Table 1, it is easy to check that $\bar{\rho}$, $\bar{\sigma}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. Thus by Proposition 2.1, $\rho$, $\sigma$ and $\tau$ lift. Since $S_5 = \langle\rho, \sigma, \tau\rangle$ is 2-arc-transitive, it follows that $\mathrm{Aut}(\widetilde{X})$ contains a 2-arc-transitive subgroup lifted by $\langle\rho, \sigma, \tau\rangle$. Therefore, $\widetilde{X}$ is 2-arc-transitive.

Finally, if $r = j - 1$, then by $r = -1$, we have $j = 0$. So by (6), $i = 1$. Also by (7), $k = 0$. Now by (2), $1 = -1$, and so $n = 2$, a contradiction.

*Subcase II.* $o(a_0) = o(a_1) = o(a_3) \neq 2$.

By considering $a_1^{\sigma^*} = a_4a_0$, we have $o(a_4a_0) \neq 2$. It follows that $o(a_4) \neq 2$. Since $a_4^{\rho^*} = b_0^{-1}a_0$, we have $o(b_0^{-1}a_0) \neq 2$. So $o(b_0^{-1}) \neq 2$, and hence $o(a_2) \neq 2$ by $a_2^{\rho^*} = b_0^{-1}a_3$. Now we may assume that $a_0 = a^i$, $a_1 = a^j$, $a_2 = a^k$, $a_3 = a^l$, $a_4 = a^m$ and $b_0 = a^n$, where $0 \leq i, j, k, l, m, n \leq n - 1$. Since $K_5 \times_\phi D_{2n}$ is connected, we have a contradiction. □

**Lemma 3.2.** *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* $\rho$ *and* $\tau$*, say* $L$*, lifts. Under the assumption* (I), $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(2n)$ *for* $n > 3$.

*Proof.* Since $\rho, \tau \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. We denote these extended automorphisms by $\rho^*$ and $\tau^*$, respectively. In this case $o(a_0) = o(a_1)$. Now we consider the following two subcases:

*Subcase I.* $o(a_0) = o(a_1) = 2$.

By considering $a_0^{\tau^*} = a_2^{-1}a_1^{-1}$, we have $o(a_2^{-1}a_1^{-1}) = 2$. It follows that $o(a_2) \neq 2$. Since $a_2^{\rho^*} = b_0^{-1}a_3$, we have either $o(b_0) = o(a_3) = 2$ or $o(b_0) \neq 2$ and $o(a_3) \neq 2$. First suppose that $o(b_0) \neq 2$ and $o(a_3) \neq 2$. Since $a_3^{\rho^*} = a_4b_0$, we have $o(a_4) \neq 2$. Also since $a_4^{\rho^*} = b_0^{-1}a_0$, it follows that $o(a_0) \neq 2$, a contradiction.

Now suppose that $o(b_0) = o(a_3) = 2$. Since $a_3^{\rho^*} = a_4b_0$, it implies that $o(a_4) \neq 2$. Now we may assume that $a_0 = a^ib$, $a_1 = a^jb$, $a_3 = a^kb$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^lb$, where $0 \leq i, j, k, l \leq n - 1$ and $0 < r, s \leq n - 1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$ and $0 < r, s \le n-1$. Since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, r, s\}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get the same results. First suppose that $(i, n) = 1$. Since $\sigma\colon a \mapsto a^i, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^r$, $a_4 = a^s$ and $b_0 = a^j b$, where $0 \le i, j \le n-1$ and $0 < r, s \le n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = ab$, $a_1^{\rho^*} = (ab)^{\rho^*} = a^{\rho^*} b^{\rho^*} = a^{r+j} b$. Thus $a^{\rho^*} = a^{r+j-1}$. By considering the image of $a_2 = a^r$, $a_3 = a^i b$ and $b_0 = a^j b$ under $\rho^*$, we conclude that $a^{r(r+j-1)} = a^{j-i}$, $a^{i(r+j-1)} ab = a^{s+j} b$ and $a^{j(r+j-1)} ab = a^j b$. Also $a_0^{\tau^*} = b^{\tau^*} = a^{-r+1} b$, $a_1^{\tau^*} = (ab)^{\tau^*} = a^{\tau^*} b^{\tau^*} = a^{i-s} b$. Thus $a^{\tau^*} = a^{i-s+r-1}$. By considering the image of $a_2 = a^r$ and $b_0 = a^j b$ under $\tau^*$, we conclude that $a^{r(i-s+r-1)} = a^{i-s-j+r}$ and $a^{j(i-s+r-1)} a^{-r+1} = a^{-r+1-s+i}$.

Therefore, we have the following:

(1) $r(r+j-1) = j - i$, (2) $i(r+j-1) + 1 = s + j$,
(3) $j(r+j-1) + 1 = j$, (4) $r(i-s+r-1) = i - s - j + r$,
(5) $j(i-s+r-1) = i - s$.

By (4) and (5), $(j-r)(i-s+r-2) = 0$. Thus $j = r$ or $i - s + r = 2$. If $i - s + r = 2$, then by (4) $j = i - s$. Now by (1), $r(r+i-s-1) = -s$. So by considering (4) $i + r = j$. Thus $r = -s$ by $j = i - s$. So $i = 2s + 2$, and hence $j = s + 2$. Now by (2), $1 = 0$, a contradiction. If $j = r$, then $r(2r-1) = r - i$ by (1). Also by (3), $r(2r-1) = r - 1$. So $i = 1$, and hence by (2), $s = r$. Now by (5), $s = r = j = 1$. Thus by (1), $1 = 0$, a contradiction.

Now suppose that $(r, n) = 1$. Since $\sigma\colon a \mapsto a^r, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a$, $a_4 = a^r$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$ and $0 < r \le n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = a^i b$, $a_2^{\rho^*} = (a)^{\rho^*} = a^{k-j}$. By considering the image of $a_1 = a^i b$, $a_3 = a^j b$, $a_4 = a^r$ and $b_0 = a^k b$ under

$\rho^*$, we conclude that $a^{i(k-j)}a^ib = a^{k+1}b$, $a^{j(k-j)}a^ib = a^{k+r}b$, $a^{r(k-j)} = a^k$ and $a^{k(k-j)}a^ib = a^kb$. Also $a_0^{\tau^*} = b^{\tau^*} = a^{i-1}b$, $a_2^{\tau^*} = a^{\tau^*} = a^{j-r-k+1}$. By considering the image of $a_1 = a^ib$, $a_3 = a^jb$ and $b_0 = a^kb$ under $\tau^*$, we conclude that $a^{i(j-r-k+1)}a^{i-1}b = a^{j-r}b$, $a^{j(j-r-k+1)}a^{i-1}b = a^{j-1}b$ and $a^{k(j-r-k+1)}a^{i-1}b = a^{i-1-r+j}b$.

Therefore, we have the following:

(1) $ik - ij + i = k + 1,$ (2) $jk - j^2 + i = r + k,$

(3) $rk - rj = k,$ (4) $k^2 - kj + i = k,$

(5) $i(j - r - k + 1) + i - 1 = -r + j,$ (6) $j(j - r - k + 1) = j - i,$

(7) $k(j - r - k + 1) = j - r.$

By (6), $j^2 - jr - jk + i = 0$. Also by (6) and (7), we have $kj(j - r - k + 1) = kj - ki$ and $kj(j - r - k + 1) = j^2 - rj$. Thus $j^2 - jr = kj - ki$. Thus $i(k - 1) = 0$, and so $i = 0$ or $k = 1$. If $i = 0$, then by (1), we have $k = -1$. Also by (4), $j = -2$. Now by (2), $r = -1$. Therefore,

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2}b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. By Proposition 2.1, $\rho$ and $\tau$ lift. Clearly, $AGL(1,5) = \langle \rho, \tau \rangle$ is 1-regular. Thus $\mathrm{Aut}(\widetilde{X})$ contains a 1-regular subgroup lifted by $\langle \rho, \tau \rangle$.

Now if $k = 1$, then by (3) and (4), $r - rj = 1$ and $i - j = 0$. Since $i = j$, it follows that $i(i - r) = -r + 1$ by (5). So $i^2 - ir = -r + 1 = -1 - rj + 1$. Thus $i = j = 0$, and so $r = 1$. Now by (2), $2 = 0$, a contradiction.

*Subcase II*. $o(a_0) = o(a_1) \neq 2$.

By considering $a_0^{\tau^*} = a_2^{-1}a_1^{-1}$, we have $o(a_2^{-1}a_1^{-1}) \neq 2$. It follows that $o(a_2) \neq 2$. Since $a_2^{\rho^*} = b_0^{-1}a_3$, we have either $o(b_0) = o(a_3) = 2$ or $o(b_0) \neq 2$ and $o(a_3) \neq 2$. First suppose that $o(b_0) = o(a_3) = 2$. Since $a_3^{\rho^*} = a_4b_0$, it follows that $o(a_4) \neq 2$. Now by considering $a_4^{\rho^*} = b_0^{-1}a_0$, we have $o(b_0^{-1}a_0) \neq 2$ a contradiction.

Now suppose that $o(b_0) \neq 2$ and $o(a_3) \neq 2$. Since $a_3^{\rho^*} = a_4 b_0$, we have $o(a_4) \neq 2$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. $\square$

**Lemma 3.3.** *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* $\rho$, $\sigma$ *and* $\tau$, *say* $L$, *lifts. Under the assumption* (I), $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(2n)$ *for* $n \geq 3$.

*Proof.* $\rho$ and $\sigma$ lift. With the same arguments as in Cubcase I, we have $n = 3$ and

$$a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\rho}$ $\bar{\sigma}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. By Proposition 2.1, $\rho$, $\sigma$ and $\tau$ lift. Also $S_5 = \langle \rho, \sigma, \tau \rangle$ is 2-arc-transitive. Thus $\mathrm{Aut}(\widetilde{X})$ contains a 2-arc-transitive subgroup lifted by $\langle \rho, \sigma, \tau \rangle$. Thus $\widetilde{X}$ is 2-arc-transitive. Moreover, $\rho$ and $\tau$ lift. With the same arguments as in Subcase II, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2}b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\sigma}$ can be extended to automorphisms of $D_{2n}$ whenever $n = 3$. Now if $n = 3$, then by Proposition 2.1, $\sigma$ lift. Now with the same arguments as above, $\widetilde{X}$ is 2-arc-transitive. $\square$

Now suppose that $n$ is even.

**Lemma 3.4.** *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* $\rho$ *and* $\sigma$, *say* $L$, *lifts. Then there is no connected regular covering of the complete graph* $K_5$ *whose fibre-preserving group is arc-transitive.*

*Proof.* Since $\rho, \sigma \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\sigma}$ can be extended to automorphisms of $D_{2n}$. We denote these extended automorphisms by $\rho^*$ and $\sigma^*$, respectively. In this case $o(a_0) = o(a_1) = o(a_3)$. Now we consider the following two subcases:

*Subcase I*. $o(a_0) = o(a_1) = o(a_3) = 2$.

Since $o(a_0) = 2$, we may assume that $a_0 = a^{n/2}$ or $a_0 \neq a^{n/2}$ and $a_0 = a^i b$ $(0 \leq i < n)$. If $a_0 = a^{n/2}$, then $a_1 = a_3 = a^{n/2}$. By Table 1, $a_1^{\sigma^*} = a_4 a_0$ and $a_3^{\sigma^*} = a_0$. Thus $a_4 = 1$ and so by $a_4^{\rho^*} = b_0^{-1} a_0$, we have $b_0 = a^{n/2}$. Also by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $a_2 = 1$. Therefore $K_5 \times_\phi D_{2n}$ is not connected, a contradiction.

Thus we may assume that $a_0 \neq a^{n/2}$. So $a_1 \neq a^{n/2}$ and $a_3 \neq a^{n/2}$. Thus we may assume that $a_0 = a^i b$, $a_1 = a^j b$ and $a_3 = a^k b$, where $0 \leq i, j, k < n$. By considering $a_1^{\rho^*} = a_2 b_0$, we have one of the following cases:

i) $a_2 = a^l b$, $b_0 = a^t$ $\quad (0 \leq l < n,\ 0 < t < n)$;

ii) $a_2 = a^l$, $b_0 = a^t b$ $\quad (0 < l < n,\ 0 \leq t < n)$.

First suppose that $a_2 = a^l b$, $b_0 = a^t$ $(0 \leq l < n,\ 0 < t < n)$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we may suppose that $a_4 = a^s b$, where $0 \leq s < n$. Now since $b_0^{\sigma^*} = a_2^{-1} a_4 a_0$, we have a contradiction. Now suppose that $a_2 = a^l$, $b_0 = a^t b$ $(0 < l < n,\ 0 \leq t < n)$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. First suppose that $o(a_4) \neq 2$. Now by Proposition 2.2, we may assume that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^l$, $a_4 = a^k$ and $b_0 = a^t b$, where $0 \leq i, j, k, t \leq n-1$ and $0 < l, k \leq n-1$. Now with the same arguments as in Subcase I, when $n$ is odd, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = ab, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1} b.$$

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\sigma}$ can be extended to automorphisms of $D_{2n}$ when $n = 3$, a contradiction.

Now suppose that $a_4 = a^{n/2}$. Now we may assume that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^r$, $a_4 = a^{n/2}$, and $b_0 = a^l b$, where $0 \leq i, j, k, l \leq n-1$ and $0 < r \leq n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^{n/2}$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$ and $0 < r \leq n-1$. Since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where

$t \in \{i, j, k, r\}$. Without loss of generality, we may assume that $(i, n) = 1$ or $(r, n) = 1$. In fact, with the same arguments as in other cases we get same results. First suppose that $(i, n) = 1$. Since $\sigma\colon a \mapsto a^i, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^r$, $a_4 = a^{(n/2)}$ and $b_0 = a^j b$, where $0 \leq i, j \leq n-1$ and $0 < r \leq n-1$. Now with the same arguments as in Subcase I, when $n$ is odd (by replacing $s$ with $(n/2)$), we have a contradiction.

Now suppose that $(r, n) = 1$. Since $\sigma\colon a \mapsto a^r, b \mapsto b$ is an automorphism of $D_{2n}$, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a$, $a_4 = a^{(n/2)}$ and $b_0 = a^k b$, where $0 \leq i, j, k \leq n-1$. Now by replacing $r$ with $(n/2)$ in Case I, when $n$ is odd, we have $(n/2)(k-j) = k$ and $(n/2)((n/2)-j) = -k+1$ (see Equations (3) and (7) in Subcase I). So $n = 2$, a contradiction.

*Subcase II*. $o(a_0) = o(a_1) = o(a_3) \neq 2$.

By considering $a_1^{\sigma^*} = a_4 a_0$, we have $o(a_4 a_0) \neq 2$. So we have $o(a_4) \neq 2$ or $o(a_4) = 2$ and $a_4 = a^{n/2}$. If $o(a_4) \neq 2$, then $o(b_0^{-1} a_0) \neq 2$ by $a_4^{\rho^*} = b_0^{-1} a_0$. Now we have $o(b_0) \neq 2$ or $o(b_0) = 2$ and $b_0 = a^{n/2}$. If $b_0 = a^{n/2}$, then $o(a_2) \neq 2$ by $a_2^{\rho^*} = b_0^{-1} a_3$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. If $o(b_0) \neq 2$, then by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_2) \neq 2$ or $o(a_2) = 2$ and $a_2 = a^{n/2}$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. Finally, if $a_4 = a^{n/2}$, then by considering $a_3^{\rho^*} = a_4 b_0$, we have $o(b_0) \neq 2$ or $o(b_0) = 2$ and $b_0 = a^{n/2}$. Clearly, $b_0 \neq a^{n/2}$ by $a_3^{\rho^*} = a_4 b_0$. Thus $o(b_0) \neq 2$, and so by $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_2) \neq 2$ or $o(a_2) = 2$ and $a_2 = a^{n/2}$. Therefore, $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. $\square$

**Lemma 3.5.** *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* $\rho$ *and* $\tau$*, say* $L$*, lifts. Under the assumption* (I), $\widetilde{X}$ *is arc-transitive if and only if* $\widetilde{X}$ *is isomorphic to* $DK(2n)$ *for* $n > 3$.

*Proof.* Since $\rho, \tau \in L$, Proposition 2.1 implies that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. We denote these extended automorphisms by $\rho^*$ and $\tau^*$, respectively. In this case $o(a_0) = o(a_1)$. Now we consider the following two subcases:

<u>*Subcase I*</u>. $o(a_0) = o(a_1) = 2$.

Since $o(a_0) = 2$, we may assume that $a_0 = a^{n/2}$ or $a_0 \neq a^{n/2}$ and $a_0 = a^i b$ $(0 \le i < n)$. If $a_0 = a^{n/2}$, then $a_1 = a^{n/2}$. By Table 1, we have $a_0^{\tau^*} = a_2^{-1}a_1^{-1}$ and $a_1^{\rho^*} = a_2 b_0$. Therefore, $a_2 = 1$ and $b_0 = a^{n/2}$. Also by $a_2^{\rho^*} = b_0^{-1}a_3$, we have $a_3 = a^{n/2}$. Now by $a_3^{\rho^*} = a_4 b_0$, we have $a_4 = 1$. Thus $\widetilde{X}$ is not connected, a contradiction. Thus we may assume that $a_0 \neq a^{n/2}$ and $a_0 = a^i b$. So $a_1 \neq a^{n/2}$ and so we may assume that $a_0 = a^i b$, $a_1 = a^j b$, where $0 \le i, j < n$. By considering $a_0^{\tau^*} = a_2^{-1}a_1^{-1}$, we have $o(a_2) \neq 2$ or $a_2 = a^{n/2}$. First assume that $o(a_2) \neq 2$. Thus $b_0 = a^k b$ $(0 \le k < n)$ by $a_1^{\rho^*} = a_2 b_0$. Also since $a_2^{\rho^*} = b_0^{-1}a_3$, we have $o(a_3) = 2$ and $a_3 = a^l b$ $(0 \le l < n)$. Finally, since $a_4^{\rho^*} = b_0^{-1}a_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. First suppose that $a_4 = a^{n/2}$. We have $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^r$, $a_4 = a^{n/2}$ and $b_0 = a^l b$, where $0 \le i, j, k, l \le n-1$ and $0 < r \le n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^r$, $a_4 = a^{n/2}$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$ and $0 < r \le n-1$. Since $a_4^{\rho^*} = b_0^{-1}a_0$, we have $k = n/2$. Now $a_4 a_0 = b_0$, and so $(a_4 a_0)^{\rho^*} = b_0^{\rho^*}$. Thus $a_0 = a_1$, and so $i = 0$. We have $a_0^{\rho^*} = a_1^{\rho^*}$. So $a_1 = a_2 b_0$, and hence $r = n/2$. Now $a_2 = a_4$, and so $a_2^{\rho^*} = a_4^{\rho^*}$. Therefore, $a_0 = a_3$, and hence $a_3 = b$. Now $K_5 \times_\phi D_{2n}$ is not connected a contradiction.

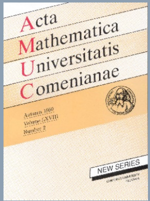
Now suppose that $o(a_4) \neq 2$. With the same arguments as in Subcase II, when $n$ is odd, we have

$$a_0 = b, a_1 = b, a_3 = a^{-2}b, a_2 = a, a_4 = a^{-1}, b_0 = a^{-1}b.$$

From Table 1, it is easy to check that $\bar{\rho}$ and $\bar{\tau}$ can be extended to automorphisms of $D_{2n}$. By Proposition 2.1, $\rho$ and $\tau$ lift. Also $AGL(1,5) = \langle \rho, \tau \rangle$ is 1-regular. Thus $\mathrm{Aut}(\widetilde{X})$ contains a 1-regular subgroup lifted by $\langle \rho, \tau \rangle$.

Now assume that $a_2 = a^{n/2}$. Thus $b_0 = a^k b$ $(0 \le k < n)$ by $a_1^{\rho^*} = a_2 b_0$. Also since $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_3) = 2$ and $a_3 = a^l b$ $(0 \le l < n)$. Finally, since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $o(a_4) \ne 2$ or $a_4 = a^{n/2}$. First suppose that $a_4 = a^{n/2}$. We have $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a_4 = a^{n/2}$ and $b_0 = a^l b$, where $0 \le i, j, k, l \le n-1$. Since $a_4^{\tau^*} = a_3 a_1 a_2$, we have $k = j$. Also since $a_2^{\tau^*} = a_3 a_4 b_0 a_2$, we have $l = k = j$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, by Proposition 2.2, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^i b$, $a_2 = a_4 = a^{n/2}$, and $b_0 = a^i b$, where $0 \le i, j, k \le n-1$. Since $a_4^{\rho^*} = b_0^{-1} a_0$, we have $i = n/2$, a contradiction.

Now suppose that $a_0 = a^i b$, $a_1 = a^j b$, $a_3 = a^k b$, $a_2 = a^{n/2}$, $a_4 = a^s$ and $b_0 = a^l b$, where $0 \le i, j, k, l \le n-1$ and $0 < s \le n-1$. Since $\mathrm{Aut}(D_{2n})$ acts transitively on involutions, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^{n/2}$, $a_4 = a^s$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$ and $0 < s \le n-1$. Since $K_5 \times_\phi D_{2n}$ is assumed to be connected, $D_{2n} = \langle a_0, a_1, a_2, a_3, a_4, b_0 \rangle$. Thus we may assume that $(t, n) = 1$, where $t \in \{i, j, k, s\}$. Without loss of generality we may assume that $(i, n) = 1$ or $(s, n) = 1$. In fact, with the same arguments the in other cases we get the same results. First suppose that $(i, n) = 1$. Therefore, we may assume that $a_0 = b$, $a_1 = ab$, $a_3 = a^i b$, $a_2 = a^{(n/2)}$, $a_4 = a^s$, and $b_0 = a^j b$, where $0 \le i, j \le n-1$ and $0 < s \le n-1$. Now with the same arguments as in Case II, when $n$ is odd we get a contradiction. Now suppose that $(s, n) = 1$. Therefore, we may assume that $a_0 = b$, $a_1 = a^i b$, $a_3 = a^j b$, $a_2 = a^{n/2}$, $a_4 = a$ and $b_0 = a^k b$, where $0 \le i, j, k \le n-1$. From Table 1, one can see that $a_0^{\rho^*} = b^{\rho^*} = a^i b$, $a_4^{\rho^*} = (a)^{\rho^*} = a^k$. By considering the image of $a_1 = a^i b$, $a_3 = a^j b$ and $a_2 = a^{n/2}$ under $\rho^*$, we conclude that $a^{ik+i} b = a^{(n/2)+k} b$, $a^{jk+i} b = a^{k+1} b$ and $a^{(n/2)k} = a^{k-j}$. Thus, we have $ik + i = n/2 + k$, $jk + i = k + 1$ and $(n/2)k = k - j$. By $(n/2)k = k - j$, we have $nk = 2k - 2j$. It follows that $2j = 2k$. Also $a^{\tau^*} = a^j b a^i b a^{(n/2)} = a^{j-i+(n/2)}$.




Thus $a_2^{\tau^*} = a^{n/2(j-i+(n/2))} = a^j b a a^k b a^{(n/2)} = a^{j-1-k+(n/2)}$. So, $2j - 2k - 2 = 0$ and so $2 = 0$, a contradiction.

<u>*Subcase II*</u>. $o(a_0) = o(a_1) \neq 2$.

By considering $a_0^{\tau^*} = a_2^{-1} a_1^{-1}$, we have $o(a_2^{-1} a_1^{-1}) \neq 2$. Thus $o(a_2) \neq 2$ or $a_2 = a^{n/2}$. First suppose that $o(a_2) \neq 2$. By considering $a_2^{\rho^*} = b_0^{-1} a_3$, we have one of the following cases:

i) $a_3 = a^i b$, $b_0 = a^j b$ $\quad (0 \leq i, j < n)$;
ii) $a_3 = a^i$, $b_0 = a^{n/2}$ $\quad (0 < i < n)$;
iii) $a_3 = a^{n/2}$, $b_0 = a^i$ $\quad (0 < i < n)$.

By $a_1^{\rho^*} = a_2 b_0$, we have a contradiction in the first case. Now consider the second case. Since $a_3^{\rho^*} = a_4 b_0$, we have $o(a_4) \neq 2$. Now $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. Now consider the last case. Since $a_3^{\rho^*} = a_4 b_0$, we have $o(a_4) \neq 2$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction.

Now suppose that $a_2 = a^{n/2}$. By $a_1^{\rho^*} = a_2 b_0$, we have $o(b_0) \neq 2$. Also since $a_2^{\rho^*} = b_0^{-1} a_3$, we have $o(a_3) \neq 2$. Finally, since $a_3^{\rho^*} = a_4 b_0$, we have $o(a_4) \neq 2$ or $a_4 = a^{n/2}$. Thus $K_5 \times_\phi D_{2n}$ is not connected, a contradiction. $\square$

**Lemma 3.6.** *Suppose that the subgroup of* $\mathrm{Aut}(\widetilde{X})$ *generated by* $\rho$, $\sigma$ *and* $\tau$, *say* $L$, *lifts. Then there is no connected regular covering of the complete graph* $K_5$ *whose fibre-preserving group is arc-transitive.*

*Proof.* $\rho$ and $\sigma$ lift. With the same arguments as in Case I, we have a contradiction. Also $\rho$ and $\tau$ lift. With the same arguments as in Subcase II, we have

$$a_0 = b, \quad a_1 = b, \quad a_3 = a^{-2} b, \quad a_2 = a, \quad a_4 = a^{-1}, \quad b_0 = a^{-1} b.$$

From Table 1, it is easy to check that $\bar{\sigma}$ can be extended to automorphisms of $D_{2n}$ whenever $n = 3$, a contradiction. $\square$

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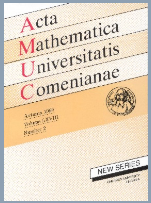


*Proof of Theorem 1.1.* This follows from Lemmas 3.1, 3.2 , 3.3, 3.4, 3.5 and 3.6. □


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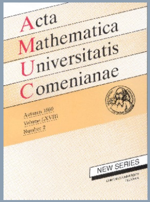


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