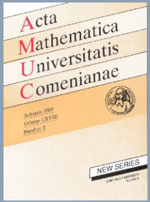

ON BROWDER-TYPE THEOREMS

F. LOMBARKIA AND H. ZARIOUH

ABSTRACT. The aim of this paper is to introduce new spectral properties as a continuation of the papers [8], [9], [11], [16], which are variants to the classical a-Browder's and Browder's theorem and we study the relationship between these properties and other Weyl-type theorems.

1. Introduction

We begin this part by some preliminary basic definitions. Let X be a Banach space and let $L(X)$ be the Banach algebra of all bounded linear operators acting on X. For $T \in L(X)$, by $N(T)$, we denote the null space of T, by $n(T)$, the nullity of T, by $d(T)$, its defect and by $R(T)$ the range of T. By $\sigma(T)$, we denote a the spectrum of T and by $\sigma_a(T)$, the approximate point spectrum of T. If $R(T)$ is closed and $n(T) < \infty$ (resp., $d(T) < \infty$), then T is called an upper semi-Fredholm operator, (resp., a lower semi-Fredholm operator). If $T \in L(X)$ is either upper or lower semi-Fredholm operator, then T is called a semi-Fredholm operator and the index of T is defined by $\operatorname{ind}(T) = n(T) - d(T)$. If both $n(T)$ and $d(T)$ are finite, then T is called a Fredholm operator. An operator $T \in L(X)$ is called a Weyl operator if it is a Fredholm operator of index zero. The Weyl spectrum $\sigma_W(T)$ of T is defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}.$$

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For a bounded linear operator T and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$, viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n, the range space $R(T^n)$ is closed and $T_{[n]}$, is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp., a lower) semi-B-Fredholm operator, and in this case, the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [7]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator, see [4]. An operator $T \in L(X)$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

The ascent $a(T)$ of an operator T is defined by

$$a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$$

and the descent $\delta(T)$ of T is defined by

$$\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$$

with $\inf \emptyset = \infty$.

According to [14], a complex number $\lambda \in \sigma(T)$ is a pole of the resolvent of T if $T - \lambda I$ has a finite ascent and finite descent, and in this case, they are equal. According to [6], a complex number $\lambda \in \sigma_a(T)$ is a left pole of T if $a(T - \lambda I) < \infty$ and $R(T^{a(T-\lambda I)+1})$ is closed. An operator T is called Drazin invertible if 0 is a pole of T and is called left Drazin invertible if 0 is a left pole of T.

Let $SF_+(X)$ be the class of all upper semi-Fredholm operators and

$$SF_+^-(X) = \{T \in SF_+(X) : \operatorname{ind}(T) \leq 0\}.$$

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The upper-semi Weyl spectrum $\sigma_{SF_+^-}(T)$ of T is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}.$$

Similarly the upper semi-B-Weyl spectrum $\sigma_{SBF_+^-}(T)$ of T is defined.

This paper is a continuation of our recent investigations in the subject of Browder-type theorems. As in [8] and [9], we investigate other new variants of Browder's and a-Browder's theorem by introducing new spectral properties (see later for Definitions) for bounded linear operators, in connection with Weyl-type theorems. The essential results obtained in this paper are summarized in the diagram presented at the end of this paper. For further definitions and symbols, we refer the reader to [9] and also to [4], [5], [7], [8], [15] and [16] for more details.

Hereafter the symbol iso$A$ means isolated points of a given subset A of $\mathbb{C}$. In addition, we recall the list of all symbols and notations we use:

$E(T)$: eigenvalues of T that are isolated in the spectrum $\sigma(T)$,
$E^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma(T)$,
$E_a(T)$: eigenvalues of T that are isolated in the approximate point spectrum $\sigma_a(T)$,
$E_a^0(T)$: eigenvalues of T of finite multiplicity that are isolated in $\sigma_a(T)$,
$\Pi(T)$: poles of T,
$\Pi^0(T)$: poles of T of finite rank,
$\Pi_a(T)$: left poles of T,
$\Pi_a^0(T)$: left poles of T of finite rank,
$\sigma_{BW}(T)$: B-Weyl spectrum of T,
$\sigma_W(T)$: Weyl spectrum of T,
$\sigma_{SBF_+^-}(T)$: upper semi-B-Weyl spectrum of T,
$\sigma_{SF_+^-}(T)$: upper-semi Weyl spectrum of T,

$\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$,
$\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$,
$\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$; $\quad \Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$,
$\Delta(T) = \sigma(T) \setminus \sigma_W(T)$; $\quad \Delta_+(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T)$,
$\Delta(T) = \Pi^0(T)$: Browder's theorem holds for T (B for brevity),
$\Delta^g(T) = \Pi(T)$: generalized Browder's theorem holds for T (gB for brevity),
$\Delta_a^g(T) = \Pi_a^0(T)$: a-Browder's theorem holds for T (aB for brevity),
$\Delta_a^g(T) = \Pi_a(T)$: generalized a-Browder's theorem holds for T (gaB for brevity),

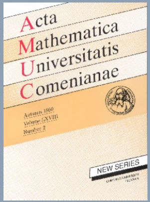
The inclusion of the following table [8], [11], [16], [17] which contains the meaning of some properties is motivated by having overview of the subject.

(z)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	(gz)	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$
(az)	$\sigma(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$	(gaz)	$\sigma(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
(b)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(w_1)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset E^0(T)$	(Bb)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$

2. Main results

Very recently [11], the property (w_1) for bounded linear operators as a variant of Weyl's theorem was introduced, and the necessary and sufficient conditions for which this property holds were established. But the next proposition shows that property (w_1) is equivalent to the property (b) introduced in [8] and studied by several authors, see, for example, [2], [8].

Proposition 2.1. *Let $T \in L(X)$. Then the following statements are equivalent.*



(i) T *satisfies property* (w_1);
(ii) T *satisfies property* (b);
(iii) $\sigma_a(T) = \sigma_{SF_+^-}(T) \cup \operatorname{iso}\sigma(T)$.

Proof. (i) $\Longleftrightarrow$ (ii) Suppose that T satisfies property (w_1), then

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset E^0(T).$$

Thus from [1, Theorem 3.77], we have

$$\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T) \Longleftrightarrow \lambda \in \operatorname{iso}\sigma(T) \cap \sigma_{SF_+^-}(T)^C \Longleftrightarrow \lambda \in \Pi^0(T),$$

where $\sigma_{SF_+^-}(T)^C$ is the complement of the upper semi-Weyl spectrum of T. So $\Delta_a(T) = \Pi^0(T)$ and T satisfies property (b). The converse is obvious since $\Pi^0(T) \subset E^0(T)$ is always true.

(ii) $\Longrightarrow$ (iii) Property (b) for T entails that $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset \operatorname{iso}\sigma(T)$. So $\sigma_a(T) \subset \sigma_{SF_+^-}(T) \cup \operatorname{iso}\sigma(T)$ and since $\sigma_a(T) \supset \sigma_{SF_+^-}(T) \cup \operatorname{iso}\sigma(T)$ is always true, then $\sigma_a(T) = \sigma_{SF_+^-}(T) \cup \operatorname{iso}\sigma(T)$. The reverse implication is obtained by using the same argument as in the proof of "(i) $\Longrightarrow$ (ii)". $\square$

In the following definition, we investigate new variants of a-Browder-type theorem and Browder's theorem.

Definition 2.2. Let $T \in L(X)$. We will say that:

- T satisfies property (gw_1) if $\Delta_a^g(T) \subset E(T)$.
- T satisfies property (w_2) if $\Delta(T) \subset E^0(T)$.
- T satisfies property (gw_2) if $\Delta^g(T) \subset E(T)$.

As a generalization of Proposition 2.1 to the general context of B-Fredholm theory, in the next result, we prove that property $(gb) \Longleftrightarrow$ property (gw_1).

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Proposition 2.3. *Let $T \in L(X)$. Then the following statements are equivalent:*

(i) *T satisfies property (gw_1);*
(ii) *T satisfies property (gb);*
(iii) $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$.

Proof. (i) $\Longleftrightarrow$ (ii) Assume that T satisfies property (gw_1), then

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subset E(T).$$

From the punctured neighborhood theorem [7, Corollary 3.2] we deduce that

$$\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \Longleftrightarrow \lambda \in \operatorname{iso}\sigma(T) \cap \sigma_{BW}(T)^C \Longleftrightarrow \lambda \in \Pi(T),$$

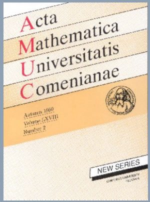
where $\sigma_{BW}(T)^C$ is the complement of the B-Weyl spectrum of T. This implies that $\Delta_a^g(T) = \Pi(T)$ and T satisfies property (gb). The converse is trivial, since $\Pi(T) \subset E(T)$ holds for every operator.

(ii) $\Longleftrightarrow$ (iii) Property (gb) for T entails that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subset \operatorname{iso}\sigma(T)$. So $\sigma_a(T) \subset \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$ and since $\sigma_a(T) \supset \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$ holds for every operator, then $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup \operatorname{iso}\sigma(T)$. The reverse implication goes similarly to the proof of “(i) $\Longrightarrow$ (ii)”. $\square$

Now, the next proposition proves that $(w_2) \Longleftrightarrow (gw_2) \Longleftrightarrow gB \Longleftrightarrow B$.

Proposition 2.4. *Let $T \in L(X)$. Then the following assertions are equivalent:*

(i) *T satisfies property (w_2);*
(ii) *T satisfies Browder's theorem;*
(iii) *T satisfies generalized Browder's theorem;*
(iv) *T satisfies property (gw_2).*



Proof. (i) $\iff$ (ii) Suppose that T satisfies property (w_2), then $\sigma(T) \setminus \sigma_W(T) \subset E^0(T)$. Thus

$$\lambda \in \sigma(T) \setminus \sigma_W(T) \iff \lambda \in \operatorname{iso} \sigma(T) \cap \sigma_W(T)^C \iff \lambda \in \Pi^0(T),$$

where $\sigma_W(T)^C$ is the complement of the Weyl spectrum of T. So $\Delta(T) = \Pi^0(T) \iff \Delta(T) \subset E^0(T)$ as desired.

(iii) $\iff$ (iv) As $\Delta(T) \subset \Delta^g(T)$, then $(gw_2) \implies (w_2) \implies B$. Since $B \iff gB$ [i.e. (ii) $\iff$ (iii)], see [3, Theorem 2.1] we conclude that $(gw_2) \iff gB \iff (w_2)$. $\square$

We investigate new variants of a-Browder's theorem and other new variants of Browder's theorem, in the two following definitions (Definitions 2.5 and 2.7).

Definition 2.5. Let $T \in L(X)$. We say that:

- T satisfies property (aw_1) if $\Delta_a(T) \subset E^0_a(T)$.
- T satisfies property (gaw_1) if $\Delta^g_a(T) \subset E_a(T)$.
- T satisfies property (ab_1) if $\Delta(T) \subset \Pi^0_a(T)$.
- T satisfies property (gab_1) if $\Delta^g(T) \subset \Pi_a(T)$.
- T satisfies property (aw_2) if $\Delta(T) \subset E^0_a(T)$.
- T satisfies property (gaw_2) if $\Delta^g(T) \subset E_a(T)$.

The next result proves that $(aw_1) \iff (gaw_1) \iff (gaB) \iff aB$ and $(ab_1) \iff (w_2) \iff (gw_2) \iff (aw_2) \iff (gaw_2) \iff (gab_1) \iff gB \iff B$.

Proposition 2.6. *Let $T \in L(X)$. Then*

1. *The assertions* (i)–(iv) *are equivalent:*
   - (i) *T satisfies property (aw_1);*
   - (ii) *T satisfies a-Browder's theorem;*
   - (iii) *T satisfies property (gaw_1);*
   - (iv) *T satisfies generalized a-Browder's theorem.*

2. *The assertions* (a)–(e) *are equivalent:*
   (a) T *satisfies property* (ab_1)*;*
   (b) T *satisfies property* (w_2)*;*
   (c) T *satisfies property* (gab_1)*;*
   (d) T *satisfies property* (aw_2)*;*
   (e) T *satisfies property* (gaw_2)*.*

Proof. 1. (i) $\Longleftrightarrow$ (ii) Suppose that T satisfies property (aw_1). Then $\sigma_a(T)\setminus\sigma_{SF_+^-}(T)\subset E_a^0(T)$. Thus

$$\lambda\in\sigma_a(T)\setminus\sigma_{SF_+^-}(T)\Longleftrightarrow\lambda\in\operatorname{iso}\sigma_a(T)\cap\sigma_{SF_+^-}(T)^C\Longleftrightarrow\lambda\in\Pi_a^0(T).$$

This implies that

$$\Delta_a(T)=\Pi_a^0(T)\Longleftrightarrow\Delta_a(T)\subset E_a^0(T).$$

as desired.

(iii) $\Longleftrightarrow$ (iv) As $\Delta_a(T)\subset\Delta_a^g(T)$, then $(gaw_1)\Longrightarrow(aw_1)\Longrightarrow aB$. Since $aB\Longleftrightarrow gaB$ [i.e., (ii)$\Longleftrightarrow$ (iv)], see [3, Theorem 2.2], we conclude that $(gaw_1)\Longleftrightarrow gaB$.

2. (a) $\Longleftrightarrow$ (b) Suppose that T satisfies property (ab_1). Then $\sigma(T)\setminus\sigma_W(T)\subset\Pi_a^0(T)$. Thus $\lambda\in\Delta(T)$ implies $a(T-\lambda I)<\infty$ and since $\operatorname{ind}(T-\lambda I)=0$, then $a(T-\lambda I)=\delta(T-\lambda I)<\infty$. Hence $\lambda\in\Delta(T)\Longleftrightarrow\lambda\in\Pi^0(T)$. So $\Delta(T)\subset E^0(T)$. Conversely, if property (w_2) holds for T, then from Proposition 2.4, we have $\Delta(T)=\Pi^0(T)\subset\Pi_a^0(T)$, i.e., property (ab_1) holds for T.

(b)$\Longrightarrow$ (d), (c) $\Longrightarrow$ (a), (e) $\Longrightarrow$ (d) and (c) $\Longrightarrow$ (e) are clear.

(e) $\Longrightarrow$ (c) Suppose that T satisfies property (gaw_2). Then $\sigma(T)\setminus\sigma_{BW}(T)\subset E_a(T)$. Thus

$$\lambda\in\sigma(T)\setminus\sigma_{BW}(T)\Longrightarrow\lambda\in\operatorname{iso}\sigma_a(T)\cap\sigma_{SBF_+^-}(T)^C\Longleftrightarrow\lambda\in\Pi_a(T),$$

where $\sigma_{SBF_+^-}(T)^C$ is the complement of the upper semi-B-Weyl spectrum of T. Then $\Delta^g(T)\subset\Pi_a(T)$ as desired. Similarly, we prove the implication (d) $\Longrightarrow$ (a).

(b) $\Longrightarrow$ (c) Property (w_2) for T implies from Proposition 2.4 that $\Delta^g(T) = \Pi(T)$. Hence $\Delta^g(T) \subset \Pi_a(T)$. This completes the proof. $\square$

Definition 2.7. Let $T \in L(X)$. We will say that:

- T satisfies property (Bw_1) if $\Delta^g(T) \subset E^0(T)$.
- T satisfies property (Bab_1) if $\Delta^g(T) \subset \Pi_a^0(T)$.
- T satisfies property (Baw_1) if $\Delta^g(T) \subset E_a^0(T)$.

Similarly to Proposition 2.6, in the next result, we show that $(Bw_1) \Longleftrightarrow (Bab_1) \Longleftrightarrow (Baw_1) \Longleftrightarrow (Bb)$, and under the assumption $\Pi(T) = \Pi^0(T)$, $(Bw_1) \Longleftrightarrow gB$.

Proposition 2.8. *Let $T \in L(X)$. Then*

1. *The following assertions are equivalent:*
   (i) *T satisfies property (Bw_1);*
   (ii) *T satisfies property (Bab_1);*
   (iii) *T satisfies property (Baw_1);*
   (iv) *T satisfies property (Bb).*
2. *T satisfies property (Bw_1) $\Longleftrightarrow$ T satisfies property (gab_1) and $\Pi(T) = \Pi^0(T)$.*

Proof. 1. (ii) $\Longleftrightarrow$ (iii) Since $\Pi_a^0(T) \subset E_a^0(T)$ holds for every operator, this proves the direct implication. Conversely, suppose that $\Delta^g(T) \subset E_a^0(T)$. Then

$$\lambda \in \sigma(T) \setminus \sigma_{BW}(T) \Longrightarrow \lambda \in \operatorname{iso}\sigma_a(T) \cap \sigma_{SF_+^-}(T)^C \Longleftrightarrow \lambda \in \Pi_a^0(T).$$

Thus $\Delta^g(T) \subset \Pi_a^0(T)$.

(i)$\Longrightarrow$ (iii) Obvious since $E^0(T) \subset E_a^0(T)$ is always true.

(ii)$\Longrightarrow$ (i) Suppose that $\Delta^g(T) \subset \Pi_a^0(T)$. Then $\Delta^g(T) \subset \Pi_a(T)$. From Proposition 2.6, we have $\Delta^g(T) \subset E(T)$. Thus $\Delta^g(T) \subset E^0(T)$.

(i) $\iff$ (iv) Suppose that $\Delta^g(T) \subset E^0(T)$. Then

$$\lambda \in \sigma(T) \setminus \sigma_{BW}(T) \Longrightarrow \lambda \in \operatorname{iso}\sigma(T) \cap \sigma_W(T)^C \iff \lambda \in \Pi^0(T).$$

Hence $\Delta^g(T) = \Pi^0(T)$ as desired. The reverse implication is clear.

2. If T satisfies property (Bw_1), then it satisfies property (Bb) that is $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T) \subset \Pi(T)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$ and this is equivalent from Proposition 2.6 to the property (gab_1). So T satisfies (gab_1) and $\Pi(T) = \Pi^0(T)$. The converse is trivial. $\square$

From Proposition 2.8 we remark that if $T \in L(X)$ satisfies property (Bw_1) then it satisfies property (gab_1). But the converse does not hold in general: for example, let $U \in L(\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$ for all $(x) = (x_i) \in \ell^2(\mathbb{N})$. Then $\sigma(U) = \sigma_a(U) = \{0, 1\}$, $\sigma_{BW}(U) = \emptyset$ and $\Pi_a(U) = \{0, 1\}$. So $\Delta^g(U) \subset \Pi_a(U)$, i.e., U satisfies property (gab_1), but it does not satisfy property (Bw_1) because $E^0(U) = \{0\}$. Here $\Pi(U) = \{0, 1\}$ and $\Pi^0(U) = \{0\}$.

In the following corollary, we combine the results obtained in Propositions 2.4, 2.6 and 2.8.

Corollary 2.9. *Let $T \in L(X)$. Then we have:*

1. *T satisfies $B \iff T$ satisfies $gB \iff T$ satisfies $(ab_1) \iff T$ satisfies $(w_2) \iff T$ satisfies $(gw_2) \iff T$ satisfies $(gab_1) \iff T$ satisfies $(aw_2) \iff T$ satisfies (gaw_2).*
2. *T satisfies $(Bw_1) \iff T$ satisfies $(Bab_1) \iff T$ satisfies $(Baw_1) \iff T$ satisfies (Bb).*
3. *T satisfies $aB \iff T$ satisfies $gaB \iff T$ satisfies $(aw_1) \iff T$ satisfies (gaw_1).*

Recall that an operator $T \in L(X)$ is said to satisfy property (SBb) if $\Delta^g_a(T) = \Pi^0(T)$ which introduced very recently in [10]. In the next definition, we investigate new variants of property (az) [16] and new variants of property (SBb) [10].

Definition 2.10. Let $T \in L(X)$. We will say that:

- T satisfies property (z_1) if $\Delta_+(T) \subset E^0_a(T)$.
- T satisfies property (gz_1) if $\Delta^g_+(T) \subset E_a(T)$.

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- T satisfies property (SBw_1) if $\Delta_a^g(T) \subset E^0(T)$.
- T satisfies property $(SBaw_1)$ if $\Delta_a^g(T) \subset E_a^0(T)$.

The next result shows that $(z_1) \iff (gz_1) \iff (az) \iff (gaz)$ and $(SBw_1) \iff (SBb) \implies (SBaw_1)$.

Proposition 2.11. *Let $T \in L(X)$. Then*

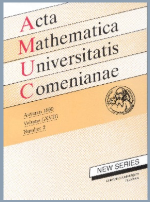
1. *The properties* (i)–(iii) *are equivalent:*
   (i) *T satisfies property (z_1);*
   (ii) *T satisfies property (gz_1);*
   (iii) *T satisfies property (az).*
2. *The properties* (a)–(b) *are equivalent:*
   (a) *T satisfies property (SBw_1);*
   (b) *T satisfies property (SBb);*
3. *If T satisfies property (SBb) then it satisfies property $(SBaw_1)$.*

Proof. 1. The equivalence (i) $\iff$ (iii) is clear.
(ii) $\implies$ (iii) Suppose that $\Delta_+^g(T) \subset E_a(T)$. Then

$$\begin{aligned}\lambda \in \sigma(T) \setminus \sigma_{SF_+^-}(T) &\Longrightarrow \lambda \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)\\ &\iff \lambda \in \operatorname{iso}\sigma_a(T) \cap \sigma_{SF_+^-}(T)^C \iff \lambda \in \Pi_a^0(T).\end{aligned}$$

Hence $\Delta_+(T) = \Pi_a^0(T)$ as desired. The equivalence of (az) and (gaz) follows from [16, Corollary 3.5]. Hence this proves the reverse implication (iii) $\implies$ (ii).

2. Since it was already mentioned we have $E^0(T) \subset E_a^0(T)$ and $\Pi^0(T) \subset E^0(T)$. Then immediately we obtain the equivalence of two properties (a) and (b). The third assertion is obvious. $\square$



As a conclusion, we give a summary of the results obtained in this paper. In the following diagram, arrows signify implications between the properties studied in this paper and other Browder-type theorems. The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (numbers in square brackets).

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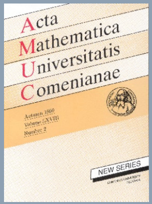

$$
\begin{tikzcd}
 & (gz_1) \arrow[d, Leftrightarrow, "2.11"] & & & \\
 & (z_1) \arrow[d, Leftrightarrow, "2.11"] & (w_2) \arrow[d, Leftrightarrow, "2.6"] & & \\
 & (gaz) \arrow[d, Leftrightarrow, "{[16]}"] & (aw_2) \arrow[d, Leftrightarrow, "2.6"] & & \\
(w_1) \arrow[d, Leftrightarrow, "2.1"] & (az) \arrow[d, "{[16]}"] & (ab_1) \arrow[d, Leftrightarrow, "2.9"] & (Bab_1) \arrow[d, Leftrightarrow, "2.8"] & \\
(b) \arrow[r, "{[8]}"] & aB \arrow[r, "{[12]}"] \arrow[d, Leftrightarrow, "{[3]}"] & B \arrow[d, Leftrightarrow, "{[3]}"] & (Bw_1) \arrow[d, Leftrightarrow, "2.8"] & (SBw_1) \arrow[d, Leftrightarrow, "2.11"] \\
(gb) \arrow[u, "{[8]}"] \arrow[r, "{[8]}"] \arrow[d, Leftrightarrow, "2.3"] & gaB \arrow[r, "{[6]}"] \arrow[d, Leftrightarrow, "2.6"] & gB \arrow[d, Leftrightarrow, "2.9"] & (Bb) \arrow[l, "{[17]}"'] \arrow[d, Leftrightarrow, "2.8"] & (SBb) \arrow[l, "{[10]}"'] \arrow[d, Leftrightarrow, "2.11"] \\
(gw_1) & (gaw_1) \arrow[d, Leftrightarrow, "2.6"] & (gab_1) \arrow[d, Leftrightarrow, "2.6"] & (Baw_1) & (SBaw_1) \\
 & (aw_1) & (gaw_2) \arrow[d, Leftrightarrow, "2.9"] & & \\
 & & (gw_2) & &
\end{tikzcd}
$$

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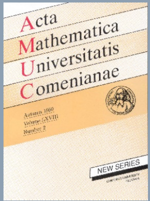

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