Acta Math. Univ. Comenianae
Vol. LX, 1(1991), pp. 23–29

TYPICAL MEASURABLE FUNCTION IN THE TOPOLOGY OF CLOSE APPROXIMATION

J. TIŠER AND L. ZAJÍČEK

Abstract. We show that the typical Lebesgue or Baire measurable function with respect to the topology of close approximation has the range of second category and that there are non empty open sets of the space of measurable functions where the typical function has G_δ dense range and the preimage of every point of the range contains a perfect set.

Let Σ be a σ-field of subsets of a set X and let S be the set of all real-valued Σ-measurable functions on X. Write for $f, g \in S$

$$G(f,g) = \{\, h \in S \mid f(x) < h(x) < g(x) \quad \text{for } x \in X \,\}.$$

These sets form a base of topology called topology of **close approximation** τ_c. We will use also the following notation: if $M \in \Sigma$ and $J \subset \mathbb{R}$ is any interval we denote

$$S_{M,J} = \{ f \colon M \to J \mid \ f \ \text{ is } \ \Sigma\text{-measurable} \}.$$

If Σ is the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ the topology of closed approximation was investigated in [**DS**]. A. H. Stone and D. Maharam proved that the space S is always Baire under the topology τ_c [cf. **F1**]. D.Fremlin [**F**],[**F1**] studied this topology in general setting. We are interested in the behavior of the typical function in the space (S, τ_c). As usual, we shall say that the typical function has a given property P provided the set of the functions possessing P forms a residual set. D. Fremlin and (independently) D. Preiss [cf. **DS**] proved that the typical Lebesgue measurable function is **not** injective. This is even true for functions defined on so called proto-perfect measure space [cf. **F1**] (which implies that the typical Baire measurable function is not injective too). During the Meeting on Real Analysis and Measure Theory (Capri 1990) A. H. Stone posed in his lecture the following questions: whether for the typical Lebesgue measurable function f there exists $y \in \mathbb{R}$ such that $f_{-1}(y)$ is a singleton and whether the typical Lebesgue measurable function has range of the first category. We answer these two questions in the negative also in the case where Σ is the σ-field of sets with the Baire property (this case was brought to our attention by D. Fremlin).

Received January 29, 1991; revised March 21, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 26A99, 54C30.

We shall call each product $B \times I$ of a Σ-measurable set B and a closed non degenerate interval $I \subset \mathbb{R}$ a **box**. If $\mathcal{B}$ is a family of boxes we say that $\mathcal{B}$ is **regular** provided $\mathcal{B}$ is countable and the system $\{\pi_1(B)\}_{B\in\mathcal{B}}$ forms a decomposition of X.(Symbols π_1, π_2 denote projections onto the first and the second coordinate, respectively). Also, we define for regular $\mathcal{B}$

$$G(\mathcal{B}) = \{ f \in S \mid \operatorname{graph}(f) \subset \cup\mathcal{B} \}.$$

Remark. If the sequence of sets $G(\mathcal{B}_n)$ is decreasing and $\operatorname{diam}\pi_2(B) \leq \frac{1}{n}$ whenever $B \in \mathcal{B}_n$ then, clearly,

$$\bigcap_{n=1}^{\infty} G(\mathcal{B}_n) = \{f\}$$

and the function f is Σ-measurable.

We describe now the Banach-Mazur game which will be used below. Let Y be an arbitrary topological space, and let $\mathcal{G}$ be a class of subsets of Y such that each member of $\mathcal{G}$ has a non empty interior and such that each non empty open set contains a member of $\mathcal{G}$. Let $C \subset Y$ be any set. Two players (I) and (II) alternately choose sets $G_n \in \mathcal{G}$ such that $G_{n+1} \subset G_n$ for $n = 1, 2, \dots$. So the player (I) chooses the sets with odd index, and the player (II) those with even index. Player (II) wins in case $\bigcap G_n \subset C$. Otherwise (I) wins.

We shall use the following proposition from [**O**] which is a straightforward generalization of the result due to S. Banach [cf. **O1**].

Lemma 1. *There is a winning strategy in the Banach-Mazur game for the second player if and only if the set C is residual in Y.*

It will be, also, convenient to use the following standard notation.

Definition. Let $\sigma \in \{0,1\}^{\mathbb{N}}$. The first n terms of the sequence $\sigma = \{\sigma_1, \sigma_2, \dots\}$ will be denoted by

$$\sigma|n = (\sigma_1, \sigma_2, \dots, \sigma_n).$$

Before stating the main result we prove the following easy lemma.

Lemma 2. *Let (I_k) be a family of all closed non degenerate intervals with the rational endpoints contained in a given interval I_0. Suppose that for each I_k there is a non degenerate interval J_k such that $J_k \subset I_k$ for every k. Then the set D of $y \in I_0$ belonging to at most one J_k is nowhere dense.*

Proof. Let U be any open subinterval of I_0. There are two numbers k, l such that $U \supset I_k \supset J_k \supset I_l \supset J_l$. So the set $\operatorname{Int} J_l$ does not meet the set D, hence D is nowhere dense in I_0. $\square$

Proposition. *Let $J \subset \mathbb{R}$ be any interval. Let (X, Σ) be a measurable space of the following type: either $X \subset \mathbb{R}$ is a second category set with the Baire property and Σ is the σ-field of all subsets of X with the Baire property (with respect to $\mathbb{R}$) or $X \subset \mathbb{R}$ is a Lebesgue measurable set of positive measure and Σ is the σ-field of all Lebesgue measurable subsets of X. Then there is an open non empty subset $G \subset (S_{X,J}, \tau_c)$ such that the typical function $f \in G$ has G_δ range $f(X)$ dense in J and for each $y \in f(X)$ the set $f_{-1}\{y\}$ contains a non empty perfect set.*

Proof. We start with the case Σ is the family of the sets with the Baire property. Let $\mathcal{B}_0$ be a regular family of boxes such that

(*) the set $\pi_1(B)$ is of the second category with the Baire property for each $B \in \mathcal{B}_0$ and

(**) the system $\{\pi_2(B) \mid B \in \mathcal{B}_0\}$ contains all closed subintervals of J with the rational endpoints.

Such family $\mathcal{B}_0$ clearly exists. Now put

$$G = \operatorname{Int} G(\mathcal{B}_0).$$

Let C be the set of all functions of G with G_δ range $f(X)$ dense in J and such that the preimage of any point of the range contains a non empty perfect set. By Lemma 1 we are obliged to find a winning strategy for the second player of the Banach-Mazur game corresponding to the space $Y = (G, \tau_c)$, the set C and

$$\mathcal{G} = \{G(\mathcal{B}) \subset G \mid \mathcal{B} \text{ is regular family of boxes }\}.$$

It is easy to prove that $\mathcal{G}$ has the desired properties for the Banach-Mazur game.

Suppose the first player will choose sets U_m and the second one the sets V_m such that

$$G \supset U_1 \supset V_1 \supset U_2 \supset V_2 \supset \dots .$$

The second player will construct in each his m-th move not only the set V_m but also the following auxiliary objects

$$\mathcal{B}_m,\ \mathcal{M}_m,\ \widetilde{\mathcal{B}}_m,\ \widetilde{\mathcal{M}}_m,\ \mathcal{E}_m, \text{ and } \quad \mathcal{D}_m$$

such that

(1) $\mathcal{E}_m = \big(E_i^{(m)}\big)_{i=1}^\infty$ and $\mathcal{D}_m = \big(D_i^{(m)}\big)_{i=1}^\infty$ are the families of closed nowhere dense subsets of X and J, respectively,

(2) $\widetilde{\mathcal{B}}_m \cup \widetilde{\mathcal{M}}_m$ and $\mathcal{B}_m \cup \mathcal{M}_m$ are the regular families of boxes such that

$$V_m = G(\mathcal{B}_m \cup \mathcal{M}_m) \subset G(\widetilde{\mathcal{B}}_m \cup \widetilde{\mathcal{M}}_m) \subset U_m\,,$$

(3) If $B \in \widetilde{\mathcal{B}}_m$ then there is a compact non degenerate interval $I_B \supset \pi_1(B)$ such that $\operatorname{diam} I_B < \frac{1}{m}$ and $I_B \setminus \pi_1(B) \subset \bigcup_{i=1}^\infty E_i^{(m)}$. (Note that I_B is uniquely determined.) Further

$$\bigcup \{I_B \mid B \in \widetilde{\mathcal{B}}_m\} \cap \bigcup_{\substack{1 \le k < m \\ 1 \le j \le k}} E_j^{(k)} = \emptyset\,,$$

(4) Whenever $B \in \widetilde{\mathcal{B}}_m \cup \widetilde{\mathcal{M}}_m$ then $\pi_2(B)$ is a closed interval of diameter at most $\frac{1}{m}$ and

$$\pi_2(B) \cap \bigcup_{\substack{1 \le k < m \\ 1 \le j \le k}} D_j^{(k)} = \emptyset\,,$$

(5) For each $\widetilde{B} \in \widetilde{\mathcal{B}}_{m-1}$ and $y \in \pi_2(\widetilde{B}) \setminus \bigcup_{i=1}^{\infty} D_i^{(m)}$

$$\operatorname{card}\{B \in \widetilde{\mathcal{B}}_m \mid B \subset \widetilde{B},\, y \in \pi_2(B)\} \ge 2\,,$$

(6) Every $B \in \mathcal{B}_m$ has the projection $\pi_1(B)$ of the second category with the Baire property and it is contained in some box $\widetilde{B} \in \widetilde{\mathcal{B}}_m$. Also, for each $\widetilde{B} \in \widetilde{\mathcal{B}}_m$ the system $\{\pi_2(B) \mid B \in \mathcal{B}_m,\, B \subset \widetilde{B}\}$ is the system of all closed intervals $I \subset \pi_2(\widetilde{B})$ with the rational endpoints.

We see that (*) and (**) express the validity of (6) for $m = 0$ if we formally denote $\widetilde{\mathcal{B}}_0 = \{X \times J\}$. Suppose now that $n \ge 1$ and all $\mathcal{B}_m$, $\mathcal{M}_m$, $\widetilde{\mathcal{B}}_m$, $\widetilde{\mathcal{M}}_m$, $\mathcal{E}_m$, $\mathcal{D}_m$, U_m, V_m are constructed for all $1 \le m < n$ such that (1) – (6) hold and let the first player choose a set U_n. We know that

$$U_n \subset G(\mathcal{B}_{n-1} \cup \mathcal{M}_{n-1}) \tag{$\diamondsuit$}$$

and that $U_n = G(\mathcal{S})$ where $\mathcal{S}$ is a regular family of boxes. Our aim is to construct $\mathcal{B}_n$, $\mathcal{M}_n$, $\widetilde{\mathcal{B}}_n$, $\widetilde{\mathcal{M}}_n$, $\mathcal{E}_n$, $\mathcal{D}_n$ such that (1) – (6) hold with $m = n$.

Let $B \in \mathcal{B}_{n-1}$ be fixed. There must be a box $F_B \in \mathcal{S}$ such that $\pi_1(F_B) \cap \pi_1(B)$ is a second category set with the Baire property. From ($\diamondsuit$) we infer that $\pi_2(F_B) \subset \pi_2(B)$. Now we can take a closed interval T_B of the diameter less than $\frac{1}{n}$ such that $T_B \subset \pi_2(F_B) \setminus \bigcup_{\substack{1 \le m < n \\ 1 \le j \le m}} D_j^{(m)}$. Further we choose an open interval K such that the set $K \setminus \big(\pi_1(F_B) \cap \pi_1(B)\big)$ is of the first category and, subsequently, we can choose another interval K_B, this time a closed one with $\operatorname{diam} K_B < \frac{1}{n}$, such that

$$K_B \subset K \setminus \bigcup_{\substack{1 \le m < n \\ 1 \le j \le m}} E_j^{(m)}\,.$$

Now we put

$$P_B = [K_B \cap \pi_1(F_B) \cap \pi_1(B)] \times T_B \tag{$\heartsuit$}$$

and denote $\widetilde{\mathcal{B}}_n = \{P_B \mid B \in \mathcal{B}_{n-1}\}$. As for the family $\widetilde{\mathcal{M}}_n$, let us choose for each box $F \in \mathcal{S}$ a closed interval $J_F \subset \pi_2(F) \setminus \bigcup_{\substack{1 \le m < n \\ 1 \le j \le m}} D_j^{(m)}$ with $\operatorname{diam} J_F < \frac{1}{n}$ and put

$$Q_F = [\pi_1(F) \setminus \bigcup\{\pi_1(B) \mid B \in \widetilde{\mathcal{B}}_n\}] \times J_F\,.$$

Then we put $\widetilde{\mathcal{M}}_n = \{Q_F \mid F \in \mathcal{S}\}$. Obviously,

$$G(\widetilde{\mathcal{B}}_n \cup \widetilde{\mathcal{M}}_n) \in \mathcal{G} \quad \text{and} \quad G(\widetilde{\mathcal{B}}_n \cup \widetilde{\mathcal{M}}_n) \subset U_n. \tag{$\sharp$}$$

Also (4) is satisfied. Take $\widetilde{B} \in \widetilde{\mathcal{B}}_{n-1}$ arbitrary. Then, in view of (6), the set

$$\{\pi_2(B) \mid B \in \mathcal{B}_{n-1},\, B \subset \widetilde{B}\}$$

forms a family of all closed subintervals of $\pi_2(\widetilde{B})$ with the rational endpoints. Further, each $B \in \mathcal{B}_{n-1}$ contains the corresponding set P_B, so $\pi_2(P_B) \subset \pi_2(B)$. Using Lemma 2 with $I_0 = \pi_2(\widetilde{B})$, $(I_k)=\{\pi_2(B) \mid B \subset \widetilde{B},\, B \in \widetilde{\mathcal{B}}_{n-1}\}$ and $(J_k)=\{\pi_2(P_B) \mid B \subset \widetilde{B},\ B \in \widetilde{\mathcal{B}}_{n-1}\}$ we conclude that the set $D_{\widetilde{B}}$ of those $y \in \pi_2(\widetilde{B})$ belonging to at most one of $\pi_2(P_B)$ is nowhere dense. Let $\mathcal{D}_n = (D_i^{(n)})_{i=1}^{\infty}$ be a sequence of all closed nowhere dense sets which contain any set of the form $\overline{D}_{\widetilde{B}}$, $\widetilde{B} \in \widetilde{\mathcal{B}}_{n-1}$. Hence the (5) holds. Further, recall that every box in $\widetilde{\mathcal{B}}_n$ is of the form P_B where $B \in \mathcal{B}_{n-1}$ (see ($\heartsuit$)). Consequently, if we put $I_{P_B} = K_B$ and define a sequence $\mathcal{E}_n = (E_i^{(n)})_{i=1}^{\infty}$ of closed nowhere dense sets such that

$$\bigcup_{B\in\mathcal{B}_{n-1}} \big(K_B \setminus (\pi_1(F_B) \cap \pi_1(B))\big) \subset \bigcup_{i=1}^{\infty} E_i^{(n)}$$

we obtain (3). Such family $\mathcal{E}_n$ exists since all sets $K_B \setminus (\pi_1(F_B) \cap \pi_1(B))$ are of the first category. Now it is easy to choose the countable set of boxes $\mathcal{B}_n$ with disjoint projections onto the first coordinate such that (6) holds. It remains to define family $\mathcal{M}_n$. But proceeding similarly as in the above definition of $\widetilde{\mathcal{M}}_n$ we can obviously construct a set of boxes $\mathcal{M}_n$ such that $\mathcal{B}_n \cup \mathcal{M}_n$ is a regular family and $\mathcal{B}_n \cup \mathcal{M}_n \subset \widetilde{\mathcal{B}}_n \cup \widetilde{\mathcal{M}}_n$. Putting $V_n = G(\mathcal{B}_n \cup \mathcal{M}_n)$ and using ($\sharp$) we get the condition (2).

The strategy for the second player is now well defined. By (4) and Remark we obtain that $\bigcap V_n = \{f\}$ where f is Baire measurable. Also, the condition (4) gives that the range $f(X)$ is contained in G_δ set $J \setminus \bigcup_{j,m=1}^{\infty} D_j^{(m)}$, residual in J. Consider an arbitrary $y \in J \setminus \bigcup_{j,m=1}^{\infty} D_j^{(m)}$. The condition (5) enables to assign by induction to each $\sigma \in \{0,1\}^{\mathbb{N}}$ a system of sets $(B_{\sigma|n})_{n=1}^{\infty}$ with the following properties:

(i) $B_{\sigma|n} \in \widetilde{\mathcal{B}}_n$ and $y \in \pi_2(B_{\sigma|n})$ for each σ and n.
(ii) If $\sigma_1|n \neq \sigma_2|n$ then $B_{\sigma_1|n} \cap B_{\sigma_2|n} = \emptyset$.
(iii) $B_{\sigma|n+1} \subset B_{\sigma|n}$ for each n.

Let $\sigma \in \{0,1\}^{\mathbb{N}}$ be fixed and let $I_{B_{\sigma|n}}$ be the interval from (3). Since clearly $I_{B_{\sigma|n+1}} \subset I_{B_{\sigma|n}}$ we conclude that

$$\bigcap_{n=1}^{\infty} I_{B_{\sigma|n}} = \{x_\sigma\}.$$

But in view of (3) $x_\sigma \notin \bigcup_{j,m=1}^{\infty} E_j^{(m)}$ and consequently $x_\sigma \in \bigcap_{i=1}^{\infty} \pi_1(B_{\sigma|n})$. By (i) we see that $f(x_\sigma) = y$. Since the correspondence $\sigma \to x_\sigma$ is 1-1 (see (ii)) and continuous (see (3) and (iii)), hence a homeomorphism, the preimage of y contains a perfect set.

For the case of σ-field of Lebesgue measurable subsets of X the proof is even simpler. The conditions (1) – (6) will change in the following way:

> Firstly, it is not neccessary to introduce the families $\mathcal{E}_n$. Formally we can put $\mathcal{E}_n = \emptyset$.
> Instead of (3) we consider the condition (3*):
> If $B \in \widetilde{\mathcal{B}}_m$ then $\pi_1(B)$ is a compact set of positive measure and $\operatorname{diam} \pi_1(B) \le \frac{1}{m}$.
> The last change is at the point (6) where the requirement on $\pi_1(B)$ will be now that this projection is of positive measure.

The further corresponding changes during proof are obvious. □

Theorem. *Let Σ be either the σ-field of all subsets of $\mathbb{R}$ with the Baire property or the σ-field of all Lebesgue measurable subsets of $\mathbb{R}$ and consider the space S of all Σ-measurable functions endowed with topology τ_c. Then*

(1) *there is a non empty open set $G \subset S$ such that the typical function $f \in G$ has G_δ dense range and for each $y \in f(\mathbb{R})$ the set $f_{-1}(y)$ contains a non empty perfect set*
(2) *the typical function in S has the range of the second category.*

Proof. The first assertion follows immediately from the Proposition if we put $J = X = \mathbb{R}$. As for the second one consider an open non empty set $H \subset (S, \tau_c)$. There is a regular family $\mathcal{B}$ of boxes such that $G(\mathcal{B}) \subset H$. Let $B \in \mathcal{B}$ be of the form $B = X \times J$ where X is a second category set with the Baire property (or a Lebesgue measurable set of positive measure) and $J \subset \mathbb{R}$ a non degenerate interval. Let $\Phi\colon (G(\mathcal{B}), \tau_c) \to (S_{X,J}, \tau_c)$ be a restriction of a function f to a given set X. Note that the preimage of a residual set under the map Φ is again a residual set. Since the Proposition asserts that there is an open set in $S_{X,J}$ where the typical function has the range of the second category so the same is valid for some non empty open set $G_H \subset G(\mathcal{B})$. Zorn's lemma offers a system $\mathcal{H}$ of pairwise disjoint non empty open sets of the form G_H, such that $\bigcup \mathcal{H}$ is dense in S. And (2) easily follows. □

We conclude with the observation that the typical function $f \in (S, \tau_c)$ even does not possess the property that for each $y \in f(\mathbb{R})$ the set $f_{-1}(y)$ is infinite. In fact, take an arbitrary $x_0 \in \mathbb{R}$ and put

$$\mathcal{B} = \{(\mathbb{R} \setminus \{x_0\}) \times (0,1),\ \{x_0\} \times (2,4)\}.$$

Then **every** function $f \in G(\mathcal{B})$ satisfies $f_{-1}(3) = \{x_0\}$.

References

[O] Oxtoby J.C., *The Banach-Mazur game and Banach category theorem*, Contribution to the Theory of Games, vol III, Annals of Math. Studies **39** (1957), Princeton, 159–163.

[O1] Oxtoby J. C., *Measure and Category*, Springer-Verlag, New York Heidelberg Berlin, 1980.
[DS] van Douwen E. K. and Stone A. H., *The topology of close approximation*, Topology and its Application **35** (1990), 261–275.
[F] Fremlin D., *The topology of close approximation*, in Sem. d'Analyse Fonctionelle 1983–84 (Beauzamy B., Krivine J.L., Maurey B., Pisier G., eds.), Univ. Paris VII-VI, 1985, pp. 33–44.
[F1] Fremlin D., *Proto-perfect spaces and the topology of close approximation*, Preprint, University of Essex, 1990.

J. Tišer, Katedra matematiky FEL ČVUT, Suchbátarova 2, 166 27 Praha 6, Czechoslovakia

L. Zajíček, Katedra matematické analýzy MFF UK, Sokolovská 83, 186 00 Praha 8, Czechoslovakia