Acta Math. Univ. Comenianae
Vol. LX, 2(1991), pp. 163–183

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS ON AN UNBOUNDED STRIP

P. BRUNOVSKÝ, X. MORA[1], P. POLÁČIK AND J. SOLÀ–MORALES[1]

1. Introduction

This paper is devoted to the study of semilinear elliptic equations of the form

$$-u_{tt} + 2\alpha u_t - u_{xx} = f(t, x, u), \tag{1.1}$$

where α is a nonzero constant, $f : \mathbb{R}^3 \to \mathbb{R}$ is of class C^2 and $f(t + \tau, x, u) \equiv f(t, x, u)$ for some $\tau > 0$. We consider solutions $u(t, x)$ on a strip

$$Q := (0, +\infty) \times (0, 1)$$

(or on $\mathbb{R} \times (0, 1)$), which also satisfy one of the following boundary conditions

$$\begin{aligned} &u_x(t, 0) \equiv u_x(t, 1) \equiv 0, \quad \text{(Neumann)},\\ &u(t, 0) \equiv u(t, 1) \equiv 0, \quad \text{(Dirichlet)},\\ &u_x(t, 0) - \beta_0 u(t, 0) \equiv u_x(t, 1) + \beta_1 u(t, 1) \equiv 0,\ \beta_0, \beta_1 > 0 \quad \text{(Robin)}. \end{aligned} \tag{1.2}$$

We are interested in the asymptotic behavior of $u(t, x)$, as $t \to \pm\infty$.

Our motivation for studying such problems is twofold. First, (1.1) arises as the travelling wave equation for certain parabolic equations on the spatial domain $\mathbb{R} \times (0, 1)$. More specifically, consider the semilinear reaction–diffusion equation

$$Z_s = \Delta_{t,x} Z + f(t + 2\alpha s, x, Z). \tag{1.3}$$

In this equation s plays the role of time, $\Delta_{t,x} Z = Z_{tt} + Z_{xx}$ is a standard diffusion term and f represents a nonlinear source (reaction term), which travels in the direction of the unbounded variable with the constant speed -2α.

Among the solutions of (1.3) of a particular interest are solutions of the form $Z(s, t, x) = u(t + 2\alpha s, x)$ (the travelling waves with velocity -2α). The profile

Received March 20, 1991.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35J50.
[1]With partial support from the DGICYT project PB86–0306.

$u(t,x)$ of any such travelling wave satisfies equation (1.1). It also inherits the boundary conditions that one imposes on Z at $x = 0, 1$. Thus, our aim in this paper can be rephrased as to describe the profile of the travelling waves of (1.3) for t near $\pm\infty$.

Our second motivation stems from a comparison of (1.1) with the semilinear parabolic equation obtained from (1.1) with $\alpha > 0$ by omitting the term $-u_{tt}$. For equations with $f = f(u)$ independent of t and x, such comparison has been given in [**4**]. The parabolic equation

$$2\alpha u_t - u_{xx} = f(u) \tag{1.4}$$

is considered there as the singular limit when $\varepsilon \to 0$ of the elliptic equation

$$-\varepsilon^2 u_{tt} + 2\alpha u_t - u_{xx} = f(u) \tag{1.5}$$

As proved in [**4**] there is a reasonable way of considering (1.5) with boundary conditions, say (1.2), as an evolution problem (with t playing the role of time). The dynamical system defined by $(1.5), (1.2)$ on an appropriate space is gradient–like and, under a dissipativity condition, it has a compact attractor consisting of all entire (i.e. defined for all $t \in \mathbb{R}$) and bounded solutions.

Obviously, (1.5) and (1.4) have the same equilibria. Much more can be said about the relation between the dynamical systems defined by $(1.5), (1.2)$ and $(1.4), (1.2)$, if ε is small. Namely, if all the equilibria are hyperbolic, then the flows on the attractors of $(1.5), (1.2)$ and $(1.4), (1.2)$ are conjugate to one another.

As an interesting consequence one observes that the attractor of $(1.5), (1.2)$ consists of the orbits connecting equilibria and the pattern of connections between equilibria is the same for (1.5) and (1.4). So if we again interprete (1.5) as a travelling wave equation, then the asymptotic profiles of all bounded travelling waves are determined by the connection pattern in the well–understood problem $(1.4), (1.2)$ (see [**2**]).

An important step in the proof of the above results of [**4**] was to prove transversal intersection of stable and unstable manifolds of any two equilibria. For this, similarly as in parabolic equations, the study of the nodal properties of solutions of the linearization of (1.4) appears to be crucial. The assumption that all equilibria are hyperbolic (which implies convergence of all trajectories) allowed the authors of [**4**] to prove certain properties of the zero number functional Z considered along solutions of the linearization. (For a function $w(x), Z(w(\cdot)) \leq +\infty$ is the number of the sign changes of $x \to w(x)$, as x increases from 0 to 1). These properties are weaker than in parabolic equations but still sufficient for the transversality.

It is not clear whether, without the a–priori knowledge that all trajectories are convergent, one can effectively use Z to obtain other results known in parabolic equations. The convergence itself can be chosen as one of such results (see [**16,**

12] for convergence results in autonomous parabolic equations). Yet, the maximum principle –the "cause of the nice behavior" of Z in parabolic equations– is applicable to the linearization of (1.5), at least for α/ε^2 sufficiently large. This motivated our effort to derive convergence of all trajectories directly from the maximum principle. Taking up this problem, we had also in mind that, similarly as in parabolic equations [**5, 3**], the maximum principle proof of convergence should extend to nonautonomous, time–periodic equations (1.1) (when our equation is not considered as a singular perturbation problem, there is no reason to introduce the additional parameter ε). We thus prove that if $f_u \leq \alpha^2$ (which makes the maximum principle applicable) then any bounded solution of (1.1), (1.2) approaches, as $t \to +\infty$, a τ–periodic (i.e., periodic with period τ) solution of (1.1), (1.2). As we show on an example, this result is not true without any restriction on α. An exception is the autonomous case (f independent of t), where convergence to equilibria holds without the restriction on α. In Section 3, we give a proof of this result not using the maximum principle.

We now give precise formulations of our main results. For definiteness, we choose boundary conditions (1.2), but the results are valid for all the other separated boundary conditions listed above.

First we need some preparation. Denote $Y := L_2(0,1)$, $X := H^1(0,1)$ and $H^2_B := \{w \in H^2(0,1) : w \text{ satisfies } (1.2)\}$. By a solution of (1.1), (1.2) on an interval $I \subset \mathbb{R}$ we understand a function $w(t,x)$ such that $t \to w(t,\cdot)$ is in $C(I, H^2_B(0,1)) \cap C^1(I,X) \cap C^2(I,Y)$ and (1.1) holds a.e.. We say that a solution is bounded on I if the X–norm $\|w(t,\cdot)\|_X$ is bounded by a constant independent of $t \in I$.

Our main results are the following two theorems.

Theorem 1. *Let $f(t,x,w)\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^2–function τ–periodic in t ($\tau > 0$). Let $\alpha \neq 0$ and*

$$f_w(t,x,w) \leq \alpha^2 \quad \text{for any} \quad (t,x,w) \in \mathbb{R}^3 \tag{1.6}$$

Then for any solution of (1.1), (1.2) *bounded on* $[0,+\infty)$ (*resp.* $(-\infty,0]$) *there exists a τ–periodic solution $q(t,x)$ such that*

$$\|w(t,\cdot) - q(t,\cdot)\|_{C^1} \to 0, \quad \text{as} \quad t \to +\infty \ (\text{resp. } t \to -\infty).$$

Here $\|\cdot\|_{C^1}$ denotes the C^1–supremum norm, which makes sense because $H^2_B \hookrightarrow C^1[0,1]$.

Theorem 2. *Suppose that in equation* (1.1) $f = f(x,u)$ *is a C^2–function independent of t and $\alpha \neq 0$. Then for any solution of* (1.1), (1.2) *bounded on* $[0,+\infty)$ (*resp.* $(-\infty,0]$) *there exists an equilibrium $q(x)$ such that*

$$\|w(t,\cdot) - q(\cdot)\|_{C^1} \to 0 \quad \text{as} \quad t \to +\infty \ (\text{resp. } t \to -\infty).$$

The two counterparts, $t \to +\infty$ and $t \to -\infty$, in both theorems are mutually symmetric. One can pass from one to the other by the time reversal. The latter affects only the sign of α which is of no relevance here. Below we therefore consider only solutions bounded on $(0, +\infty)$.

As was already mentioned, under the additional assumption that the equilibria are hyperbolic, the conclusion of Theorem 2 follows from [**4**]. Another convergence result for autonomous elliptic equations of a similar type as considered here has been proved by Chen et al. [**6, 7**]. They do not assume hyperbolicity (in fact, continua of equilibria occur in their paper), but instead they use specific properties of their equation.

Throughtout the paper, we use several results of [**4**]. We will summarize these results in a moment. Before doing so, some preliminary remarks are necessary.

For [**4**] to be applicable, we need our function f to satisfy certain dissipativity condition. However, since we always deal with solutions bounded in X, a space continuosly embedded in $C[0,1]$, we can modify our function outside a set $\{(t,x,u) : |u| < R\}$, without affecting solutions in question, in such a way that the modified function satisfies the dissipativity condition. Below, when listing results of [**4**], we may thus without loss of generality assume that this condition holds. Specifically, we assume that f can be expressed in the form

$$f(t,x,u) = Ku + h(t,x,u) + b(t,x,u),$$

where $K < 0$, h and b are τ–periodic in $t, h_u \leq 0$, and b is bounded (see [**4,** sect. 1.4] for a discussion of this condition).

It should be pointed out also that nonautonomous equations (1.1) are not considered in [**4**]. However, the abstract results of that paper allow for using a simple trick by which one embeds (1.1) into an autonomous equation of the type studied in [**4**]. This trick is carried out in the Remark below.

The dynamical system approach to elliptic equations was initiated by Kirchgässner and his coworkers (see [**13**] for some recent results). Another remarkable work in this approach is [**6, 7**]. This dynamical system approach uses $X \times Y$ as the space for the initial states (initial values of u and u_t). However, in order to cope with the fact that the initial–value problem for elliptic equations is in general "ill– posed", one has to restrict the class of admissible initial states. Denote

$$\begin{aligned} \mathcal{X} := \{(u_0, v_0) \in X \times Y :\ & \text{there exists a bounded solution} \\ & u(t,x) \quad \text{of} \quad (1.1), (1.2) \quad \text{on} \quad (0, +\infty) \quad \text{such that} \\ & (u(t,\cdot), \dot{u}(t,\cdot)) \to (u_0, v_0) \quad \text{as} \quad t \to 0^+\} \end{aligned}$$

Here $\dot{u}(t,\cdot) = u_t(t,\cdot)$ and the convergence is in $X \times Y$.

The following results follow from [**4**]: The set $\mathcal{X}$ is closed in $X \times Y$. For any $(u_0, v_0) \in \mathcal{X}$ there exists a unique solution $u(t,x)$ of $(1.1), (1.2)$ with (u_0, v_0)

as the initial state (at $t = 0$). The trajectory of this solution, i.e. the set $\{(u(t,\cdot), u_t(t,\cdot)) : t > 0\}$ is relatively compact in $X \times Y$. Moreover, $\{(u(t,\cdot) : t > 0\}$ is relatively compact in $C^1[0,1]$ and $u(t,x)$ is a classical solution of (1.1), (1.2). The latter means that for any domain Ω such that $\overline{\Omega} \subset (0,+\infty) \times [0,1]$, u is in $C^1(\overline{\Omega})$ and has continuous derivatives u_{tt}, u_{xx} in Ω. The equation (1.1) is then of course satisfied everywhere in Ω. The mapping $(u_0, v_0) \mapsto (u(t,\cdot), \dot{u}(t,\cdot))$, where $u(t,x)$ is the solution with the initial condition (u_0, v_0), is continuous from $\mathcal{X}$ into $X \times Y$, uniformly for t in compact subintervals of $[0,+\infty)$. For the particular choice $t = \tau$, the period map $(u_0, v_0) \mapsto (u(\tau,\cdot), \dot{u}(\tau,\cdot))$ has the additional property that its image is contained in $\mathcal{X}$. This follows from the fact that, due to periodicity, $t \mapsto u(t+\tau, x)$ is a bounded solution of (1.1), (1.2) with the initial condition $(u(\tau,\cdot), \dot{u}(\tau,\cdot))$. In the autonomous case $(u(t,\cdot), \dot{u}(t,\cdot)) \in \mathcal{X}$ for any $t > 0$ and (1.1), (1.2) defines a (semi) dynamical system on $\mathcal{X}$.

Finally, we would like to emphasize that if $u(t,x)$ is a solution, with $(u(0,\cdot),\ \dot{u}(0,\cdot)) \in \mathcal{X}$ and $t_n \to +\infty$ is a sequence such that $(u(t_n,\cdot),\ \dot{u}(t_n,\cdot))$ converges in $X \times Y$ to a $(\overline{u}, \overline{v})$, then $u(t_n,\cdot) \to \overline{u}$ in $C^1[0,1]$. This is a trivial consequence of relative compactness of $\{u(t,\cdot)\}$ in $C^1[0,1]$.

Remark. To embed (1.1) into an autonomous equation of the type considered in [**4**] one can proceed as follows.

Suppose, to be simple, that $\tau = 2\pi$. It is easy to see that there is no restriction in considering f of the following form: $f(t,x,u) = \widetilde{f}(\cos t, \sin t, x, u)$ where $\widetilde{f}$ is of class $\mathcal{C}^2$ in its (four) arguments (ξ, η, x, u) and it is such that $\widetilde{f} = \widetilde{K}u + \widetilde{h} + \widetilde{b}$ where $\widetilde{K} \in \mathbb{R}$, $\widetilde{K} < 0$, $\widetilde{h} = \widetilde{h}(\xi,\eta,x,u)$ satisfies $\widetilde{h}_u \le 0$ and $\widetilde{b} = \widetilde{b}(\xi,\eta,x,u)$ is bounded.

Then we see that our original solutions $u(x,t)$ give rise to a triplet $(u,\xi,\eta) = (u(x,t), \cos t, \sin t)$ that is a solution of the system

$$\begin{aligned}
&-u_{tt} + 2\alpha u_t - u_{xx} = \widetilde{f}(\xi,\eta,x,u)\\
&-\xi_{tt} + 2\alpha \xi_t + \xi = -2\alpha\eta + \xi + \varphi(\xi)\\
&-\eta_{tt} + 2\alpha\eta_t + \eta = 2\alpha\xi + \eta + \varphi(\eta)
\end{aligned}$$

where φ is a smooth real function such that $\varphi(r) = r$ for $|r| \le 1$ and $\varphi(r) = -r$ for large $|r|$. Observe that $r\varphi(r) \le -r^2 + R$ for some $R > 0$.

It is clear that this system satisfies the smoothness assumption $(F1)$ of [**4**] in $E = L_2(0,1) \times \mathbb{R}^2$ with $A(u,\xi,\eta) = (-\Delta u, \xi, \eta)$ and $F(u,\xi,\eta) = (\widetilde{f}, -2\alpha\eta + \xi + \varphi(\xi), 2\alpha\xi + \eta + \varphi(\eta))$. To see that it also satisfies the dissipativity condition $(F2)$ of [**4**] observe that

$$\begin{aligned}
&\langle A(u,\xi,\eta) - F(u,\xi,\eta), A(u,\xi,\eta)\rangle_E =\\
&= \langle -\Delta u - \widetilde{f}, -\Delta u\rangle_{L_2} + (2\alpha\eta - \varphi(\xi))\xi + (-2\alpha\xi - \varphi(\eta))\eta =\\
&= \langle -\Delta u - \widetilde{f}, -\Delta u\rangle_{L_2} - \varphi(\xi)\xi - \varphi(\eta)\eta \ge\\
&\ge \langle -\Delta u - \widetilde{f}, -\Delta u\rangle_{L_2} + \xi^2 + \eta^2 - 2R
\end{aligned}$$

and $\langle -\Delta u - \widetilde{f}, -\Delta u\rangle_{L_2} \geq \delta(\langle u, -\Delta u\rangle_{L_2} - R_*^2)$ for some $\delta > 0$ (see [**4**], Lemma 1.4).

Acknowledgement. This work was done while P. Brunovský and P. Poláčik were visiting the Centre de Recerca Matemàtica, of the Institut d'Estudis Catalans. They would like to express their gratitude to the Centre for excellent working conditions and a very friendly atmosphere.

2. Convergence to Periodic Solutions

In this section we prove Theorem 1. The general scheme of the proof is similar to the one used by Brunovský at al. (see [**3**]) in the proof of an analogous convergence result for parabolic equations. In the first step, we examine the behavior of a given solution at the boundary point $x = 0$. We prove that as $t \to +\infty$, $u(t, 0)$ approaches a τ–periodic function. Then, in the second step, we extend the convergence from $x = 0$ to the entire interval $[0, 1]$ and we prove that the limit function is τ–periodic in t. This outlines the proof for Neumann conditions, which will be carried out here. We do not give the proof for Dirichlet or Robin boundary conditions, since they require only straight–forward modifications (for Dirichlet, the boundary behavior of u_x instead of u has to be considered).

In both steps outlined above our argumentation is based on certain properties of solutions of a linear elliptic equation. These properties are to be applied to the differences of particularly chosen solutions of (1.3). It is a standard observation that any such difference satisfies a linear equation. This is also similar to the parabolic case considered in [**3**]. However, when dealing with elliptic equations one has to find a substitute for the properties of the zero–number functional, which is the basic tool in [**3**] (and most of the other results in parabolic equations on interval). It appears that the maximum principle, used in conjunction with other results for elliptic equations (uniqueness for the Cauchy problem and unique continuation theorems), provides for sufficient equipment for our proof.

We now prove two lemmas which deal with the linear elliptic problem

$$-w_{tt} + 2\alpha w_t - w_{xx} = l(t, x)w, \quad t > 0,\ x \in (0, 1) \tag{2.1}$$

$$w_x(t, 0) = w_x(t, 1) = 0 \tag{2.2}$$

Lemma 2.1. *Let $l(t, x)$ be continuous and bounded on $[0, \infty) \times [0, 1]$ and $l(t, x) \leq \alpha^2$ everywhere. Let $w(t, x)$ be a classical solution of* (2.1), (2.2)*. Suppose that there is a sequence $s_n \to +\infty$ such that for any $n = 1, 2, \ldots$ the function $w(s_n, \cdot\,)$ has at most m zeros in $[0, 1]$, where m is a positive integer independent of n. Then there is a $t^* > 0$ such that $w(t, 0) \neq 0$ for any $t > t^*$.*

Proof. First observe that $w(t, 0)$ cannot vanish identically on any interval $I \subset (0, +\infty)$. Indeed, the contrary would mean that both w and w_x vanish identically on the segment $I \times \{0\}$ (recall (2.2)). By uniqueness for the Cauchy problem for elliptic equations [**14**], this would imply that w vanishes on a neighbourhood of this segment in $[0,\infty)\times[0,1]$. By the unique continuation theorem [**14**], this is possible only if $w\equiv 0$ on $[0,\infty)\times[0,1]$, which would contradict our assumption.

Next we need the following claim:
Let $\mathcal{G}\neq\emptyset$ be a connected component of the set $Q^+ := \{(t,x)\in[0,\infty)\times[0,1] : w(t,x)>0\}$. Then either $\mathcal{G}$ is unbounded, or else $\mathcal{G}\cap\{(0,x) : x\in[0,1]\}$ is not empty.

Roughly speaking, the role of this claim is to show that for a point t such that $w(t,0)=0$, there is a nodal curve (i.e. a curve on which $w=0$) containing $(t,0)$, which is either unbounded (hence intersects any segment $(s,x),\ x\in[0,1]$, for s sufficiently large) or contains a point $(0,x),\ x\in[0,1]$. Since an infinite number of zeros of w in $(0,\infty)\times\{0\}$ would give us infinitely many of such curves, which we show to be impossible, the conclusion of Lemma 2.1 will follow.

To prove the claim we apply the elliptic maximum principle to the function

$$\widetilde{w}(t,x) := e^{-\alpha t} w(t,x).$$

Note that $\widetilde{w}$ satisfies Neumann boundary conditions and

$$-\widetilde{w}_{tt}-\widetilde{w}_{xx} = (l(t,x)-\alpha^2)\widetilde{w}\ , \quad t>0\ ,\ x\in(0,1)\ .$$

Moreover, $\widetilde{w}$ has the same sign as w. Since $l(t,x)-\alpha^2\le 0$ by assumption, the maximum principle applies to $\widetilde{w}$ [**15**].

Suppose that the component $\mathcal{G}$ is bounded. Decompose the boundary $\partial\mathcal{G}$ of $\mathcal{G}$ into three parts $\Gamma_1,\Gamma_2,\Gamma_3$ which are the intersections of $\partial\mathcal{G}$ with the sets $\{0\}\times[0,1]$, $(0,\infty)\times\{0,1\}$ and $(0,\infty)\times(0,1)$ respectively. By the maximum principle, $\widetilde{w}:\overline{\mathcal{G}}\to\mathbb{R}$ attains its maximum (which is of course positive) at a point $M\in\partial\mathcal{G}$. We prove that $M\in\Gamma_1$, which will in particular imply that $\mathcal{G}\cap(\{0\}\times[0,1])\neq\emptyset$. Obviously, $\widetilde{w}=0$ on Γ_3, so $M\notin\Gamma_3$. Suppose that $M\in\Gamma_2$. Since $\widetilde{w}(M)>0$, there is a neighbourhood of M in $(0,\infty)\times[0,1]$ in which $\widetilde{w}>0$. Therefore, near $M,\mathcal{G}$ is a domain with smooth boundary. Applying the Hopf boundary principle [**15**], we obtain that $\widetilde{w}_x(M)\neq 0$, contradicting Neumann boundary conditions. Thus $M\in\Gamma_2$ is also impossible. Therefore, $M\in\Gamma_1$ which proves the claim.

The above arguments applied to the function $-w$ show that the claim also holds for any connected component of the set $Q^- := \{(t,x)\in[0,\infty)\times[0,1] : w(t,x)<0\}$.

Now suppose that the conclusion of the lemma fails. We derive a contradiction. Since $w(t,0)$ does not vanish identically on any interval, there exist intervals $(t_n,\overline{t}_n)$ such that $\overline{t}_n<t_{n+1}, t_n\to+\infty,\ w(t_n,0)=w(\overline{t}_n,0)=0$ and either $w(t,0)$ is positive on each of these intervals or it is negative on each of them. Replacing w by $-w$ if necessary, we may assume that $w(t,0)>0$ for $t\in(t_n,\overline{t}_n)$.

We prove that for $n < k$, the segments $(t_n, \overline{t}_n) \times \{0\}$ and $(t_k, \overline{t}_k) \times \{0\}$ do not lie in the same component of Q^+. Indeed, if they did then there would exist a simple curve Γ in $Q^+ \cap ((0,\infty) \times (0,1))$ joining two points $M_n \in (t_n, \overline{t}_n) \times \{0\}$ and $M_k \in (t_k, \overline{t}_k) \times \{0\}$. Consider the domain Ω bounded by this curve and the segment $[M_n, M_k]$. It is obvious that any curve in $[0,\infty) \times [0,1]$ joining a point from Ω to a point on $\{0\} \times [0,1]$ must cross Γ. Since $w > 0$ on Γ, the above claim on the components of Q^- implies that $w \geq 0$ in Ω. Therefore the minimum of w in $\overline{\Omega}$ is 0, and it is achieved at the boundary point $(\overline{t}_n, 0)$. The Neumann boundary condition $w_x(\overline{t}_n, 0) = 0$ now contradicts the Hopf boundary principle. This contradiction shows that the connected components of Q^+ containing $(t_n, \overline{t}_n) \times \{0\}$ are pairwise disjoint. Denote these components by $\mathcal{G}_n, n = 1, 2, ...$

Next we prove that all the $\mathcal{G}_n$ are bounded. Suppose that some $\mathcal{G}_k$ is unbounded. Then also $\mathcal{G}_k$ is unbounded for any $k > n$. This follows from the above claim: If $\mathcal{G}_k$ was bounded then it would have to contain a point $(0, x)$. It is obvious that this would imply that $\mathcal{G}_k$ intersects $\mathcal{G}_n$, which is impossible. So $\mathcal{G}_n$ unbounded implies that $\mathcal{G}_{n+1}, ..., \mathcal{G}_{n+m+1}$ (with m as in the hypotheses) are unbounded. Therefore all these sets intersect any segment $\{s\} \times [0,1]$ with s sufficienty large. This implies that the intersection of Q^+ with this segment has at least $m+1$ components. As a consequence, we obtain that $w(s, \cdot\,)$ has at least $m+1$ zeros in the interval $[0,1]$. Since this is true for any large s, we obtain a contradiction to the hypotheses. We have thus proved that the $\mathcal{G}_n$ are indeed bounded.

By the claim, any $\mathcal{G}_n$ contains a point $(0, x)$. Therefore it must intersect any segment $\{t\} \times [0,1]$ with $t < t_n$. Since for any given t there are infinitely many such t_n's, we conclude that for any $t > 0$, the function $w(t, \cdot\,)$ has infinitely many zeros in $[0,1]$. This contradicts the assumption. □

Lemma 2.2. *Let $l(t,x)$ be continuous. Let $w \neq 0$ be a classical solution of* (2.1), (2.2). *Then there is a dense subset $\mathcal{H} \subset \mathbb{R}$ such that for any $t \in \mathcal{H}$ the function $w(t, \cdot\,)$ has only simple zeros in* $[0,1]$.

Proof. We use [**10,** Cor. 1], according to which the set of all common zeros of the functions w, w_t, w_x in $(0,+\infty) \times (0,1)$ consists of isolated points. This result, combined with the fact that $\{t > 0 : w(t,0) = 0 \text{ or } w(t,1) = 0\}$ is a closed set not containing an interval (as shown above), implies the following property. For any $\widetilde{t}$, arbitrarily close to $\widetilde{t}$ there are $t_1 < t_2$ such that $(t_1, t_2) \times [0,1]$ does not contain any common zero of these three functions. In other words, 0 is a regular value of the function $w : (t_1, t_2) \times [0,1] \to \mathbb{R}$. A well–known "transversality" result now implies that for almost all t (hence for a dense set of t's) in (t_1, t_2) 0 is a regular value of the function $w(t, \cdot\,) : [0,1] \to \mathbb{R}$ (see [**11,** Ch. 3, Theorem 2.7]). This implies the conclusion of Lemma 2.2. □

Proof of Theorem 1. Consider a solution $u(t,x)$ of (1.1), (1.2) bounded on $[0,+\infty)$. By definition of $\mathcal{X}$, boundednes of u implies $(u(0, \cdot\,), \dot{u}(0, \cdot\,)) \in \mathcal{X}$, hence the trajectory $\{(u(t,\cdot),\dot{u}(t,\cdot):t\geq 0\}$ is relatively compact in $X\times Y$. We suppose that $u(t,\cdot)$ is not τ–periodic (otherwise the conclusion is trivial).

Our first aim is to prove that for any $t\geq 0$ the sequence

$$\beta_n(t):=u(t+n\tau,0),\quad n=1,2,... \tag{2.3}$$

is monotone, hence convergent (we know that $\beta_n(t)$ is bounded as $n\to+\infty$). To this end, we want to apply Lemma 2.1 to the function

$$w(t,x):=u(t+\tau,x)-u(t,x)\not\equiv 0.$$

Unfortunately, it is not always possible to verify that w satisfies the last hypotheses of that lemma. Difficulties arise if the following situation occurs:

(S1) Any limit point of the sequence $(w(n\tau,\cdot),\dot{w}(n\tau,\cdot))$ is τ–periodic (i.e. its trajectory is τ–periodic).

We shall therefore proceed as follows. First we suppose that (S1) does not hold. We show this to imply that $\beta_n(t)$ is convergent. From this we then obtain that the conclusion of Theorem 1 holds. This, in fact, will be a contradiction to the assumption that (S1) does not hold. Then we consider the opposite possibility, i.e. that (S1) does hold, and we show it to imply the conclusion of Theorem 1.

Suppose that (S1) fails. We show that Lemma 2.1 applies to $w(t,x)$. Since $u(t+\tau,x)$, as well as $u(t,x)$, are solutions of (1.1), (1.2) (due to periodicity), $w(t,x)$ is a classical solution of (2.1), (2.2) with

$$l(t,x):=\int_0^1 f_u(t,x,u(t,x)+s(u(t+\tau,x)-u(t,x)))ds.$$

Obviously $l(t,x)$ is continuous, bounded and, by assumption, $l(t,x)\leq\alpha^2$. In order to verify the last hypothesis of Lemma 2.1, choose a sequence of integers k_n such that $k_n\to+\infty$ and $(u(k_n\tau,\cdot),\dot{u}(k_n\tau,\cdot))$ converges in $X\times Y$ to a $\xi_0=(\overline{u}_0(\cdot),\overline{v}_0(\cdot))$. In addition we may assume that ξ_0 is not τ–periodic ((S1) is supposed not to hold). By closedness of $\mathcal{X},\xi_0\in\mathcal{X}$. Clearly, $t\to u(t+k_n\tau,x)$ is the solution of (1.1), (1.2) with the initial condition $(u(k_n\tau,\cdot),\dot{u}(k_n\tau,\cdot))$ at $t=0$. Hence, by the continuity with respect to initial conditions, $(u(t+k_n\tau,\cdot),\dot{u}(t+k_n\tau,\cdot))$ converges, as $n\to+\infty$, to $(\overline{u}(t,\cdot),\dot{\overline{u}}(t,\cdot))$ where $\overline{u}(t,x)$ is the solution of (1.1), (1.2) with the initial condition $\xi_0=(\overline{u}_0(\cdot),\overline{v}_0(\cdot))$ at $t=0$. As mentioned in the introduction, this convergence implies the convergence of $u(t+k_n\tau,\cdot)$ in the $C^1[0,1]$ norm. Now consider the difference

$$\overline{w}(t,x)=\overline{u}(t+\tau,x)-\overline{u}(t,x).$$

We have $\overline{w} \not\equiv 0$, because $\overline{u}$ is not τ–periodic. Since $\overline{w}$ satisfies an equation of the form (2.1), Lemma 2.2 implies that for some $t_0, \overline{w}(t_0, \cdot\,)$ has only simple zeros in $[0, 1]$. Therefore any function sufficiently close to $\overline{w}(t_0, \cdot\,)$ in $C^1[0, 1]$ has the same number of zeros as $\overline{w}(t_0, \cdot\,)$. Since

$$w(t_0 + k_n\tau, x) = u(t_0 + \tau + k_n\tau, x) - u(t_0 + k_n\tau, x)$$

converges to $\overline{w}(t_0, x)$ in $C^1[0, 1]$, we see that the last hypothesis of Lemma 2.1 is satisfied with $t_n = t_0 + k_n\tau$ and n sufficiently large.

By Lemma 2.1, there is a t^* such that $w(t, 0)$ is of a constant nonzero sign in $(t^*, +\infty)$. This implies that for any $t \geq 0$, the sequence $\beta_n(t)$ is eventually monotone (because $\beta_{n+1}(t) - \beta_n(t) = w(t + n\tau, 0)$), hence convergent.

Denote

$$\beta_\infty(t) := \lim_{n\to+\infty} \beta_n(t) \tag{2.4}$$

We now prove that the sequence $(u(n\tau, \cdot\,), \dot{u}(n\tau, \cdot\,))$ is convergent in $X \times Y$. Since this sequence is relatively compact, it suffices to prove that it has a unique limit point. To see this, consider arbitrary two convergent subsequences $(u(n_i\tau, \cdot\,), \dot{u}(n_i\tau, \cdot\,))$ and $(u(\overline{n}_i\tau, \cdot\,), \dot{u}(\overline{n}_i\tau, \cdot\,))$ with limits $\xi, \overline{\xi}$, respectively. As was already shown in this proof, for any $t > 0$, the functions $u(t + n_i\tau, \cdot\,), u(t + \overline{n}_i\tau, \cdot\,)$ converge to $p(t, \cdot\,), \overline{p}(t, \cdot\,)$, where $p(t, x)$ and $\overline{p}(t, x)$ are solutions of (1.1), (1.2) with the initial data $(p(0, \cdot\,), \dot{p}(0, \cdot\,)) = \xi, (\overline{p}(0, \cdot\,), \dot{\overline{p}}(0, \cdot\,)) = \overline{\xi}$. By (2.3), (2.4), we have

$$p(t, 0) \equiv \overline{p}(t, 0) \equiv \beta_\infty(t) \tag{2.5}$$

Therefore the difference

$$\widetilde{w}(t, x) := p(t, x) - \overline{p}(t, x) \tag{2.6}$$

satisfies

$$\widetilde{w}(t, 0) \equiv 0,$$

together with the Neumann boundary condition

$$\widetilde{w}_x(t, 0) \equiv 0.$$

Since $\widetilde{w}$ solves a linear elliptic equation, uniqueness for the Cauchy problem for such equations [**14**] implies $w \equiv 0$ hence $p \equiv \overline{p}$. In particular, $\xi = \overline{\xi}$. This shows that $(u(n\tau, \cdot\,), \dot{u}(n\tau, \cdot\,))$ is convergent. As a consequence, we also obtain that its limit point ξ is τ–periodic (because the value of its trajectory at τ is the limit of $(u(n\tau + \tau, \cdot\,), \dot{u}(n\tau + \tau, \cdot\,))$ which is ξ again).

We have thus shown that the assumption that (S1) fails leads to a contradiction. It remains to prove that (S1) implies the conclusion of Theorem 1. For this it suffices to prove that the sequence in (S1) has a unique limit point. Suppose there are more than one such points. Then, since all limit points are τ–periodic, there are at least three of them (in fact, there have to be infinitely many). This follows from the continuity of the period map $\Pi : \mathcal{X} \to \mathcal{X}$ (By definition, $\Pi(\xi) = (u(\tau, \cdot), \dot{u}(\tau, \cdot))$ where u is the solution of (1.1), (1.2) with the initial condition ξ). Indeed if ξ_1, ξ_2 are two τ–periodic points, i.e. fixed points of Π, then they have neighbourhoods V_1, V_2 such that no trajectory of Π can "jump" from one to the other (i.e., $\Pi(V_1) \cap V_2 = \emptyset = \Pi(V_2) \cap V_1$). Thus if two ξ_1, ξ_2 are limit points of the sequence in (S1), which is a trajectory of Π and is relatively compact, then there is another limit point in $\mathcal{X} \setminus (V_1 \cup V_2)$.

Next we show that the existence of three such limit points $\xi_i, i = 1, 2, 3$, leads to a contradiction. Let $p_i(t, x)$ be the τ–periodic solution of (1.1), (1.2) with the initial condition ξ_i. We claim that there exists a t^* such that

$$u(t, 0) - p_i(t, 0) \neq 0, \quad \text{for any } \ t > t^* \text{ and } i = 1, 2, 3. \tag{2.7}$$

Admit for a while that this claim is proved. Then at least two of the differences in (2.7) have the same nonzero sign for any $t > t^*$. Let us consider the case

$$u(t, 0) - p_i(t, 0) > 0, \quad \text{for } t > t^* \text{ and } i = 1, 2 \tag{2.8}$$

(the other cases are analogous). By definition of the p_i and by the continuity with respect to initial conditions, there is a sequence n_j such that for any $s \geq 0$

$$u(s + n_j\tau, \cdot) \to p_2(s, \cdot), \text{ as } j \to +\infty$$

with the convergence in $C^1[0, 1]$. In particular we have the convergence at $x = 0$. Using the latter, in conjunction with (2.8) for $i = 1$, we obtain

$$p_1(s, 0) = p_1(s + n_j\tau, 0) < u(s + n_j\tau, 0) \to p_2(s, 0)$$

Hence,

$$p_1(s, 0) \leq p_2(s, 0)$$

Interchanging the roles of p_1, p_2 in the above arguments, we obtain $p_1(s, 0) \geq p_2(s, 0)$, hence $p_1(s, 0) \equiv p_2(s, 0)$. As we have already demostrated for the functions in (2.6), such identity implies $p_1 \equiv p_2$, hence $\xi_1 = \xi_2$, a contradiction.

We have thus seen that the sequence in (S1) having more than one limit point is impossible, provided the claim is correct.

In order to prove the claim, we apply Lemma 2.1 again, this time to the difference

$$w_1(t, x) := u(t, x) - p_j(t, x)$$

The verification of the hypotheses of Lemma 2.1 is quite analogous to the verification for $w(t, x) = u(t + \tau, x) - u(t, x)$, that we have made of at the beginning of the proof. The only exception is the verification of the last hypothesis, where the following fact has to be used: There is a sequence k_n of positive integers such that $k_n \to +\infty$ and for any $t > 0, u(t + k_n\tau, x)$ converges to a $p_i(t, x)$ with $i \neq j$. Then

$$w(t + k_n\tau, x) = u(t + k_n\tau, x) - p_j(t, x)$$

(recall that $p_j(t, \cdot\,)$ is τ–periodic) converges to

$$p_i(t, x) - p_j(t, x) \not\equiv 0$$

which, being a solution of a linear equation, has to have simple zeroes for some fixed $t > 0$ by Lemma 2.2. We leave the details to the reader. □

We conclude this section with a remark concerning the hypothesis

$$f_u(t, x, u) \leq \alpha^2$$

of Theorem 1. One easily observes that it can be slightly weakened (for the convergence of a $u(t, x)$, it is only necessary that it holds in a neighbourhood of the closure of $\{(t, x, u(t, x)) : \ t \geq 0, \ x \in [0, 1]\}$ in $\mathbb{R}^3$). However without any restriction on α, Theorem 1 does not hold. This can be seen from the following example. Let

$$f(t, x, u) = g(u) - \varepsilon r(t).$$

Then spatially homogeneous (x–independent) solutions of (1.1), (1.2) are solutions of the ODE

$$\ddot{u} - \alpha\dot{u} + g(u) = \varepsilon r(t).$$

For $g(u) = u^3 - u$ and $r(t) = \cos\omega t$, we recognize the forced Duffing equation. It is known [**8**] that the period map of this equation exhibits a fairly complicated dynamics if the parameters $\alpha, \varepsilon, \omega$ are appropriately chosen. In particular, it has periodic orbits with arbitrarily large prime periods. So no convergence can be established.

3. Convergence to Equilibria

In this section we prove Theorem 2. From the Introduction we recall that:

(1) It can be extended to other than Neumann boundary conditions.
(2) It is sufficient to prove the part of the Theorem concerning $t \to \infty$.
(3) We consider the problem (1.1), (1.2) on the closed subset $\mathcal{X}$ of $X \times Y$ on which (1.1), (1.2) generates a (semi–)dynamical system (which, in accord to [**4**] we denote by ψ^t), all trajectories of which are relatively compact in $X \times Y$. By $\|\cdot\|$ we denote a norm of $X \times Y$.

As mentioned in [**4,** section 9], the dynamical system ψ^t is gradient–like: the function

$$V(u, u_t) = \text{sign}\,\alpha \int_0^1 \left(\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_t^2 - \int_0^u f(x,s)ds \right) dx \tag{3.1}$$

is strictly decreasing along non–constant trajectories. This, in addition to relative compactness of the trajectories, implies that the ω–limit set of each trajectory is a non–empty compact continuum of equilibria, [**9,** Lemma 3.8.2].

Since maximum principle methods turned out not to be applicable for the case of general α and f, we had to look for alternative ones. Another known approach refers to the case of families of equilibria being smooth manifolds (which is true in our case, see Lemma 3.1 below). The existing theory for our problem [**13**] is not sufficiently developed to allow us to apply its results. Rather than trying to develop it further we have worked out a different proof. Instead of applying invariant manifold theory it works directly with the linearization of the equation. This leads to success due to some special features of our problem among which the fact that continua of equilibria are smooth curves is the most prominent one.

Lemma 3.1. *A compact continuum of equilibria of the dynamical system ψ^t, if not a single point, is a C^1–curve (i.e. a one dimensional C^1 submanifold) in $C^1[0,1] \times \{0\}$ which is diffeormorphic to a compact interval.*

Proof. A point $(u, \dot{u}) \in X \times Y$ is an equilibrium of ψ^t if and only if $\dot{u} = 0$ and u solves the boundary value problem

$$u'' + f(x, u) = 0, \tag{3.2}$$

$$u'(0) = u'(1) = 0. \tag{3.3}$$

The set of solutions of $(3.2), (3.3)$ can be viewed as the intersection of the set of all functions of $C[0,1]$ solving (3.2) with initial conditions

$$u'(0) = 0, \quad u(0) = l, \tag{3.4}$$

(l varying) and the closed linear subspace $u'(1) = 0$ of codimension 1 in $C^1[0,1]$. Therefore, the Lemma will be proved if we show that the set of solutions of $(3.2), (3.4)$ that extend to the interval $[0,1]$ is the image of an open subset of $\mathbb{R}$ under a proper regular C^1 injection from $\mathbb{R}$ to $C^1[0,1]$.

To this end we note that from the basic theory of ordinary differential equations (existence, uniqueness, intervals of existence, dependence of solutions on initial data) it follows that the set J of those $l \in \mathbb{R}$ for which the solution of $(3.2), (3.4)$ is defined on $[0,1]$ is open and the map $\sigma : J \to C^1[0,1]$ defined by

$$\sigma(l) := u(x, l),$$

where $u(x,l)$ is the solution of (3.2), (3.4), is C^1 and one–to–one. We have

$$\sigma'(l)(x) = y(x)$$

where y is the solution of the linearized equation

$$y'' + f_u(x, u(x,l))y = 0, \qquad y'(0) = 0, \;\; y(0) = 1$$

and therefore is not identically zero. This proves that σ is regular. The fact that σ is proper is immediate and left to the reader. □

The proof of Theorem 2 requires certain local exponential estimates of the components of the linear parts of the deviation of a trajectory from its limit point. The absence of a semigroup for our problem forced us to work with a special norm for which the estimates could be obtained from the equation directly. This norm is provided by the lemma below for which we have to introduce some notation.

For elements of $X \times Y$ we shall use the letter ξ and we shall write the system of first order (in t) equations associated with (1.1), (1.2) as

$$\dot{\xi} + \Phi(\xi) = 0 \tag{3.5}$$

where $\xi = (u,v),\ u \in X,\ v \in Y$ and

$$\Phi(u,v) = (-v, -Au + F(u) - 2\alpha v),$$

where $Au(x) = -u''(x) \quad$ for $u \in H^2_B(0,1)$, and $F(u)(x) = f(x, u(x))$. Recall that by ψ^t we denote the semiflow on $\mathcal{X}$ generated by (3.5).

We observe that $X \times Y$ is continuously imbedded into the space $L_2(0,1) \times L_2(0,1)$ which we denote shortly by L. By $\|\cdot\|_L$ we denote the standard norm on L.

For the next lemma we fix an equilibrium $\widehat{\xi} = (\widehat{u}, 0)$ of (3.5) and denote $\Lambda := D\Phi(\widehat{\xi})$. Thus, Λ is a linear unbounded closed operator on L with domain $H^2_B(0,1) \times L_2(0,1)$. By P^s, P^u, P^c we denote the spectral projections to the invariant subspaces of Λ corresponding to its eigenvalues with positive, negative and zero real parts respectively, $S^i = \text{Range } P^i,\ i = s, u, c$.

Lemma 3.2.

(i) *The intersection of the spectrum of Λ with the imaginary axis, if non–empty, consists of 0 as an algebraically simple eigenvalue. We thus have*

$$\Lambda \eta^c = 0 \quad \textit{for} \quad \eta^c \in S^c. \tag{3.6}$$

(ii) *There exists a scalar product $\langle \cdot , \cdot \rangle$ on L generating a norm $|\cdot|$ equivalent to the norm $\|\cdot\|_L$ such that*

$$\begin{aligned} \langle -\Lambda \eta^u, \eta^u \rangle &\geq \gamma |\eta^u|^2 \quad \textit{for} \quad \eta^u \in S^u, \\ \langle -\Lambda \eta^s, \eta^s \rangle &\leq -\gamma |\eta^s|^2 \quad \textit{for} \quad \eta^s \in S^s, \end{aligned} \tag{3.7}$$

for some $\gamma > 0$.

Proof. Denote $\mu_0 < \mu_1 < \mu_2 < \ldots$ and $\varphi_0, \varphi_1, \varphi_2, \ldots$ the eigenvalues and the corresponding eigenfunctions of the Sturm–Liouville problem

$$\begin{aligned} &y'' + [f_u(x, \widehat{u}(x)) + \mu]\, y = 0, \\ &y'(0) = y'(1) = 0. \end{aligned} \tag{3.8}$$

As is well known the functions φ_j can be chosen orthonormal in $L_2(0,1)$ obviously, Λ can be written as the matrix operator

$$\begin{pmatrix} 0 & -I \\ -A + F'(u)I & -2\alpha I \end{pmatrix}.$$

It follows that the two–dimensional subspace Z_j of $X \times Y$ spanned by vectors $(\varphi_j, 0)$ and $(0, \varphi_j)$ is invariant. Under Λ in the basis consisting of these two vectors $\Lambda|_{Z_j}$ has the representation

$$\begin{pmatrix} 0 & -1 \\ -\mu_j & -2\alpha \end{pmatrix}.$$

Since the functions $(\varphi_j, 0)$ and $(0, \varphi_j)$ for $j = 0, 1, 2, \ldots$ form already an orthonormal basis in L, the spectrum of Λ is the union of the spectra of Λ/L_j, hence, it consists of the solutions of the quadratic equations

$$\lambda^2 + 2\alpha\lambda - \mu_j = 0 \tag{3.9}$$

for $j = 0, 1, 2, \ldots$. The algebraic multiplicities of these eigenvalues coincide with their multiplicities as roots of (3.9). This is because the eigenvalues associated with different μ_j's are different. Indeed, for each $\mu = \mu_j$, the equation (3.9) has either a pair of simple roots

$$\lambda^j_{1,2} = -\alpha \pm \sqrt{\alpha^2 + \mu_j} \tag{3.10}$$

if $\alpha^2 + \mu_j \neq 0$, or a double root

$$\lambda^j = \alpha \tag{3.11}$$

if $\alpha^2 + \mu_j = 0$. From (3.10) it follows that if $\lambda^j_{1,2}$ are not real then $\mathrm{Re}\lambda^j_{1,2} \neq 0$ and, from (3.11) it follows that zero cannot be a double eigenvalue . Therefore, S^c consists of the multiples of an eigenvector of 0, which implies (3.6). This proves (i).

To prove (ii) note that the subspaces Z_j are orthogonal with respect to the scalar product in L. Hence, it is sufficient to find for each j a scalar product $\langle \cdot, \cdot \rangle_j$ on Z_j such that for some $\gamma > 0$ independent of j (3.7) holds with $\eta^u \in S^u \cap Z_j$, $\eta^s \in S^s \cap Z_j$ (this pair of inequalities we label by $(3.7)_j$) and

(3.12) $\langle \xi, \xi' \rangle$ equals their scalar product in L for all $\xi, \xi' \in Z_j$ and all but a finite number of j's .

Indeed, such partially defined products allow us to define $\langle \xi, \xi' \rangle = \sum_{j=1}^{\infty} \langle \xi_j, \xi'_j \rangle$, for $\xi, \xi' \in L$ with $\xi = \sum_{n=0}^{\infty} \xi_j$, $\xi' = \sum_{n=0}^{\infty} \xi'_j$, $\xi_j, \xi'_j \in Z_j$, $|\xi| = \langle \xi, \xi \rangle^{1/2}$.

Then, the subspaces Z_j are orthogonal with respect to $\langle \cdot\, , \cdot \rangle$ as well as in L. This orthogonality together with (3.12) implies that $\langle \cdot\, , \cdot \rangle$ is well defined for all $\xi, \xi' \in L$ and generates a norm equivalent to $\| \cdot \|_L$. In addition, it makes (3.7) an immediate consequence of $(3.7)_j$ for all j.

It remains to be shown that $\langle \cdot\, , \cdot \rangle_j$ can be defined so as to satisfy (3.12) and $(3.7)_j$ for all j.

Since $\mu_j \to \infty$ for $j \to \infty$, (3.10) implies that $|\mathrm{Re} \lambda_{1,2}^j| \to \infty$ for $j \to \infty$. Hence,

$$\gamma = \tfrac{1}{2} \inf(\{\alpha\} \cup \{|\mathrm{Re}\, \lambda_{1,2}^j| : \lambda_{1,2}^j \neq 0\}) > 0. \tag{3.13}$$

Moreover, for almost all j one has $\lambda_1^j > 0$ and $\lambda_2^j < 0$. For such j, $S^u \cap Z_j$ and $S^s \cap Z_j$ consist of multiples of eigenvectors of λ_1^j, λ_2^j respectively. Therefore $(3.7)_j$ is satisfied if we take $\langle \xi, \xi \rangle_j$ equal to the scalar product of L on Z_j. Since the number of the remaining subspaces Z_j is finite, (3.12) will be satisfied no matter how we define $\langle \cdot\, , \cdot \rangle_j$ on them. To satisfy $(3.7)_j$ in the case of a pair of real eigenvalues of the same sign or a double eigenvalue λ_j we take the scalar product in the coordinates in which $\Lambda|_{Z_j}$ is in the Jordan canonical form with a sufficiently small off–diagonal entry. In the case where $\lambda_{1,2}^j$ is a pair of complex conjugate eigenvalues one can refer to the real canonical form

$$\begin{pmatrix} -\alpha & -\sqrt{\mu_j + \alpha^2} \\ \sqrt{\mu_j + \alpha^2} & -\alpha \end{pmatrix}$$

for $\Lambda|_{Z_j}$. This completes the proof of Lemma 3.2. □

To prove Theorem 2 we shall apply the estimates of Lemma 3.2 to follow the evolution of the components of the deviation of a trajectory from a relatively interior point of its ω–limit set. Any neighbourhood of such a point has to be crossed by the trajectory following the curve of equilibria within an arbitrary small distance for an arbitrary long time. These two requirements will be shown to contradict the way the components of the deviation evolve during this crossing.

Proof of Theorem 2. Assume that $\xi(t)$ is a bounded trajectory the ω–limit set of which is not a single equilibrium. We show that this assumption leads to a contradiction.

By Lemma 3.1, $\Gamma := \omega(\xi(\cdot))$ is a curve diffeomorphic to an interval. Let $\sigma : [0,1] \to X \times Y$ be a diffeomorphic parametrization of this curve; denote $\xi_0 := \sigma(0)$, $\xi_1 := \sigma(1)$, $\widehat{\xi} := \sigma(1/2)$, $\eta := \xi - \widehat{\xi}$, $\Lambda := D\Phi(\widehat{\xi})$. Since Γ consists of equilibria, we have $\Phi(\sigma(s)) = 0$. Differentiating we obtain

$$D\Phi(\widehat{\xi})\sigma'(1/2) = \Lambda \sigma'(1/2) = 0,$$

which means that Γ is tangent to the eigenvector of 0 at $\widehat{\xi}$.

The function $\eta(t) := \xi(t) - \widehat{\xi}$ satisfies the differential equation

$$\dot{\eta} + \Lambda\,\eta = R(\eta)\eta,$$

where, if $\eta = (h,k)$ and $\zeta = (y,z)$, we have

$$R(\eta)\zeta = (0, -\int_0^1 (F'(\widehat{u} + \vartheta h) - F'(\widehat{u}))y\,d\vartheta),$$

where $\widehat{\xi} = (\widehat{u}, 0)$. If $\xi = (u,v)$, we have $[F'(\xi)y](x) = f_u(x,u(x))y(x)$, hence

$$\|R(\eta)\zeta\|_L \leq \sup_{\substack{0\leq\vartheta\leq 1\\ 0\leq x\leq 1}} f_{uu}(x, \widehat{u}(x) + \vartheta h(x)) \sup_{0\leq x\leq 1} |h(x)| \|\zeta\|_L. \tag{3.14}$$

If $\eta \to 0$ in $X \times Y$, then $\|h\|_{H_1(0,1)} \to 0$ and, consequently, also

$$\sup_{0\leq x\leq 1} |h(x)| \to 0.$$

Hence and from (3.14) it follows that

$$\|R(\eta)\|_L \to 0 \quad \text{for } \|\eta\| \to 0. \tag{3.15}$$

Since, by Lemma 3.2, the norms $\|\cdot\|_L$ and $|\cdot|$ are equivalent, (3.15) implies that

$$|R(\eta)| \to 0 \qquad \text{for } \|\eta\| \to 0 \tag{3.16}$$

($|R(\eta)|$ is understood as the operator norm of R associated with $|\cdot|$). Since $\overline{\xi(\mathbb{R}^+)} = \xi(\mathbb{R}^+) \cup \omega(\mathbb{R}^+)$ is compact, the topologies on $\overline{\xi(\mathbb{R}^+)}$ generated by the norms $\|\cdot\|$ and $|\cdot|$ are equivalent. Hence, we have

$$|R(\eta)| \to 0 \quad \text{for } |\eta| \to 0\,, \quad \eta \in \overline{\xi(\mathbb{R}^+)} - \widehat{\xi} \tag{3.17}$$

Denote $a(t) = |\eta^s(t)|\,, b(t) = |\eta^u(t)|\,, c(t) = |\eta^c(t)|$. We have

$$a(t)\dot{a}(t) = \langle \eta^s(t)\,, -\Lambda\eta^s(t) + P^s R(\eta(t))\eta(t)\rangle. \tag{3.18}$$

By (3.17), given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\eta(t) \in U_\delta := \{\eta : \max\{|\eta^s|, |\eta^u|, |\eta^c|\} \leq \delta\}$, we have

$$|R(\eta(t))| < \varepsilon/(|P^s| + |P^u| + |P^c|). \tag{3.19}$$

From Lemma 3.2, (3.18) and (3.19) we infer that

$$\dot{a}(t) \leq -\gamma a(t) + \varepsilon(a(t) + b(t) + c(t)) \tag{3.20}$$

for almost all t such that $\eta(t) \in U_\delta$. Similarly, we conclude that for almost all t such that $\eta(t) \in U_\delta$ we have

$$\dot{b}(t) \geq \gamma b - \varepsilon(a(t) + b(t) + c(t)) \tag{3.21}$$

and

$$|\dot{c}(t)| \leq \varepsilon(a(t) + b(t) + c(t)) \tag{3.22}$$

□

Having established the inequalities (3.20) − (3.22), we are prepared for the crucial step of the proof which consists in an estimate of the relation of the values of a, b, c at an exit point of U_δ in terms of their relation at an entry point. This estimate is provided by

Lemma 3.3. *Let a, b, c be nonnegative absolutely continuous functions satisfying for almost all $t \in [t_0, T_0]$ the system of inequalities* (3.20) − (3.22) *as well as*

$$|c(t_0)| = |c(T_0)| = \delta_0, \quad \textit{and} \quad c(t_1) = 0 \tag{3.23}$$

where $\gamma > 0\,, \quad \delta_0 > 0\,, \quad t_0 < t_1 < T_0$, *and*

$$\varepsilon < \frac{\gamma}{9}\,. \tag{3.24}$$

Then,

$$a(t_0) + b(t_0) \leq c(t_0) \tag{3.25}$$

implies

$$a(T_0) + c(T_0) \leq b(T_0). \tag{3.26}$$

Assume for a moment that this lemma holds true. Choose $\delta_1 > 0$ so small that (3.20) − (3.22) hold for ε satisfying (3.24) and for almost all t such that $\eta(t) \in U_{\delta_1}$. Since Γ is tangent to S^c at $\widehat{\xi}$, there exists a $\delta_2 \leq \delta_1$ such that

$$|\eta^s| + |\eta^u| \leq \delta/2\,, \qquad \text{if} \quad \delta \leq \delta_2 \quad \text{and} \quad \eta \in U_\delta \cap (\Gamma - \widehat{\xi})\,. \tag{3.27}$$

Choose $\delta_0 \leq \delta_2$ so small that none of the boundary points ξ_1, ξ_2 of Γ is in $\widehat{\xi} + U_{\delta_0}$. Further, choose a neighbourhood V of Γ such that

$$|\eta^s| + |\eta^u| \leq 2/3\delta_0 \quad \text{if} \quad \eta \in (V - \widehat{\xi}) \cap U_{\delta_0}\,, \tag{3.28}$$

$V \setminus (\widehat{\xi} + U_{\delta_0}) = V_1 \cup V_2$, $\xi_1 \in V_1$, and $\xi_2 \in V_2, \overline{V}_1 \cap \overline{V}_2 = \emptyset$ with $\overline{V}_1, \overline{V}_2$ intersecting different faces $|\eta^c| = \delta_0$ of $\widehat{\xi} + U_{\delta_0}$ (this is possible due to (3.27)).

Next, we recall that the distance of $\xi(t)$ to Γ converges to zero as $t \to \infty$ [9, Lemma 2.1.2]. This implies the existence of a T_1 such that for $t > T_1$ we have $\xi(t) \in V$. We now show that the hypotheses of Lemma 3.3 are satisfied for some t_0, T_0, t_1. Since both ξ_1 and ξ_2 belong to the ω–limit set of the trajectory $\xi(t)$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, for $t > T_2$ the point $\xi(t)$ has to pass from V_1 to V_2 through $\widehat{\xi} + U_{\delta_0}$. More precisely, there exist $T_0 > t_0 > T_2$ such that

$$\xi(t_0) \in \overline{V}_1, \qquad \xi(T_0) \in \overline{V}_2, \tag{3.29}$$

$$|P^c(\xi(t_0) - \widehat{\xi})| = |P^c(\xi(T_0) - \widehat{\xi})| = \delta_0, \tag{3.30}$$

and $|P^\nu(\xi(t) - \widehat{\xi})| \le \delta_0$ for $t_0 \le t \le T_0$ and $\nu = s, u, c$. Since $\overline{V}_1, \overline{V}_2$ meet different faces $|\eta^c| = \delta_0$ of $\widehat{\xi} + U_{\delta_0}$, (3.30) implies the existence of a $t_1 \in (t_0, T_0)$ satisfying the last equality in (3.23). The first two equalities are simply the equalities (3.30). The inequality (3.25) is an immediate consequence of (3.30) and (3.28). This completes the verification of the hypotheses of Lemma 3.3.

Applying the lemma, from (3.26) we obtain

$$\delta_0 = |\eta^c(T_0)| = c(T_0) \le a(T_0) + c(T_0) \le b(T_0) = |\eta^u(T_0)|.$$

On the other hand (3.29) implies

$$|\eta^u(T_0)| \le |\eta^u(T_0)| + |\eta^s(T_0)| \le \tfrac{2}{3}\delta_0,$$

which is impossible. This contradiction proves Theorem 2 provided Lemma 3.3 holds true. □

Proof of Lemma 3.3. Denote

$$p(t) := \frac{a(t)}{b(t) + c(t)},$$

$$r(t) := \frac{a(t) + c(t)}{b(t)}$$

for $t \in [t_0, T_0]$ such that $b(t) + c(t) \neq 0$, $b(t) \neq 0$, respectively.

Each of the functions $p(t), r(t)$ is absolutely continuous on any interval on which it is defined. By (3.20) - (3.22) we have the inequality

$$\dot{p} = \frac{\dot{a}}{b + c} - \frac{\dot{b} + \dot{c}}{b + c}\, p \le (-\gamma + 7\varepsilon)\, p + \varepsilon, \tag{3.31}$$

respectively

$$\dot{r} = \frac{\dot{a} + \dot{c}}{b} - \frac{\dot{b}}{b} r \leq (-\gamma + 4\varepsilon) r + 4\varepsilon. \tag{3.32}$$

which holds almost everywhere on any interval where $p \leq 2$, respectively $r \leq 2$, we have $p^2 \leq 2p$ on such an interval.

Notice that (3.31) and (3.24) imply that $\dot{p}(t) < 0$ whenever it exists and $p(t)$ is close to 1. A similar property holds for r as well.

By (3.25) we have $p(t_0) \leq 1$. From (3.31) it follows that $p(t) \leq 1$ for all $t_0 \leq t \leq t_1$ hence

$$r(t_1) = \frac{a(t_1 + c(t_1)}{b(t_1} = \frac{a(t_1)}{b(t_1) + c(t_1)} = p(t_1) \leq 1 \tag{3.33}$$

(note $c(t_1) = 0$ by (3.23)). From (3.32) it follows that $r(t) \leq 1$ for all $t \in [t_1, T_0]$, hence

$$\frac{a(T_0) + c(T_0)}{b(T_0)} = r(T_0) \leq 1.$$

□

References

1. Aulbach B., *Continuous and discrete dynamics near manifolds of equilibria*, Lecture Notes in Math. 1058, Springer, 1984.

2. Brunovský P. and Fiedler B., *Connecting orbits in scalar reaction–diffussion equations II: The complete solution*, J. Diff. Equations **81** (1989), 106–135.

3. Brunovský P., Poláčik P. and Sanstede B., *Convergence in general parabolic equations in one space dimension*, Nonlinear Analysis, TMA (to appear).

4. Calsina A., Mora X. and Solà–Morales J., *The dynamical approach to elliptic problems in cylindrical domains, and a study of their parabolic singular limit*, J. Diff. Equations (to appear).

5. Chen X. Y. and Matano H., *Convergence, asymptotic periodicity, and finite point blow up in one dimensional semilinear parabolic equations*, J. Diff. Equations **78** (1989), 160–190.

6. Chen X. Y., Matano H. and Véron L., *Singularités anisotropes d'équations elliptiques semi–linéaires dans le plan*, C.R. Acad. Sci. Paris **I 303** (1986), 963–966.

7. Chen X. Y. and Matano H., *Anisotropic singularities of solutions of nonlinear elliptic equations in* $\mathbb{R}^2$, J. Funct. Anal. **83** (1989), 50–97.

8. Guckenheimer J. and Holmes P., *Nonlinear oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.

9. Hale J. K., *Asymptotic Behavior of Dissipative systems*, Amer. Math. Soc. Surveys and Monographs **25,** Providence 1988.

10. Hartman P. and Wintner A., *On the local behaviour of solutions of non–parabolic partial differential equations*, Amer. J. Math. **75** (1953), 449–476.

11. Hirsch M. W., *Differential Topology*, Springer, 1976.

12. Matano H., *Convergence of solutions of one dimensional semilinear parabolic equations*, J. Math. Kyoto Univ. **18** (1978), 221–227.

13. Mielke A., *Inertial manifolds for elliptic problems in infinite cylinders*, Preprint.

14. Miranda C., *Partial Differential Equations of Elliptic type*, Springer, 1955, 1970.

15. Protter M. H. and Weinberger H. F., *Maximum Principles in Differential Equations*, Prentice–Hall and Springer, 1984.

16. Zelenyak T. I., *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differential Equations **4** (1968), 17–22.

P. Brunovský, Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia

X. Mora, Departament de Matemàtiques, Universitat Autónoma de Barcelona, 08193–Bellaterra, Barcelona, Spain

P. Poláčik, Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia

J. Solà–Morales, Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, ETSEIB Diagonal 647, 08028–Barcelona, Spain