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RADICALS COMMUTING WITH BANDS OF SEMIGROUPS

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It is known that ring and semigroup theories have evolved many similar methods. The exchange of ideas between these theories has enriched both of them. Among ring concepts transfered to semigroups, the notion of a radical deserves mentioning. The analogy with radicals of rings suggests that semigroup radicals may be applicable in studying the structure of semigroups. On the other hand, many structure results involve bands of semigroups. Therefore in order to apply radicals it seems important to know what are the radicals interacting with bands well in a sense. Let ρ be a radical, B a band. Defering complete definitions to §1, now we only say that one of the most convenient cases arises when $\rho(S) = \cup_{b\in B}\rho(S_b)$ for every $S = \cup_{b\in B}S_b$, band of semigroups. The aim of the present paper is to describe radicals with this property.

1. The Main Theorem

We use the standard notation [**2**]. For any semigroup, the universal congruence and the equality relation will be designated by ω and ι, respectively. Assume that, for each semigroup S in an abstract class A of semigroups, a congruence $\rho(S)$ is defined. The mapping $\rho\colon S \to \rho(S)$ is called a radical (or a radical on A) if

(R1) the $(x,y) \in \rho(S) \implies (f(x), f(y)) \in \rho(T)$ for any homomorphism $f\colon S \to T$;
(R2) $\rho(S/\rho(S)) = \iota$ for any $s \in A$.

Throughout, we shall consider radicals on the class of all semigroups. The congruence $\rho(S)$ is called the ρ-**radical** of S. A semigroup S is said to be ρ-**semisimple** (ρ-**radical**) if $\rho(S) = \iota$ (or $\rho(S) = \omega$). If it is clear which radical is under consideration, then we shall omit the prefix 'ρ-'. The class $\mathcal{S}$ of all semisimple semigroups is called the **semisimple class** of ρ. A radical ρ will be called **trivial** if every semigroup is ρ-**semisimple**. Each radical is uniquely determined by its semisimple class (see [**10**]). Therefore there is only one trivial radical.

Let B be a band. For possible applications of radicals to the study of semigroups which are represented as a band B of semigroups the most convenient case arises

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when for any $S = \cup_{b\in B} S_b$ the equality $\rho(\cup_{b\in B} S_b) = \cup_{b\in B}\rho(S_b)$ holds. In this case we shall say that ρ **commutes with** B. Note that in ring theory an analogous concept of radicals commuting with bands was introduced in [**3**]. Radicals of this sort were described in [**3**], [**2**] and [**5**]. Similar problem concerning radicals and bands in the class of semigroups with zero was solved in [**6**].

Recall that ρ is said to be **hereditary** if $\rho(I) = \rho(S)\cap(I\times I)$ for every semigroup S with an ideal I. We say that ρ is **strict** if $\rho(T) \subseteq \rho(S)$ for any subsemigroup T of S. This concept is analogous to that of strict ring radicals, see [**11**]. The strict radicals of groups and monoids are defined in a somewhat different manner (cf. [**8**]). If S is a semigroup and $I \subseteq S$, then $I\rho(S)$ denotes the set of all pairs (is, it) where $i \in I$, $(s,t) \in \rho(S)$. We say that ρ is **right strict** (**right weakly hereditary**; **right transitive**) if and only if for any right ideal I of S the inclusion $\rho(I) \subseteq \rho(S)$ ($I\rho(S) \subseteq \rho(I)$; $\rho(I)S \subseteq \rho(I)$) holds. Left strict, left weakly hereditary and left transitive radicals are defined dually.

Let $\mathcal{S}$ be a class of semigroups, B a band. We say that $\mathcal{S}$ is B-closed if and only if for every $S = \cup_{b\in B} S_b$ all S_b belong to $\mathcal{S}$ implies $S \in \mathcal{S}$. We say that $\mathcal{S}$ is 0-closed if for each $S \in \mathcal{S}$ the semigroup S^0 with zero adjoined belongs to $\mathcal{S}$. Recall that B is said to be a **semilattice** (**left zero band**; **right zero band**; **rectangular band**) if it satisfies the identity $xy = yx$ ($xy = x; xy = y; xyx = x$). We say that B is **nonsingular** if $|B| > 1$.

Theorem. *Let B be a nonsingular band, and let $\mathcal{S}$ be the semisimple class of a radical ρ. Then ρ commutes with B if and only if one of the following conditions holds:*

(i) *B is a semilattice, ρ is strict and hereditary, $\mathcal{S}$ is 0-closed;*
(ii) *B is a left zero band, ρ is right strict, right weakly hereditary and right transitive, $\mathcal{S}$ is B-closed;*
(iii) *dual to* (2)*;*
(iv) *B is a rectangular band, ρ is right and left strict, right and left weakly hereditary, right and left transitive, $\mathcal{S}$ is B-closed.*

This result was announced in [**4**]. As it has been mentioned above, the analogous problems on the commutation of bands and radicals in the classes of rings (cf. [**1**], [**3**], [**5**]) and semigroups with zero (cf. [**6**]) are solved too. There are essential differences between the answers in the three classes considered. For instance, in [**5**] an example of a non-hereditary ring radical commuting with every semilattice is constructed. Our Theorem yields that a radical of this sort does not exist in the case of semigroups.

2. Proof of the Main Theorem

We need a few known results on the structure of bands (cf. [**2**] §4.2).

Lemma 1. *Each rectangular band is isomorphic to a direct product of a left zero band and a right zero band.*

Lemma 2. *Each band is uniquelly represented as a semilattice of rectangular bands.*

Lemma 2 easily gives us

Lemma 3. *Let B be a nonsingular band. If B is a semilattice, then B contains a two-element subsemilattice. If B is not a semilattice, then B contains a two-element left zero band or a two-element right zero band, as a subsemigroup.*

We say that a radical ρ is B-homogeneous if for every semigroup $S = \cup_B S_b$ the radical $\rho(S)$ is contained in $\mathcal{S} = \cup_{b\in B} S_b \times S_b$. Obviously every radical commuting with B is B-homogeneous.

Lemma 4. *A radical ρ is B-homogeneous if and only if B is ρ-semisimple.*

Proof. The semigroup B is a band B of one-element semigroups. Therefore if ρ is B-homogeneous, then $\rho(B) \subseteq \cup_{b\in B} \{(b,b)\}$, i.e. B is semisimple.

Conversely, let B be semisimple. Suppose that ρ is not B-homogeneous, i.e. there is a semigroup $S = \cup_{b\in B} S_b$ with elements x, y such that $(x,y) \in \rho(S)$, $x \in S_a$, $y \in S_b$, $a \neq b$. Consider the map f defined by the rule $f(s) = b$ whenever $s \in S$, $b \in B$, $s \in S_b$. Evidently f is a homomorphism, and $f(x) = a$, $f(y) = b$. Hence condition (R1) implies $(a,b) \in \rho(B)$, giving the contradiction. This completes the proof. □

Let us say that a radical ρ is **weakly hereditary** if and only if for every semigroup S with an ideal I the inclusion $\rho(S)I \cup I\rho(S) \subseteq \rho(I)$ holds. A radical ρ will be called **transitive** if and only if for any semigroup S with an ideal I the union $\rho(I) \cup \iota$ is a congruence of S.

Lemma 5. *Let B be a non-rectangular band, ρ a radical commuting with B. Then ρ is strict, weakly hereditary and transitive.*

Proof. By Lemma 2, B is a semilattice Y of rectangular bands R_y. Since B is not rectangular, Y must be nonsingular.

First, we prove that ρ is strict. Take any semigroup S and any subsemigroup T in S. Consider elements $y_0 < y_1$ of Y. Let $b_1 \in R_{y_1}$, $b_2 \in R_{y_0}$, $b_0 = b_1 b_2 b_1$. Then the set $Z_2 = \{y_0, y_1\}$ is a semilattice. Let $Z_1 = \{y \in Y \mid y \geq y_1\}$, $Z_= Y \setminus Z_1$, $X_i = \cup_{y\in Z_i} R_y$, where $i =, 1$. There are pairwise disjoint semigroups W_b, $b \in B$, such that

$$W_b \cong \begin{cases} T & \text{when } b \in X_1; \\ S & \text{when } b \in X_0. \end{cases}$$

These isomorphisms and the operation of the semigroup S naturally induce an operation on the union $W = \cup_{b\in B} W_b$. Thus W is a semigroup being a band B of the subsemigroups W_b. Denote by π the homomorphism of W onto S expanding the isomorphisms of the components W_b on T or S. Fix any γ in X_1. Since ρ commutes with B, we get $\rho(W_\gamma) \subseteq \rho(W)$. Hence $\rho(S) \supseteq \pi(\rho(W)) \supseteq \pi(\rho(W_\gamma)) = \rho(T)$. Thus ρ is strict.

Now we prove that ρ is hereditary and transitive. Let S be a semigroup and I be an ideal in S. There are pairwise disjoint semigroups W_b, $b \in B$, such that

$$W_b \cong \begin{cases} S & \text{when } b \in X_1; \\ I & \text{when } b \in X_0. \end{cases}$$

Denote by π the homomorphism of W onto S defined by the isomorphisms of the components W_b on T or S. Let b_0, b_1 be the elements of B introduced at the beginning of the proof. Since ρ is strict we have $\rho(W_{b_1}) \subseteq \rho(W)$. Therefore $W_{b_0}\rho(W_{b_1}) \cup \rho(W_{b_1})W_{b_0} \subseteq (W_{b_0} \times W_{b_0}) \cap \rho(W)$. Keeping in mind that $W_{b_1} \cong S$, $W_{b_0} \cong I$ we get $I\rho(S) \cup \rho(S)I \subseteq \rho(I)$, because ρ commutes with B. Thus ρ is weakly hereditary.

Also, for the same semigroup S and ideal I the strictness of ρ implies $\rho(W_{b_0}) \subseteq \rho(W)$. Hence $W_{b_1}\rho(W_{b_0}) \cup \rho(W_{b_0})W_{b_1} \subseteq \rho(W_{b_0})$, and so $S\rho(I) \cup \rho(I)S \subseteq \rho(I)$. Therefore $\rho(I) \cup \iota$ is a congruence on S, yielding that ρ is transitive. □

Lemma 6. *Let ρ be a radical, B a semilattice S of rectangular bands Q_s, $s \in S$, where Q_s is a direct product of a left zero band L_s and a right zero band R_s. If ρ commutes with B and some L_s (or R_s) is not a singleton, then ρ is right (left) strict, right (left) weakly hereditary and right (left) transitive.*

Proof. We will consider only the case where some L_s is nonsingular. Let c and d be distinct elements of L_s.

Take any semigroup S with a right ideal I. Set $Z_1 = \{y \in Y \mid y \geq s\}, Z_0 = Y \backslash Z_1, X_i = \cup_{y \in Z_i} R_y$.
There are pairwise disjoint semigroups W_b, $b \in B$, such that

$$W_b \cong \begin{cases} S & \text{when } b \in X_0; \\ I & \text{when } b \in X_1 \backslash Q_s; \\ I & \text{when } b = (c, r), r \in R_s; \\ S & \text{when } b = (g, r), c \neq g \in L_s, r \in R_s. \end{cases}$$

These isomorphisms and the operation of the semigroup S naturally induce an operation on the union $W = \cup_{b \in B} W_b$. Then W is a semigroup being a band B of the subsemigroups W_b. Therefore $\rho(W) = \cup_{\alpha \in B} \rho(W_\alpha)$.

Fix some $r \in R_s$ and set $e = (c, r), f = (d, r)$. Then $\rho(W_e) \supseteq W_e\rho(W_f)$. Since $W_e \cong I$ and $W_f \cong S$ we get $\rho(I) \supseteq I\rho(S)$. Thus ρ is right weakly hereditary.

Since $\rho(W)$ is a congruence on W, it follows that $\rho(W_e)W_f \subseteq \rho(W_e)$, implying $\rho(I)S \subseteq \rho(I)$. Thereby ρ is right transitive.

Denote by π the homomorphism of W onto S continuing the isomorphisms of the components W_b on T or S. Then $\rho(S) \supseteq \pi(\rho(W_e)) = \rho(I)$. Thus ρ is right strict. $\square$

Now we will consider the interaction of radicals and semilattices.

Lemma 7. *Let ρ be a nontrivial strict and weakly hereditary radical, U_2 a commutative semigroup with generators u, v and relations $u^2 = uv = vu = v^2$. Then $(u, v) \in \rho(U_2)$.*

Proof. Let N_2 denote the two-element semigroup with zero multiplication. First we prove that N_2 is radical.

Since ρ is nontrivial, there exists a semigroup S which is not semisimple. By S^e we denote a semigroup S with an identity element e adjoined. Let P be the direct product of S^e and the additive semigroup of non-negative integers. Identify S with $S \times 0$. Choose s, t in S such that $s \neq t$, $(s, t) \in \rho(S)$. Let I be the ideal generated in S by $(e, 1)$. Since ρ is strict and weakly hereditary, we get $((s, 1), (t, 1)) = (s(e, 1), t(e, 1)) \in \rho(I)$. Clearly, the set $J = \{(x, 1) | x \in S, x \neq s\} \cup I^2$ is an ideal in S. Further, $I/J \cong N_2$ and the images of the elements $(s, 1)$ and $(t, 1)$ in I/J are not equal. Therefore N_2 is radical.

Now let P be a commutative semigroup with generators x, y, z and relations $x^2 = y^2 = xy = y$. Denote by T the ideal of P generated by z. The set $\{x, y\}$ is a semigroup isomorphic to N_2. It follows that $(x, y) \in \rho(T)$, since ρ is strict. Then by weak heredity $(xz, yz) \in \rho(T)$. Consider on T the equivalence relation μ whose classes are $\{zy\}, \{z, zx\}, \{z^n, z^n x, z^n y\}, n = 2, 3, \ldots$. It is routine to verify that μ is a congruence on T (not on P!). The quotient semigroup T/μ is isomorphic to U_2, and this isomorphism maps zx to u and zy to v. Hence $(u, v) \in \rho(U_2)$. $\square$

Lemma 8. *Let ρ be a strict radical and let Y be a nonsingular semilattice. If Y is semisimple, then all semilattices are semisimple.*

Proof. Suppose the contrary: let there exist a semilattice B and $(a, b) \in \rho(B)$, $a \neq b$. Then $a \neq ab$ or $b \neq ab$. Without loss of generality we may assume that $a \neq ab$. Obviously $(a, ab) \in \rho(B)$. Set $Z_1 = \{c \in B \,|\, c \geq a\}, Z_0 = B \backslash Z_1$. Denote by μ the congruence on B whose classes are Z_0 and Z_1. Let $Y_2 = B/\mu$. Then Y_2 is a semilattice consisting of two elements. Hence Y_2 is radical, because the images of a and ab in Y_2 do not coincide with each other. However, Y contains a subsemilattice isomorphic to Y_2. Since ρ is strict, we get a contradiction with the semisimplicity of Y. $\square$

Lemma 9. *Let $\mathcal{S}$ be a semisimple class of a strict weakly hereditary radical ρ. If $\mathcal{S}$ is 0-closed, then $\mathcal{S}$ is closed under every semilattice.*

Proof. The two-element semilattice Y_2 is semisimple, since it is isomorphic to 0^0. Hence Lemma 8 implies that all semilattices are semisimple.

Now let Y be a semilattice, and $S = \cup_{y\in Y} S_y$, where $S_y \in \mathcal{S}$. Suppose the contrary: let S be not semisimple, $(s,t) \in \rho(S), s \neq t$. By Lemma 4 there is $y \in Y$ such that $(s,t) \in \rho(S_y)$. Set $Z_1 = \{z \in Y \mid z \geq y\}, Z_0 = Z \backslash Z_1, S_i = \cup_{z\in Z_i} S_z$, $i = 0, 1$. Then $S/S_0 \cong S_1^0$. By (R1) we get $(s,t) \in \rho(S_1^0)$. Since S_y^0 is a semisimple ideal of S_1, weak heredity implies $xs = xt, sx = tx$ for every $x \in S_y$. Hence $s^2 = st = ts = t^2$. Therefore the semigroup T, generated in S_y by s, t, is a homomorphic image of the semigroup U_2 with generators u, v and relations $u^2 = uv = vu = v^2$. Lemma 7 yields $(s,t) \in \rho(T)$. This contradicts the semisimplicity of S_y and strictness of ρ. $\square$

Lemma 10. *If ρ is a strict transitive and weakly hereditary radical, then ρ is hereditary.*

Proof. Take any semigroup S and an ideal I of S. We are to prove that $\rho(I) = (I \times I) \cap \rho(S)$. The strictness of ρ yields $\rho(I) \subseteq (I \times I) \cap \rho(S)$. Now we will prove the converse inclusion.

Set $\zeta = \rho(I) \cup \iota$. Then ζ is a congruence relation on S, since ρ is transitive. Therefore we can pass to the quotient semigroup S/ζ. To simplify the notation we assume that $S = S/\zeta$, i.e. I is semisimple, which is equivalent to saying that $\zeta = \iota$. It remains to show that in this case $(I \times I) \cap \rho(S) \subseteq \iota$.

Suppose the contrary: let there be elements s, t in I such that $(s,t) \in \rho(S)$, $s \neq t$. Denote by T the semigroup generated in S by s and t. As earlier, U_2 denotes the commutative semigroup on generators u, v with relations $u^2 = uv = v^2$. Since ρ is weakly hereditary, $\rho(I)$ contains (sx, tx) and (xs, xt). Hence $sx = tx, xs = xt$. Therefore the mapping $u \mapsto s$, $v \mapsto t$ can be expanded to a homomorphism of U_2 on T. This and Lemma 7 imply $(s,t) \in \rho(T)$. The strictness of ρ yields $(s,t) \in \rho(I)$ giving a contradiction with the semisimplicity of I. $\square$

Now we can describe radicals commuting with a semilattice.

Lemma 11. *Let B be a nonsingular semilattice. A radical ρ commutes with B if and only if ρ is strict and hereditary, and the semisimple class $\mathcal{S}$ of ρ is* 0*-closed.*

Proof. If ρ commutes with B, then obviously $\mathcal{S}$ is 0-closed. Therefore the 'only if' part follows from Lemmas 5 and 10.

The 'if' part. Let ρ be strict and hereditary, and let $\mathcal{S}$ be 0-closed. First we will prove that ρ commutes with the two-element semilattice Y_2. Take any semigroup S which is a semilattice Y_2 of a subsemigroup S_e and an ideal S_o. Then $\mu = \rho(S_e) \cup \rho(S_o)$ is a congruence on S, and $\mu \subseteq \rho(S)$. By Lemma 9, $\mathcal{S}$ is closed under every semilattice. Clearly S/μ is a semilattice Y_2 of semisimple semigroups $S_e/\rho(S_e)$ and $S_o/\rho(S_o)$. Hence S/μ is semisimple, and so $\mu \supseteq \rho(S)$. Since ρ is strict, $\mu = \rho(S)$. We have shown that ρ commutes with Y_2. An easy induction yields that ρ commutes with every finite semilattice.

Take any semilattice B and any semigroup S which is a semilattice B of subsemigroups S_b. Set $\mu = \cup_{b\in B} \rho(S_b)$. Pick $x \in S_a$, and $(s,t) \in \rho(S_b)$, where

$a, b \in B$. Clearly, the semigroup $T = S_a \cup S_b \cup S_{ab}$ is a semilattice of its subsemigroups S_c, where $c \in \{a, b, ab\}$. Given that ρ is strict and commutes with finite semilattices, pairs (xs, xt) and (sx, tx) are in $\rho(S_{ab})$. So μ is a congruence on S. By Lemma 9, $\mathcal{S}$ is closed under every semilattice. This, as above, yields that S/μ is semisimple. Hence $\mu \supseteq \rho(S)$. The reverse inclusion follows from the strictness of ρ. $\square$

Now let us consider radicals commuting with rectangular bands.

Lemma 12. *A radical ρ commutes with a nonsingular left (right) zero band B if and only if it is right (left) strict, right (left) weakly hereditary, right (left) transitive, and the semisimple class $\mathcal{S}$ of ρ is closed under B.*

Proof. Necessity follows from Lemma 6.

Sufficiency. We consider only the case where B is a left zero band. Take any semigroup S which is a band B of its subsemigroups S_b. Since all S_b are right ideals in S and ρ is right strict, right weakly hereditary and right transitive, we obtain that $\mu = \cup_{b\in B}\rho(S_b)$ is a congruence on S and $\mu \subseteq \rho(S)$. Further, S/μ is a band B of semisimple semigroups $S_b/\rho(S_b)$. Therefore $S/\mu \in \mathcal{S}$, whence $\mu \supseteq \rho(S)$. So $\mu = \rho(S)$ and ρ commutes with B. $\square$

Lemma 13. *Let B be a rectangular band isomorphic to a direct product $L \times R$ where L is a left zero band, R is a right zero band. A radical ρ commutes with B if and only if it commutes with L and R.*

Proof. Necessity. Assume that ρ commutes with B. If L is a singleton, then ρ commutes with L. Let $|L| > 1$.

Then Lemma 6 says that ρ is right strict, right weakly hereditary and right transitive. We claim that the semisimple class $\mathcal{S}$ of ρ is closed under L.

Suppose the contrary: let there exist a non-semisimple semigroup S which is a band L of subsemigroups S_b. If R is a singleton, then $B = L$ gives a contradiction. Let R be nonsingular. Then ρ is left strict by Lemma 6.

There exist semigroups $W_b (b \in B)$ such that $W_a \cap W_b = \emptyset$ whenever $a \neq b$, and $W_b \cong S_l$ when $b = (l, r), l \in L, r \in R$. Expanding the multiplication of S to the union $W = \cup_{b\in B}W_b$ we get the semigroup W which is a band B of semisimple semigroups W_b. Hence W is semisimple. However W contains a left ideal $\cup_{l\in L}W_{(l,r)}$ isomorphic to S. Since ρ is left strict, it follows that S is semisimple, a contradiction. Thus $\mathcal{S}$ is closed under L.

Lemma 12 tells us that ρ commutes with L. One can dually prove that ρ commutes with R.

Sufficiency. Assume that ρ commutes with L and R. Take a semigroup S which is a band B of its subsemigroups S_b. For $l \in L$ by T_l we denote the union of semigroups $S_{(l,r)}$, where r runs over R. Then T_l is a band R of its subsemigroups $S_{(l,r)}$, and therefore $\rho(T_l) = \cup_{r\in R}\rho(S_{(l,r)})$. Besides, $\rho(S) = \cup_{l\in L}\rho(T_l)$ because S

is a band L of subsemigroups T_l. Hence $\rho(S) = \cup_{b\in B}\rho(S_b)$ which completes the proof. $\square$

Lemmas 1, 12 and 13 immediately give us

Lemma 14. *Let B be a rectangular band which is neither a left nor a right zero band. Then ρ commutes with B if and only if ρ is right and left strict, right and left weakly hereditary, right and left transitive and the semisimple class $\mathcal{S}$ of ρ is closed under B.*

Proof of Theorem. Sufficiency follows from Lemmas 11, 12 and 14.

Necessity. Suppose that ρ commutes with B but all the conditions (1)–(4) are not valid. Then Lemmas 11, 12 and 14 show that B is neither a semilattice nor a rectangular band. By Lemma 3, B contains a two-element band B_o consisting entirely of left zeros or right zeros. Without loss of generality we consider the case where B_o is a left zero band. Then ρ is right weakly hereditary by Lemma 6. Lemma 5 yields that ρ is strict. As in the proof of Lemma 7, strictness and weak heredity imply that the two-element semigroup N_2 with zero multiplication is radical.

Consider semigroup R with zero 0 defined by generators e, n and relations $e^2 = e, n^2 = 0, ene = e$. Set $T = \{0, ne, nen\}$. Since $\{0, n\}$ is isomorphic to N_2 and ρ is strict, $(0, n) \in \rho(R)$. The pair $(0, n)$ generates a universal congruence in R, and so R is radical. Clearly T is a right ideal in R. By right weak heredity $\rho(T) \supseteq T\rho(R) = T \times T$. The two-element semilattice Y_2 is isomorphic to $T/\{0, nen\}$. Hence Y_2 is radical. By Lemma 3, B contains a copy of Y_2, and so B is not semisimple or ρ is not strict. The contradiction completes our proof. $\square$

3. Examples of Radicals

Here we will give examples of radicals satisfying the conditions of main theorem. Let us consider the radical η whose semisimple class $\mathcal{S}$ consists of all semilattices. It is known and easy to prove that $\mathcal{S}$ is closed under every semilattice. The inner characterization of η due to [**12**] easily yields that η is strict and hereditary. So by the Theorem we get

Proposition 1. *The radical η commutes with every semilattice.*

Now we will construct a radical commuting with rectangular bands. For a semigroup S, by $K(S)$, we denote the set of subgroups T in S such that TST is contained in T. Let μ denote the mapping taking S to the relation $\mu(S) = \cup_{T\in K(S)} T \times T$.

If groups P, T of $K(S)$ have a common element x, then $P = xP = xPx \subseteq T$, and similarly $T \subseteq P$ implying $T = P$. It means that μ is an equivalence relation.

For $x \in S$, $T \in K(S)$ let us consider xT. First, xT is a semigroup, because $x(TxT) \subseteq xT$. Given that T is a group, for any xy in xT we get $x(Txy) =$

$xT, x(yxT) = xT$. So xT is right and left simple, and therefore it is a group. Further, $xTSxT \subseteq xT$, i.e. $xT \in K(S)$. Analogously, $Tx \in K(S)$. Therefore $\mu(S)$ is a congruence.

Condition (R1) obviously holds for μ. We are to prove (R2). Taking quotient semigroup $F = S/\mu(S)$ we claim that F is semisimple. If not, then there is $G \in K(F), |G| > 1$. Set $H = f^{-1}(G)$ where f is the natural homomorphism of S onto F. Pick any Q in $K(S)$. Since $QSQ \subseteq Q$ and $f(H)f(S)f(H) \subseteq f(H)$, we see that either Q contains H or Q does not intersect H. The former case is impossible, because H is not a singleton. Therefore $H \cap Q = \emptyset$. Hence $H \cong f(H)$, and so H is a group. Given that $G \in K(F)$, we get $H \in K(S)$. It follows that $H \times H \subseteq \mu(S)$ and so G is a singleton, giving a contradiction. Therefore F is semisimple and condition (R2) holds too. Thus μ is a radical.

Proposition 2. *The radical μ commutes with every rectangular band.*

Proof. Let B be a rectangular band. Take any semigroup S which is a band B of semisimple semigroups S_b. If S is not semisimple, then there exists a nonsingular group T in $K(S)$. Since T does not have a one-sided ideal, it follows that $T \subseteq S_b$ for some $b \in B$. By the definition of a rectangular band $S_bS_aS_b \subseteq S_{bab} = S_b$ for any $a \in B$. Hence $S_bSS_b \subseteq S_b$, and so $T \in K(S_b)$, giving a contradiction with the semisimplicity of S_b. We have proved that the semisimple class of μ is closed under every rectangular band.

Consider any semigroup S with a right ideal R. If $T \cap R \neq \emptyset$ for some $T \in K(S)$, then $T \subseteq R$. So $\mu(R)$ contains $\zeta = \mu(S) \cap (R \times R)$. Further, for any $T \in K(R)$ we get $TST = T(TS)T \subseteq T$, i.e. T is in $K(S)$. Therefore $\rho(R) = \zeta$. Hence it is clear that μ is right strict, right weakly hereditary and right transitive. Dually one can prove that μ is left strict, left weakly hereditary and left transitive. By the Theorem, μ commutes with every rectangular band. □

Among the conditions in our main theorem the most difficult for tests is the B-closeness of $\mathcal{S}$ in the case of a rectangular band B. If ρ is an M-radical (cf. [**9**]), this condition can be simplified. However the interaction of bands and M-radicals deservers a separate investigation, and we do not present that result here.

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