Acta Math. Univ. Comenianae
Vol. LXI, 1(1992), pp. 41–55

MONOTONICITY OF THE LAGRANGIAN FUNCTION IN THE PARAMETRIC INTERIOR POINT METHODS OF CONVEX PROGRAMMING

M. HAMALA AND M. HALICKÁ

ABSTRACT. Monotonicity of the Lagrangian function corresponding to the general root quasibarrier as well as to the general inverse barrier function of convex programming is proved. It is shown that monotonicity generally need not take place. On the other hand for LP-problems with some special structure monotonicity is proved for a very general class of interior point transformation functions.

1. INTRODUCTION

Current interest in interior point methods for linear programming was sparked by the 1984 algorithm of N. Karmarkar [**7**] that is claimed to be much faster than the simplex method for practical problems. The equivalence of Karmarkar's projective scaling method to interior point methods was pointed out by Gill and others in 1986 [**4**]. Since then the interior point methods for convex programming have been intensively studied again. Recently some attention has been given to the monotonicity of the corresponding Lagrangian function (with respect to the considered parameter). The known results concern only the logarithmic barrier function for convex programming problems [**1**] and the inverse barrier function for LP-problems [**2**]. In this paper we analyse the above mentioned monotonicity for convex programming and for general parametric interior point transformation function (including quasi-barrier and barrier functions). We prove monotonicity for the general root quasibarrier function as well as for the general inverse barrier function and we show that monotonicity generally need not take place. On the other hand we prove monotonicity for LP-problems with some special structure for a very general class of transformation functions.

Consider the convex programming problem

$$\text{Min}\ \{f(\mathbf{x})\,|\,g_i(\mathbf{x}) \geq 0 \quad (i = 1, 2, \dots, m)\} \tag{1}$$

Received February 17, 1992; revised April 1, 1992.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 90C25; Secondary 90C05, 90C30.
Key words and phrases. Convex programming, Lagrangian function, barrier function, quasi-barrier function, interior point methods, dual problem.

where f, $-g_i$ are convex on $\mathbb{R}^n$ with continuous first partial derivatives,

$$\mathbf{K}^0 = \{\mathbf{x} \in \mathbb{R}^n \mid g_i(\mathbf{x}) > 0 \quad (i = 1, 2, \dots, m)\} \neq \boldsymbol{\emptyset},$$

and the set of optimal solutions is nonempty and bounded.

In parametric interior point methods (i.e. barrier and quasibarrier ones) the problem (1) is transformed into a sequence of unconstrained-type problems

$$\text{Min } \left\{T(\mathbf{x}, r_k) \mid \mathbf{x} \in \mathbf{K}^0\right\}, \quad k = 1, 2, \dots \tag{2}$$

where

$$T(\mathbf{x}, r_k) = f(\mathbf{x}) + r_k \sum_{i=1}^{m} \Gamma[g_i(\mathbf{x})], \quad r_k > 0, \tag{3}$$

$\{r_k\} \to 0$ and $\Gamma\colon \mathbb{R}_{++} \to \mathbb{R}$ ($\mathbb{R}_{++}$ being the set of all positive numbers) has continuous derivative Γ' with additional properties:

$$\lim_{\xi \to \infty} \Gamma'(\xi) = 0, \tag{A}$$

$$\lim_{\xi \to 0_+} \Gamma'(\xi) = -\infty, \tag{B}$$

$$\Gamma'(\xi) \text{ is increasing.} \tag{C}$$

Accordingly, it can be proved that for any $r_k > 0$ there exists an optimal solution $\mathbf{x}^k = \mathbf{x}(r_k)$ of (2) (see [**3,** Theorem 25] for barrier functions and [**5**] for quasibarrier functions).

Remark 1. Using (C) we see that $\Gamma(\xi)$ is strictly convex. Combining (A), (B), (C) we have $\Gamma'(\xi) < 0$, for $\xi > 0$ i.e. $\Gamma(\xi)$ is decreasing. These two properties imply that the function $T(\mathbf{x}, r_k)$ defined by (3) is convex (under the assumption that $f(\mathbf{x})$ is convex, and $g_i(\mathbf{x})$ are concave).

Let us introduce the following notation:

$$f_k = f(\mathbf{x}^k), \quad \mathbf{G}_k = \sum_{i=1}^{m} \Gamma[g_i(\mathbf{x}^k)], \quad \mathbf{T}_k = \mathbf{T}(\mathbf{x}^k, r_k)\ .$$

Then the following monotonicity and convergence can be proved (e.g. [**3, 5**]):

Proposition 1. *Suppose* $0 < r_2 < r_1$. *Then*

$$f_2 \le f_1, \tag{a}$$

$$G_2 \ge G_1, \tag{b}$$

$$G_1 > 0 \Rightarrow T_2 < T_1, \tag{c}$$

$$G_2 < 0 \Rightarrow T_2 > T_1. \tag{d}$$

Proposition 2. *Let* $\{r_k\} \to 0$, *where* $r_k > 0$ *for all* k. *Then*

(a) $$\lim_{k\to\infty} T_k = \inf_{\mathbf{x}\in\mathbf{K}^0} f(\mathbf{x}),$$

(b) $$\lim_{k\to\infty} f_k = \inf_{\mathbf{x}\in\mathbf{K}^0} f(\mathbf{x}),$$

(c) $$\lim_{k\to\infty} r_k G_k = 0.$$

Obviously, for the minimum point $\mathbf{x}^k$ of (3) we have

(4) $$\nabla T(\mathbf{x}^k, r_k) = \nabla f(\mathbf{x}^k) + r_k \sum_{i=1}^{m} \Gamma'[g_i(\mathbf{x}^k)] \nabla g_i(\mathbf{x}^k) = \mathbf{0}.$$

Denote

(5) $$\Pi(\xi) = -\frac{1}{\Gamma'(\xi)}\ , \qquad \text{i.e.} \quad \Gamma'(\xi) = -\frac{1}{\Pi(\xi)}\ .$$

($\Gamma'(\xi) < 0$ because of properties (A), (C)). Then (4) can be rewritten as

(6) $$\nabla f(\mathbf{x}^k) - r_k \sum_{i=1}^{m} \frac{1}{\Pi[g_i(\mathbf{x}^k)]} \nabla g_i(\mathbf{x}^k) = \mathbf{0}$$

or equivalently as

(7) $$\nabla f(\mathbf{x}^k) - \sum_{i=1}^{m} u_i^k . \nabla g_i(\mathbf{x}^k) = \mathbf{0}$$

(8) $$u_i^k \cdot \Pi[g_i(\mathbf{x}^k)] = r_k \qquad (i = 1, 2, \ldots, m).$$

The above (A), (B), (C) properties of Γ' are equivalent with the following properties of Π:

(A*) $$\lim_{\xi\to\infty} \Pi(\xi) = +\infty\,,$$

(B*) $$\lim_{\xi\to 0_+} \Pi(\xi) = 0\,,$$

(C*) $\Pi(\xi)$ is increasing.

So, the optimal solution $\mathbf{x}^k$ of the problem (2) is equal to the x-part of the solution $(\mathbf{x}^k, \mathbf{u}^k)$ of the system (7), (8), where $\Pi(\xi)$ is given by (5). The advantage of this reformulation consists in the fact that the solution $(\mathbf{x}^k, \mathbf{u}^k)$ of (7), (8) forms a feasible solution of the Wolfe dual problem associated with the primal problem (1):

(9) $$\text{Max}\ \{L(\mathbf{x}, \mathbf{u}) \,|\, \nabla_x L(\mathbf{x}, \mathbf{u}) = \mathbf{0}, \quad \mathbf{u} \geq 0\}$$

where

$$L(\mathbf{x}, \mathbf{u}) = f(\mathbf{x}) - \sum_{i=1}^{m} u_i g_i(\mathbf{x}) \tag{10}$$

is the Lagrangian function of (1). Indeed, if $\mathbf{x}^k$ is the solution of (6) and we put

$$u_i^k = \frac{r_k}{\Pi[g_i(\mathbf{x}^k)]} \quad (i = 1, 2, \dots, m), \tag{11}$$

which are positive entities (since from (B*), (C*) it follows that for $\xi > 0$ we have $\Pi(\xi) > 0$), then (6) is equivalent to $\nabla_x L(\mathbf{x}^k, \mathbf{u}^k) = \mathbf{0}$.

2. Problem Formulation and Review of Results

Let $\mathbf{x}^k = \mathbf{x}(r_k)$ be an optimal solution of the problem (2), (i.e. $\mathbf{x}^k$ is a feasible solution of the problem (1)), $\mathbf{u}^k = \mathbf{u}(r_k)$ be defined by (11) (i.e. $(\mathbf{x}^k, \mathbf{u}^k)$ is a feasible solution of the Wolfe dual problem (9)) and let $\mathbf{x}^*$ be an optimal solution of the convex programming problem (1). Then by the weak duality theorem of convex programming we have

$$L(\mathbf{x}^k, \mathbf{u}^k) \le f(\mathbf{x}^*) \le f(\mathbf{x}^k). \tag{12}$$

Let now $\{r_k\} \to 0_+$. Then by Proposition 1(a) the upper bound in (12) is monotonically decreasing. The following question arises: **Under what assumptions is the lower bound in** (12) **monotonically increasing?**

It was proved by Fiacco and McCormick [**3**] that the lower bound in (12) is converging to the optimal value $f(\mathbf{x}^*)$. In the case of linear programming, and for the logarithmic barrier function

$$\Gamma(\xi) = -\ln \xi, \qquad \Pi(\xi) = \xi \tag{13}$$

monotonicity of $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$ was proved by Megiddo [**9**]. Recently this result was extended by Den Hertog, Roos and Terlaky [**1, 2**]. In [**1**] they proved monotonicity of $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$ for the logarithmic barrier function (13) in the general case of convex programming. In [**2**] they proved monotonicity of $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$ for the inverse barrier function

$$\Gamma(\xi) = 1/(p\xi^p), \qquad \Pi(\xi) = \xi^{p+1}, \quad \text{where } p > 0, \tag{14}$$

in the linear programming case.

In Section 3, as **the first result** of this paper, we prove monotonicity of $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$ in the general case of convex programming for the class of interior point transformation functions

$$\Gamma(\xi) = 1/(p\xi^p) = \frac{1}{p}\xi^{-p}, \qquad \Pi(\xi) = \xi^{p+1}, \quad \text{where } p > -1. \tag{15}$$

Note that (15) includes the inverse barrier function (14), if $p > 0$, the logarithmic barrier function (13), if $p = 0$, and the general root quasi-barrier function, if $-1 < p < 0$. (The notion of the quasibarrier function was introduced in [**5**].) So, our generalization consists in the following:

a) Compared with [**1**] (i.e. general convex programming, case $p = 0$) we extend the monotonicity to the intervals $p > 0$, and $-1 < p < 0$.

b) Compared with [**2**] (i.e. linear programming, case $p > 0$) we extend the monotonicity to general convex programming, as well as to the interval $-1 < p < 0$.

In Section 4 we show, as **the second result** of this paper, that there exists $\Pi(\xi)$, satisfying (A^*), (B^*), (C^*), for which $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$ does not have to be monotonic (even for linear programming problems). We give a counter-example with the barrier function $\Gamma(\xi) = -\ln(1 - e^{-\xi})$, i.e. $\Pi(\xi) = e^{\xi} - 1$ and a very simple LP-problem which shows that it is not possible to prove the monotonicity of $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$ without additional assumptions on the function $\Pi(\xi)$ or on the convex programming problem (1).

In Section 5, as **the third result** of this paper, we prove that for linear programming with some special structure, any $\Pi(\xi)$ satisfying (A^*), (B^*), (C^*) implies monotonicity of $\{L(\mathbf{x}^k, \mathbf{u}^k)\}$. We will consider LP-problems with the set of feasible solutions forming either a simplex or a parallelepiped.

3. Monotonicity of $L[\mathbf{x}(r), \mathbf{u}(r)]$ for $\Pi(\xi) = \xi^h$, $h > 0$

In the first section the problem of monotonicity was treated for a discrete sequence $\{r_k\}$ of values of parameter $r > 0$, and Proposition 1 was also proved (e.g. [**3, 5**]) using discrete values only. On the other hand, in proving monotonicity of Lagrangean function it is more convenient to use the classical calculus. Note that the previous results [**1, 2**] were obtained in the same way. This is the reason why in this section we switch from the discrete case to the continuous one. However, this technique requires uniqueness of the minimum-point $\mathbf{x}(r)$ of the function $T(\mathbf{x}, r)$ for all the values of parameter $r > 0$ as well as smoothness of the trajectory $\mathbf{x}(r)$. Both these requirements will be satisfied by the technical assumption of the regularity of Hessian matrix $\nabla^2 T(\mathbf{x}, r)$.

Lemma 1. *Let $f \in \mathbf{C}^2$, and $-g_i \in \mathbf{C}^2$ $(i = 1, 2, \dots, m)$ be convex on $\mathbb{R}^n$, $\Gamma\colon \mathbb{R}_{++} \to \mathbb{R}$ have properties* (A), (B), (C), *and $\Gamma \in \mathbf{C}^2$. Further, let for*

$$T(\mathbf{x}, r) = f(\mathbf{x}) + r \sum_{i=1}^{m} \Gamma[g_i(\mathbf{x})], \quad r > 0$$

its Hessian matrix $\nabla^2 T(\mathbf{x}, r)$ be nonsingular on the set $\mathbf{K}^0$. Then for any $r > 0$ the minimum point $\mathbf{x}(r) \in \mathbf{K}^0$ of $T(\mathbf{x}, r)$ is unique and the mapping $\mathbf{x}(r)\colon \mathbb{R}_{++} \to \mathbb{R}^n$ is $\mathbf{C}^1$.

Proof. The assumptions of the lemma imply that $T(\mathbf{x}, r)$ is strictly convex on $\mathbf{K}^0$ and therefore the minimum point $\mathbf{x}(r)$ of $T(\mathbf{x}, r)$ is unique. The system of equations (4) has nonsingular Jacobian matrix (it is, the Hessian of $T(\mathbf{x}, r)$) and therefore, by the implicit function theorem, $\mathbf{x}(r) \in \mathbf{C}^1$. □

Lemma 2. *Let the assumptions of Lemma* 1 *be satisfied and let* $\Gamma''(\xi) > 0$ *for any* $\xi > 0$. *Then*

$$\Pi(\xi) = -\frac{1}{\Gamma'(\xi)} \in \mathbf{C}^1, \qquad \Pi'(\xi) > 0$$

and for

$$u_i(r) = \frac{r}{\Pi[g_i(\mathbf{x}(r))]} \qquad (i = 1, 2, \ldots, m),$$

we have

$$\sum_{i=1}^{m} \frac{\Pi[g_i(\mathbf{x}(r))]}{\Pi'[g_i(\mathbf{x}(r))]} \cdot \frac{du_i(r)}{dr} \geq 0. \tag{16}$$

Proof. By Remark 1 we have $\Gamma'(\xi) < 0$, so $\Pi(\xi)$ is well defined. The definition of $\Pi(\xi)$ and the assumption $\Gamma''(\xi) > 0$ imply that $\Pi'(\xi) > 0$. From the assumptions we obtain the **continuous version** of the equations (7), (8), namely

$$\nabla f(\mathbf{x}(r)) - \sum_{i=1}^{m} u_i(r) \cdot \nabla g_i(\mathbf{x}(r)) = 0 \tag{17}$$

$$(i = 1, 2, \ldots, m): \ u_i(r) \cdot \Pi[g_i(\mathbf{x}(r))] = r. \tag{18}$$

Taking the derivatives of (17), (18) with respect to $r > 0$ we obtain

$$\left[\nabla^2 f - \sum_{i=1}^{m} u_i \nabla^2 g_i\right] \cdot \frac{d\mathbf{x}}{dr} - \sum_{i=1}^{m} \frac{du_i}{dr} \nabla g_i = 0, \tag{19}$$

$$(i = 1, 2, \ldots, m): \ \frac{du_i}{dr} \cdot \Pi_i + u_i \left(\nabla g_i^\top \frac{d\mathbf{x}}{dr}\right) \cdot \Pi_i' = 1, \tag{20}$$

where the following **short notation** is used:

$$\mathbf{x} = \mathbf{x}(r), \quad u_i = u_i(r), \quad f = f(\mathbf{x}), \quad g_i = g_i(\mathbf{x}), \quad \nabla^2 f = \nabla^2 f(\mathbf{x}),$$
$$\nabla^2 g_i = \nabla^2 g_i(\mathbf{x}), \text{ and } \Pi_i = \Pi[g_i(\mathbf{x})], \quad \Pi_i' = \Pi'[g_i(\mathbf{x})].$$

Multiplying (19) from the left by row vector $\dfrac{d\mathbf{x}}{dr}^\top$ we get

$$\sum_{i=1}^{m} \frac{du_i}{dr} \left(\nabla g_i^\top \frac{d\mathbf{x}}{dr}\right) \geq 0, \tag{21}$$

as the matrix $\left[\nabla^2 f - \sum_{i=1}^m u_i \nabla^2 g_i\right]$ is positive definite. From (20) we get

$$\left(\nabla g_i^\top \frac{d\mathbf{x}}{dr}\right) = \frac{1}{u_i \Pi_i'}\left[1 - \frac{du_i}{dr}\cdot \Pi_i\right] \tag{22}$$

which, when substituted into (21) gives

$$\sum_{i=1}^m \frac{du_i}{dr}\frac{1}{u_i\Pi_i'} \geq \sum_{i=1}^m \left(\frac{du_i}{dr}\right)^2 \frac{\Pi_i}{u_i\Pi_i'}.$$

But the right hand side is evidently nonnegative, and thus by (18) we have

$$\sum_{i=1}^m \frac{du_i}{dr}\frac{1}{u_i\Pi_i'} = \frac{1}{r}\sum_{i=1}^m \frac{du_i}{dr}\frac{\Pi_i}{\Pi_i'} \geq 0$$

which imply (16). The lemma is proved. □

Theorem 1. *Let the assumptions of Lemma* 1 *be satisfied. Assume the special form of function* $\Pi(\xi)$

$$\Pi(\xi) = \xi^h, \qquad h > 0. \tag{23}$$

Then for the Lagrangian function (10), (11), *i.e.* $L(r) = L[\mathbf{x}(r), \mathbf{u}(r)]$ *we have:*

$$\frac{dL}{dr} = -\sum_{i=1}^m \frac{du_i}{dr} g_i \leq 0\,, \tag{24}$$

i.e. $L[\mathbf{x}(r), \mathbf{u}(r)]$ *is monotonically nonincreasing, or*

$$0 < r_2 < r_1 \Longrightarrow L(\mathbf{x}^1, \mathbf{u}^1) \leq L(\mathbf{x}^2, \mathbf{u}^2).$$

Proof. Note that the assumptions of Lemma 2 are satisfied as for the function (23) we have $\Pi'(\xi) > 0$. Further, by (10)

$$\frac{dL}{dr} = \frac{\partial L}{\partial \mathbf{x}}\cdot\frac{d\mathbf{x}}{dr} + \frac{\partial L}{\partial \mathbf{u}}\cdot\frac{d\mathbf{u}}{dr}\,. \tag{25}$$

But from (17) we see that $\frac{\partial L}{\partial \mathbf{x}} = \mathbf{0}^\top$. Evidently $\frac{\partial L}{\partial \mathbf{u}} = -\mathbf{g}^\top$, where

$$\mathbf{g}^T = [g_1(\mathbf{x}(r)), g_2(\mathbf{x}(r)), \ldots, g_m(\mathbf{x}(r))].$$

Substituting this into (25) we get

$$\frac{dL}{dr} = -\mathbf{g}^\top\left(\frac{d\mathbf{u}}{dr}\right). \tag{26}$$

From (23) we have

$$\frac{\Pi(\xi)}{\Pi'(\xi)} = \frac{1}{h}\xi \ ,$$

which by Lemma 2 implies

$$\sum_{i=1}^{m} \frac{du_i}{dr} g_i = \mathbf{g}^\top \left(\frac{d\mathbf{u}}{dr}\right) \geq 0 \,. \tag{27}$$

Now, (26) and (27) imply (24). The theorem is proved. □

Remark 2. For function $\Pi(\xi) = \xi^h$, $h > 0$ by (5) we have

$$\Gamma(\xi) = -\ln \xi \ , \qquad \text{if } h = 1,$$
$$\Gamma(\xi) = -\frac{\xi^{1-h}}{1-h} \ , \quad \text{if } h \neq 1,$$

where for $h > 1$, $\Gamma(\xi)$ is the inverse barrier function, and for $0 < h < 1$, $\Gamma(\xi)$ is the general root quasibarrier function. (Note that in (14) and (15) we used $p = h - 1$.)

4. Counterexample for $\Pi(\xi) = e^\xi - 1$

In this section we show that for the convex programming problem (1) monotonicity of $L(r)$ cannot be proved under general assumptions (A^*), (B^*), (C^*). We give an example of one-dimensional linear programming problem with three constraints and a special function $\Pi(\xi) = e^\xi - 1$, $[\Gamma(\xi) = -\ln(1 - e^{-\xi})]$ for which the corresponding $L(r)$ is not monotonic.

Theorem 2. *There exists* $\Pi(\xi)$ *with properties* (A^*), (B^*), (C^*) *for which* $L[\mathbf{x}(r), \mathbf{u}(r)]$ *does not have to be monotonic* (*even for linear programming problems*).

Proof. For $\Pi(\xi) = e^\xi - 1$ we will construct a special linear programming problem of the form

$$\text{Min} \left\{ \mathbf{c}^\top \mathbf{x} \,|\, \mathbf{a}_i^\top \mathbf{x} - b_i \geq 0 \quad (i = 1, 2, \ldots, m), \quad \mathbf{x} \in \mathbb{R}^n \right\} \tag{28}$$

with one variable and three constraints, i.e. with $n = 1$ and $m = 3$, and then we will show that for this LP-problem $L[\mathbf{x}(r), \mathbf{u}(r)]$ is not monotonic.

LP-problem (28) is a special case of the convex programming problem (1) for which the equations (17), (18), (19), (20) take the form

$$\sum_{i=1}^{m} u_i \mathbf{a}_i = \mathbf{c} \tag{29}$$

$$u_i \Pi_i = r \tag{30}$$

$$\sum_{i=1}^{m} \frac{du_i}{dr} \mathbf{a}_i = \mathbf{0} \tag{31}$$

$$\frac{du_i}{dr} \cdot \Pi_i + u_i \left(\mathbf{a}_i^\top \frac{d\mathbf{x}}{dr} \right) \cdot \Pi_i' = 1 \qquad (i = 1, 2, \dots, m) \tag{32}$$

respectively, and the equation (26) (if we take into account (31)) takes the form

$$\frac{dL}{dr} = \sum_{i=1}^{m} b_i \frac{du_i}{dr} \ . \tag{33}$$

Multiplying (32) by $u_i \mathbf{a}_i$, then summing up the products for $i = 1, 2, \dots, m$ and considering (30), (31), and (29), we get

$$r^2 \sum_{i=1}^{m} \frac{\Pi_i'}{\Pi_i^2} \left(\mathbf{a}_i^\top \frac{d\mathbf{x}}{dr} \right) \cdot \mathbf{a}_i = \mathbf{c} \ . \tag{34}$$

Obviously, from (29) and (30) we have

$$r \sum_{i=1}^{m} \frac{1}{\Pi_i} \mathbf{a}_i = \mathbf{c} \ . \tag{35}$$

Thus we eliminated variables u_i and $\frac{du_i}{dr}$ $(i = 1, 2, \dots, m)$ from (29–32).

Consider now the case of one variable (i.e. $x \in \mathbb{R}$) and a fixed value $r > 0$. Then we can solve the equation (35) for $x(r) \in \mathbf{K}^0$, and thus from (34) we get

$$\frac{dx}{dr} = c r^{-2} \left[\sum_{i=1}^{m} \left(\frac{a_i}{\Pi_i} \right)^2 \Pi_i' \right]^{-1} . \tag{36}$$

Further, from (30) we get u_i and finally, from (32) we get $\frac{du_i}{dr}$, which is used in (33) for evaluating the derivative $\frac{dL}{dr}$.

The above sequence of computations can be simplified by introducing the following auxiliary function

$$F(x) = c - r \sum_{i=1}^{m} \frac{a_i}{\Pi_i} \quad \text{(i.e. } F(x) = T'(x, r)),$$

for which

$$F'(x) = r \sum_{i=1}^{m} \left(\frac{a_i}{\Pi_i} \right)^2 \Pi_i' > 0.$$

Let x_r be the root of (35), i.e.

$$F(x_r) = 0. \tag{37}$$

Then (36) can be rewritten as

$$\frac{dx}{dr} = \frac{c}{rF'(x_r)};$$

thus, (30) and (32) yield

$$\frac{du_i}{dr} = \frac{1}{\Pi_i} \left[1 - \frac{c}{F'(x_r)} a_i \frac{\Pi_i'}{\Pi_i} \right]. \tag{38}$$

Finally, substituting (38) into (33) we get

$$\frac{dL}{dr} = \sum_{i=1}^{m} \frac{b_i}{\Pi_i} - \frac{c}{F'(x_r)} \cdot \sum_{i=1}^{m} b_i a_i \Pi_i' \Pi_i^{-2}.$$

Let $r = 1$ and $\Pi(\xi) = e^{\xi} - 1$. Now consider the LP-problem (28) of one variable and three constraints with the following input:

$$\begin{aligned} a_1 &= 10 & b_1 &= 0 \\ a_2 &= -1 & b_2 &= -\ln 20 \\ a_3 &= -3 & b_3 &= -\ln 22000 \\ c &= 10/(10^{10} - 1) - 8/7 \end{aligned}$$

Then the root of (35) (or (37)) is: $x_r = \ln 10$. (In fact, we had first chosen x_r and then adjusted the value of c.)

Thus for $g_i = g_i(x_r) = a_i x_r - b_i$, and for $\Pi_i = \exp(g_i) - 1$ we have:

$$\begin{aligned} g_1 &= 10 \ln 10 & \Pi_1 &= 10^{10} - 1 & \Pi_1' &= 10^{10} \\ g_2 &= \ln 2 & \Pi_2 &= 1 & \Pi_2' &= 2 \\ g_3 &= \ln 22 & \Pi_3 &= 21 & \Pi_3' &= 22 \end{aligned}$$

$$F'(x_r) = \frac{10^{12}}{(10^{10} - 1)^2} + \frac{120}{49}$$

and

$$\frac{dx_r}{dr} = \frac{c}{F'(x_r)} = -\frac{7}{10}\left[\frac{4-43\varepsilon+39\varepsilon^2}{6+233\varepsilon+6\varepsilon^2}\right], \quad \text{where } \varepsilon = 10^{-10}. \tag{39}$$

From (39) it follows that

$$-\frac{7}{15} < \frac{dx_r}{dr} < \left(-\frac{7}{15}+\varepsilon_1\right), \quad \text{where } \varepsilon_1 = 100\varepsilon \tag{40}$$

and from (38), (40) we get

$$\frac{du_2}{dr} = \left(1+2\frac{dx_r}{dr}\right) > 0\,, \qquad \frac{du_3}{dr} = \frac{1}{21}\left(1+\frac{22}{7}\frac{dx_r}{dr}\right) < 0\,. \tag{41}$$

The relations (40) and (41) imply the following estimations

$$\frac{1}{15} < \frac{du_2}{dr} < \left(\frac{1}{15}+2\varepsilon_1\right), \qquad -\frac{1}{45} < \frac{du_3}{dr} < \left(-\frac{1}{45}+\varepsilon_1\right). \tag{42}$$

Obviously,

$$\begin{aligned} b_2 &= -\ln 20 \qquad = -2.995732 > -3\,, \\ b_3 &= -\ln 22000 = -9.998797 < -9.99\,. \end{aligned} \tag{43}$$

Finally, by (33), (42) and (43) we get

$$\frac{dL}{dr} = \sum_{i=1}^{3} b_i \frac{du_i}{dr} = b_2\frac{du_2}{dr} + b_3\frac{du_3}{dr} > 0.022 - 16\varepsilon_1 > 0$$

The theorem is proved. □

Note that boundedness of the set of feasible solutions and presence of the redundant inequality in the above example are substantial for the non-monotonicity of $L(r)$. Indeed, the monotonicity of $L(r)$ for the one-dimensional LP-problem with two constraints and with the bounded set of feasible solutions is the consequence of Theorem 3 of the following section.

The question whether or not the absence of redundant constraints in general can guarantee the monotonicity of $L(r)$ is still open.

5. Monotonicity of $L[\mathbf{x}(r), \mathbf{u}(r)]$ for Linear Programming

In the previous section we proved that the basic assumptions (A*), (B*), (C*) for $\Pi(\xi)$ are not generally sufficient to ensure monotonicity of the Lagrangian function. In this section we give two special classes of linear programming problems for which the Lagrangian function is strictly monotonic for any $\Pi(\xi)$ under the basic assumptions (A*), (B*), (C*). The first class consists of LP-problems with the set of feasible solutions being a simplex. The second one consists of LP-problems, where the set of feasible solutions forms a parallelepiped.

We begin with a lemma which relates the boundedness of a nonempty set

$$\mathbf{X} = \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^\top \mathbf{x} - b_i \geq 0, \quad (i = 1, 2, \dots, m)\right\} \tag{44}$$

with certain properties of the vectors $\mathbf{a}_i \in \mathbb{R}^n$ $(i = 1, 2, \dots, m)$.

Lemma 3. *Let $\mathbf{X} \subset \mathbb{R}^n$, defined by* (44)*, be nonempty and bounded. Then:*

a) The linear hull of the vectors $\mathbf{a}_i \in \mathbb{R}^n$ $(i = 1, 2, \dots, m)$ spans $\mathbb{R}^n$, i.e. n linearly independent vectors $\mathbf{a}_i$ $(i = k_1, k_2, \dots, k_n)$ can be chosen.

b) There exist $y_i > 0$, $(i = 1, 2, \dots, m)$ such that

$$\sum_{i=1}^{m} \mathbf{a}_i y_i = \mathbf{0}.$$

Proof. Consider the recession cone

$$\mathbf{Z} = \left\{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{a}_i^\top \mathbf{z} \geq 0, \quad (i = 1, 2, \dots, m)\right\}$$

of the nonempty convex set $\mathbf{X} \subset \mathbb{R}^n$ defined by (44). Since $\mathbf{X}$ is bounded, the recession cone $\mathbf{Z}$ of $\mathbf{X}$ consists of the zero vector alone [**10,** Theorem 8.4, p. 64], i.e.

(i) there does not exist $\mathbf{z} \neq \mathbf{0}$, such that $\mathbf{a}_i^\top \mathbf{z} = 0$, $(i = 1, 2, \dots, m)$, and

(ii) there does not exist $\mathbf{z} \neq \mathbf{0}$, such that $\mathbf{a}_i^\top \mathbf{z} \geq 0$, $(i = 1, 2, \dots, m)$, and at least for one index k $(1 \leq k \leq m)$ $\mathbf{a}_k^\top \mathbf{z} > 0$.

a) The statement of Lemma 3a) follows directly from (i). Indeed, if the linear hull of vectors $\mathbf{a}_i \in \mathbb{R}^n$ $(i = 1, 2, \dots, m)$ spans a proper subspace of $\mathbb{R}^n$, there would exist at least one nonzero $\mathbf{z} \in \mathbb{R}^n$, orthogonal to that subspace, which contradicts (i).

b) The statement of Lemma 3b) follows from (ii) and from Stiemke's theorem of the alternatives [**8,** p. 32]. The lemma is proved. □

Corollary. *If the assumptions of Lemma* 3 *are satisfied, then for every vector $\mathbf{z} \in \mathbb{R}^n$ there exists an $\mathbf{a}_k$ $(1 \leq k \leq m)$ such that $\mathbf{a}_k^\top \mathbf{z} \leq 0$.*

Proof. Follows directly from the proof of Lemma 3 (part (ii)). □

Remark 3. In Lemma 3 for boundedness of the nonempty set $\mathbf{X}$ (44) the conditions a), b) are necessary. However, it can also be proved by the same arguments that these conditions are even sufficient.

Now we return to the investigation of the function $L[\mathbf{x}(r), \mathbf{u}(r)]$ defined in section 3. Note that this function depends on the objective function $f(\mathbf{x})$, the constraint functions $g_i(\mathbf{x})$ $(i = 1, 2, \ldots, m)$ and the function $\Pi(\xi)$.

Consider the linear programming problem

$$\text{Min}\left\{\mathbf{c}^\top\mathbf{x} \mid \mathbf{a}_i^\top\mathbf{x} - b_i \geq 0, \quad (i = 1, 2, \ldots, m)\right\} \tag{45}$$

with the set of feasible solutions $\mathbf{X}$, defined by (44).

Theorem 3. *Suppose that for* LP*-problem* (45)*, with the set of feasible solutions* (44)*, the following assumptions hold:*

a) The system of inequalities $\mathbf{a}_i^\top\mathbf{x} - b_i \geq 0$ $(i = 1, 2, \ldots, m)$ *does not contain redundant ones.*

b) The set $\mathbf{X}$*, defined by* (44)*, is either*

  (i) *n-dimensional simplex, or*

  (ii) *n-dimensional parallelepiped.*

Suppose the function $\Pi(\xi) \in \mathbf{C}^1$ *satisfies* (A*), (B*), (C*)*, and* $\Pi'(\xi) > 0$*. Then the function* $L[\mathbf{x}(r), \mathbf{u}(r)]$ *is monotonically decreasing.*

Proof. Because of Lemma 3a, boundedness of the simplex (i) or the parallelepiped (ii) implies, that there exist n linearly independent vectors $\mathbf{a}_i$. Obviously, $\Pi'(\xi) > 0$ implies $\Gamma''(\xi) > 0$. Now, from the structure of

$$\nabla^2 T(\mathbf{x}, r) = r\sum_{i=1}^{m}\Gamma''[\mathbf{a}_i^\top\mathbf{x} - b_i]\ \mathbf{a}_i\mathbf{a}_i^\top$$

we can see that the Hessian matrix $\nabla^2 T(\mathbf{x}, r)$ is positive definite. So, the assumptions of Lemma 2 are satisfied and the properties (17)–(20) are valid. In our linear case (19) takes the form

$$\sum_{i=1}^{m}\mathbf{a}_i\frac{du_i}{dr} = \mathbf{0} \tag{46}$$

and (20) can be rewritten as

$$\frac{du_i}{dr} = \frac{1}{\Pi_i}\left[1 - r\frac{\Pi_i'}{\Pi_i}\left(\mathbf{a}_i^\top\frac{d\mathbf{x}}{dr}\right)\right]. \tag{47}$$

Since

$$\frac{dL}{dr} = -\sum_{i=1}^{m}\frac{du_i}{dr}\left(\mathbf{a}_i^\top\mathbf{x} - b_i\right),$$

it is sufficient to prove that

$$\frac{du_i}{dr} > 0, \qquad (i = 1, 2, \ldots, m).$$

(i) First, let $\mathbf{X}$ satisfy the assumption b), (i). Since $\mathbf{X}$ is an n-dimensional simplex and there are no redundant inequalities, we have $m = n+1$. The simplex is a nonempty, bounded set and so by Lemma 3a there exist n linearly independent vectors $\mathbf{a}_i$ and thus the equation

$$\sum_{i=1}^{n+1} \mathbf{a}_i \lambda_i = \mathbf{0} \tag{48}$$

has a one-dimensional set of solutions. By Lemma 3b the system (48) has a positive solution and hence every solution $\frac{du_i}{dr}$, $(i = 1, 2, \ldots, n+1)$ of (48) is either positive or negative. Now it remains to prove that there exists at least one positive component $\frac{du_i}{dr}$. So, we use the equation (47), in which obviously $r > 0$, $\Pi_i' > 0$ and $\Pi_i > 0$. From Corollary it follows that for the vector $\frac{d\mathbf{x}}{dr}$ there exists a vector $\mathbf{a}_k$, such that $\mathbf{a}_k^\top \frac{d\mathbf{x}}{dr} \leq 0$. Then by (47) we have $\frac{du_k}{dr} > 0$.

(ii) Now let $\mathbf{X}$ satisfy the assumption b), (ii). Since $\mathbf{X}$ is an n-dimensional parallelepiped and there are no redundant inequalities, we have $m = 2n$. By Lemma 3b there exist n linearly independent vectors $\mathbf{a}_i$. Without loss of generality we can assume that the first n vectors $\mathbf{a}_i$ $(i = 1, 2, \ldots, n)$ are linearly independent and $\mathbf{a}_{n+i} = -\lambda_i \mathbf{a}_i$, where λ_i $(i = 1, 2, \ldots, m)$ are positive numbers (as $\mathbf{X}$ is a parallelepiped). Thus from (46) we have

$$\sum_{i=1}^{2n} \mathbf{a}_i \frac{du_i}{dr} = \sum_{i=1}^{n} \left(\mathbf{a}_i \frac{du_i}{dr} + \mathbf{a}_{n+i} \frac{du_{n+i}}{dr} \right) = \sum_{i=1}^{n} \mathbf{a}_i \left(\frac{du_i}{dr} - \lambda_i \cdot \frac{du_{n+i}}{dr} \right) = \mathbf{0}.$$

Since $\mathbf{a}_i$ $(i = 1, 2, \ldots, n)$ are linearly independent, we have

$$\frac{du_i}{dr} = \lambda_i \cdot \frac{du_{n+i}}{dr}, \quad \lambda_i > 0, \quad (i = 1, 2, \ldots, n). \tag{49}$$

Now we will use the equation (47), in which obviously $r > 0$, $\Pi_i > 0$ and $\Pi_i' > 0$. If $\mathbf{a}_i^\top \frac{d\mathbf{x}}{dr} \leq 0$, then evidently $\frac{du_i}{dr} > 0$ and by (49) also $\frac{du_{n+i}}{dr} > 0$. If $\mathbf{a}_i^\top \frac{d\mathbf{x}}{dr} > 0$, then $(\mathbf{a}_{n+i})^\top \frac{d\mathbf{x}}{dr} < 0$ and by (47) $\frac{du_{n+i}}{dr} > 0$, so by (49) $\frac{du_i}{dr} > 0$. Theorem 3 is proved. □

Remark 4. A parallelepiped can be interpreted as a Cartesian product of n one-dimensional simplices. This fact leads directly to the generalization of Theorem 3 for the case when the set $\mathbf{X}$ is represented as a Cartesian product of s simplices of dimension n_j $(j = 1, 2, \ldots, s)$, such that $n_1 + n_2 + \ldots + n_S = n$. The proof technique is the same as in Theorem 3, but is very cumbersome, so we do not present it here.

Acknowlegdments. The authors want thank to P. Meravý and J. Plesník for stimulating discussion and helpful remarks.

References

1. Den Hertog D., Roos C. and Terlaky T., *On the classical logarithmic barrier function method for a class of smooth convex programming problems*, Report 90–28, Faculty of Technical Mathematics and Informatics, Delft University of Technology (1990).

2. ______, *Inverse barrier methods for linear programming*, Report 91–27, Faculty of Technical Mathematics and Informatics, Delft University of Technology (1991).

3. Fiacco A. V. and McCormick G. P., *Nonlinear Programming, Sequential Unconstrained Minimization Techniques*, Wiley and Sons, New York, 1968. Or the 2nd edition: SIAM, Philadelphia 1990; No. 4 in series Classics in Applied Mathematics..

4. Gill P. E., Murray W., Saunders M. A., Tomlin J. A., and Wright M. H., *On projected Newton barrier methods for linear programming and an equivalence to Karmarkar's projective method*, Math. Programming **36** (1986), 183–209.

5. Hamala M., *Quasibarrier methods for convex programming, in Survey of Mathematical Programming I,* (A. Prékopa, ed.), North-Holland, 1979, pp. 465–477.

6. Hamala M., *A general approach to interior point transformation methods for mathematical programming*, Acta Math. Universitatis Comenianae **LIV–LV** (1988), 243–266.

7. Karmarkar N., *A new polynomial-time algorithm for linear programming*, Combinatorica **4** (1984), 373–395.

8. Mangasarian O. L., *Nonlinear Programming*, McGraw-Hill, 1969.

9. Megiddo N., *Pathways to the optimal set in linear programming*, in Proceedings of the Mathematical Programming Symposium of Japan, Nagoya, Japan, 1986, pp. 1–36.

10. Rockafellar R. T., *Convex Analysis*, Princeton University Press, 1970.

M. Hamala, Department of Numerical and Optimization Methods, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia

M. Halická, Institute of Applied Mathematics, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia