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ON THE NEUMANN OPERATOR OF THE ARITHMETICAL MEAN

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We shall identify the Euclidean plane $\mathbb{R}^2$ with the set $\mathbb{C}$ of all complex numbers. If $z \in \mathbb{C}$, then $\operatorname{Re} z$, $\operatorname{Im} z$ and $\overline{z}$ denote the real part, the imaginary part and the complex conjugate of z, respectively. The scalar product of vectors $u, v \in \mathbb{R}^2$ will be denoted by $\langle u, v\rangle$ $(= \operatorname{Re} u\overline{v})$. We shall be engaged with logarithmic potentials in the plane derived from the classical kernel defined for $x, z \in \mathbb{R}^2$ by

$$h_z(x) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|z-x|}, & \text{if } x \neq z, \\ +\infty, & \text{if } x = z. \end{cases}$$

The symbol λ_k $(k \in \{1, 2\})$ will denote the k-dimensional Hausdorff measure (with the usual normalization, so that $\lambda_k([0, 1]^k) = 1$). For $M \subset \mathbb{R}^2$ we use the symbols ∂M, $\operatorname{int} M$ and $\operatorname{cl} M$ to denote the boundary, the interior and the closure of M, respectively. For $M \neq \emptyset$ we denote by $\mathcal{C}(M)$ the Banach space of all bounded continuous functions on M with the supremum norm, by 1_M the constant function equal to 1 on M, by $\operatorname{Const}(M) = \{\alpha 1_M; \alpha \in \mathbb{R}\}$ the class of all constant functions on M. $\mathcal{C}_0^{(1)}$ will stand for the class of all continuously differentiable functions with a compact support in $\mathbb{R}^2$, for bounded M we write $\mathcal{C}^{(1)}(M) = \{\varphi|_M : \varphi \in \mathcal{C}_0^{(1)}\}$ for the class of all restrictions to M of functions in $\mathcal{C}_0^{(1)}$. Throughout, $K \subset \mathbb{R}^2$ will be a fixed non-void compact set which is massive at each $z \in K$ in the sense that each disk

$$B_r(z) = \{x \in \mathbb{R}^2; |x - z| < r\}$$

with radius $r > 0$ and center z in K intersects K in a set of positive Lebesgue measure:

$$\lambda_2[B_r(z) \cap K] > 0\,.$$

This is the only à priori restriction we impose on K; it is by no means essential in connection with boundary value problems (cf. Remark 1.14 and 2.3 in [**8**]) but it will allow us to avoid some technical complications.

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Put $G = \mathbb{R}^2 \setminus K$ and denote by $\mathcal{C}^*(\partial K)$ the space of all finite signed Borel measures supported by ∂K. For each $\mu \in \mathcal{C}^*(\partial K)$ the potential

$$\mathcal{U}\mu(x) = \int_{\partial K} h_z(x)\, d\mu(z) \tag{1}$$

defines a harmonic function of the variable x on $\mathbb{R}^2 \setminus \partial K$ such that, for each bounded Borel $P \subset \mathbb{R}^2 \setminus \partial K$, the gradient of (1) is integrable over P:

$$\int_P |\mathrm{grad}\, \mathcal{U}\mu(x)|\, d\lambda_2(x) < +\infty\,.$$

This property makes it possible to introduce the so-called weak normal derivative of $\mathcal{U}\mu$, to be denoted by $N^G\mathcal{U}\mu$, which is defined as a linear functional over $\mathcal{C}_0^{(1)}$ by the formula

$$\big\langle N^G\mathcal{U}\mu, \varphi\big\rangle = \int_G \langle \mathrm{grad}\ \varphi(x), \mathrm{grad}\ \mathcal{U}\mu(x)\rangle\, d\lambda_2(x), \quad \varphi \in \mathcal{C}_0^{(1)}\,; \tag{2}$$

if the boundary $\partial G = \partial K$ is smooth and n denotes the unit normal exterior to G, and if $\mathcal{U}\mu$ extends smoothly from G to $\mathrm{cl}\, G$, then the right-hand side in (2) transforms by divergence theorem into

$$\int_{\partial K} \varphi \frac{\partial \mathcal{U}\mu}{\partial n}\, d\lambda_1$$

so that $N^G\mathcal{U}\mu$ is a natural weak characterization of the normal derivative $\frac{\partial \mathcal{U}\mu}{\partial n}$ (compare [**21**]). Transforming the integral occurring in (2) by Fubini's theorem we get, for any $\varphi \in \mathcal{C}_0^{(1)}$,

$$\big\langle N^G\mathcal{U}\mu, \varphi\big\rangle = \int_{\partial K} W\varphi(z)\, d\mu(z)\,,$$

where

$$W\varphi(z) = \int_G \langle \mathrm{grad}\ \varphi(x), \mathrm{grad}\ h_z(x)\rangle\ d\lambda_2(x)\,. \tag{3}$$

We shall consider (3) as a function of the variable $z \in K$. It is easily seen (cf. §2 in [**8**]) that, for $z \in K$, (3) depends on $\varphi\big|_{\partial K}$ only and represents a continuous function on K which is harmonic on $\mathrm{int}\, K$; this function $W\varphi$ will be called the double layer potential of density φ. Note also that, for any fixed $\mu \in \mathcal{C}^*(\partial K)$, the weak normal derivate $N^G\mathcal{U}\mu$ has support contained in ∂K in the sense that $\big\langle N^G\mathcal{U}\mu, \varphi\big\rangle = 0$ whenever ∂K does not meet the support of $\varphi \in \mathcal{C}_0^{(1)}$ (cf. 1.2 in [**8**]).

For $\emptyset \neq M \subset K$ denote by $W_M\varphi = (W\varphi)\big|_M$ the restriction to M of the double layer potential $W\varphi$. Then

$$(4_K) \qquad W_K : \varphi\big|_{\partial K} \to W_K\varphi \qquad (\mathcal{C}^{(1)}(\partial K) \to \mathcal{C}(K))$$

and

$$(4_{\partial K}) \qquad W_{\partial K} : \varphi\big|_{\partial K} \to W_{\partial K}\varphi \qquad (\mathcal{C}^{(1)}(\partial K) \to \mathcal{C}(\partial K))$$

are linear operators form $\mathcal{C}^{(1)}(\partial K)$ to $\mathcal{C}(K)$ and from $\mathcal{C}^{(1)}(\partial K)$ to $\mathcal{C}(\partial K)$, respectively. Since

$$(5) \qquad W_K 1_{\partial K} = 1_K$$

(cf. [**8,** p. 60]), we have W_K (Const $(\partial K) \subset$ Const (K) which makes it possible to consider the operators induced on the factor space $\mathcal{C}^{(1)}(\partial K)/_{\text{Const}\,(\partial K)}$ to be denoted by the same symbols

$$(6_K) \qquad W_K : \mathcal{C}^{(1)}(\partial K)/_{\text{Const}\,(\partial K)} \to \mathcal{C}(K)/_{\text{Const}\,(K)}$$

and

$$(6_{\partial K}) \qquad W_{\partial K} : \mathcal{C}^{(1)}(\partial K)/_{\text{Const}\,(\partial K)} \to \mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)} .$$

Necessary and sufficient geometric condition is known (cf. [**1**], [**9**]; see also the exposition in [**8**], [**14**]) guaranteeing extendability of the operators (4_K), $(4_{\partial K})$ to bounded linear operators defined on the whole $\mathcal{C}(\partial K)$ and of the operators (6_K), $(6_{\partial K})$ to bounded linear operators acting on $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$. As pointed out by M. Chlebík ([**6**]), results in geometric measure theory ([**3**]) permit to formulate this condition (occurring in an equivalent form in [**8**] and [**14**]; cf. proof of Lemma 3 below) in terms of the essential boundary

$$\partial_e K = \{z \in \mathbb{R}^2; \limsup_{r\to 0^+} \lambda_2[B_r(z) \cap K]/_{r^2} > 0,\ \limsup_{r\to 0^+} \lambda_2[B_r(z) \cap G]/_{r^2} > 0\}$$

as follows. Denoting for θ in

$$\Gamma \equiv \{\theta \in \mathbb{R}^2; |\theta| = 1\}$$

and fixed $z \in \mathbb{R}^2$ by $n^K(z,\theta)$ the total number of points in

$$\{z + t\theta;\ t > 0\} \cap \partial_e K$$

$(0 \le n^K(z,\theta) \le +\infty)$ we arrive at a λ_1-measurable function $\theta \mapsto n^K(z,\theta)$ which makes it possible to introduce the integral

$$v^K(z) := \frac{1}{\pi}\int_\Gamma n^K(z,\theta)\, d\lambda_1(\theta)\,.$$

Then finiteness of the quality

$$V^K := \sup\{v^K(z);\ z \in \partial K\} \tag{7}$$

is necessary and sufficient for extendability of W_K to a bounded linear operator on $\mathcal{C}(\partial K)$ to $\mathcal{C}(K)$ (or, equivalently, on $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$ to $\mathcal{C}(K)/_{\text{Const}\,(\partial K)}$) and, which is the same, extendability of $W_{\partial K}$ to a bounded operator on $\mathcal{C}(\partial K)$ (or, equivalently, on $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$). The same condition

$$V^K < +\infty \tag{8}$$

is necessary and sufficient to guarantee the existence, for each $\mu \in \mathcal{C}^*(\partial K)$, of a (uniquely determined) finite signed Borel measure $\nu_\mu \in \mathcal{C}^*(B)$ representing $N^G\mathcal{U}\mu$ in the sense that

$$\langle N^G\mathcal{U}\mu, \varphi\rangle = \int_{\partial K} \varphi\, d\nu_\mu, \quad \forall\, \varphi \in \mathcal{C}_0^{(1)}\,;$$

under the assumption (8) the arising operator $N^G\mathcal{U}\colon \mu \mapsto \nu_\mu$ is bounded on $\mathcal{C}^*(\partial K)$ and is adjoint to $W_{\partial K}$ acting on $\mathcal{C}(\partial K)$:

$$N^G\mathcal{U} = W^*_{\partial K}\,. \tag{9}$$

Assuming (8) we define the operator of the arithmetical mean, to be denoted by $T^K \equiv T$, by the equation

$$\frac{1}{2}(I + T^K) = W_{\partial K}\,, \tag{10}$$

where I is the identity operator. Then (5) implies

$$T1_{\partial K} = 1_{\partial K}\,. \tag{11}$$

The norm of T on $\mathcal{C}(\partial K)$ is precisely evaluated by

$$\|T^K\| = V^K \tag{12}$$

(cf. [**8**], 2.25; note that our normalization of $v^K(z)$ is different from that used in [**8**], so that our V^K coincides with $2V^G$ in [**8**]). The attempt to represent the solution of the Dirichlet problem for K with a prescribed boundary condition $g \in \mathcal{C}(\partial K)$

in the form of the double layer potential $W_K f$ with an unknown $f \in \mathcal{C}(\partial K)$ leads to the equation

$$(I+T)f = 2g\,. \tag{13}$$

In view of (9), the attempt to find, for a given $\nu \in \mathcal{C}^*(\partial K)$, another $\mu \in \mathcal{C}^*(\partial K)$ whose potential $\mathcal{U}\mu$ solves the weak Neumann problem $N^G\mathcal{U}\mu = \nu$ for G results in the adjoint equation

$$(I+T)^*\mu = \nu \tag{14}$$

for the unknown μ. It follows from (12) that $\|T^K\| \geq 1$ where the sign of equality holds iff K is convex (cf. [**8**], Theorem 3.1). If we consider T^K on the quotient space $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$, then the quotient norm of T^K, to be denoted by $\|T^K\|_0$, may become less that 1. Let us recall that the norm of the class containing $f \in \mathcal{C}(\partial K)$ in $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$ is given by $\frac{1}{2}\text{osc}\, f(\partial K)$, where

$$\text{osc}\, f(\partial K) = \max f(\partial K) - \min f(\partial K)\,.$$

Hence $\|T^K\|_0$ is the least constant $q \geq 0$ for which

$$\text{osc}\,(T^K f)(\partial K) \leq q\,\text{osc}\, f(\partial K)\,, \quad \forall\, f \in \mathcal{C}(\partial K)\,.$$

This constant was called the configuration constant of K by Carl Neumann who was able to prove for convex K that $\|T^K\|_0 < 1$ iff K is different from triangles and quadrangles ([**18**]) (H. Lebesgue [**12**] observed later that $\|T_K^2\| < 1$ for all convex bodies $K \subset \mathbb{R}^2$) which permitted to establish convergence (in the operator norm) of the Neumann series for the inverse of $I + T^K$ on $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$. Note that, in view of (9)–(11), $(T^K)^*$ maps the subspace

$$\mathcal{C}_0^*(\partial K) := \{\mu \in \mathcal{C}^*(\partial K) :\ \mu(\partial K) = 0\}$$

of all balanced signed measures in $\mathcal{C}^*(\partial K)$ into itself and $\mathcal{C}_0^*(\partial K)$ may be identified with the adjoint space to $\mathcal{C}(\partial K)/_{\text{Const}\,(\partial K)}$. Hence $\|T^K\|_0$ equals the norm of the operator $(T^K)^*$ restricted to $\mathcal{C}_0^*(\partial K)$. For general K no simple evaluation of $\|T^K\|_0$ comparable with the formula (12) for $\|T^K\|$ seems to be known. Nevertheless, geometric estimates of the configuration constant $\|T^K\|_0$ can be obtained which permit to establish the inequality $\|T^K\|_0 < 1$ for many concrete highly non-convex compact $K \subset \mathbb{R}^2$. We shall prove the following theorems and some of their consequences.

Theorem 1. *Let B_1, B_2 be disjoint λ_1-measurable subsets of ∂K and suppose that with each $z \in B \equiv B_1 \cup B_2$ there is associated a disk $B(z) = B_{r(z)}(\zeta(z))$ of radius $r(z) = |z-\zeta(z)|$ such that $K \cap B(z) = \emptyset$ for $z \in B_1$, $K \subset \operatorname{cl} B(z)$ for $z \in B_2$ and $z \mapsto r(z)$ is λ_1-measurable. If*

$$\lambda_1(\partial K \setminus B) = 0 \quad \text{and} \quad \int_B \frac{d\lambda_1(z)}{r(z)} < +\infty\,,$$

then

$$\|T^K\|_0 \le 1 + \frac{1}{2\pi}\left(\int_{B_1} \frac{d\lambda_1(z)}{r(z)} - \int_{B_2} \frac{d\lambda_1(z)}{r(z)}\right). \tag{15}$$

Theorem 2. *Suppose that with each $z \in B_0 \subset B$ there is associated a disk $B(z) = B_{r(z)}(\zeta(z)) \subset K$ of radius $r(z) = |z-\zeta(z)|$. If $z \mapsto r(z)$ is λ_1-measurable,*

$$\lambda_1(\partial K \setminus B_0) = 0 \quad \text{and} \quad \int_{B_0} \frac{d\lambda_1(z)}{r(z)} < +\infty\,,$$

then

$$\|T^K\|_0 \le \frac{1}{2\pi}\int_{B_0} \frac{d\lambda_1(z)}{r(z)} - 1\,. \tag{16}$$

We shall also show that the sign of equality holds in (15) and (16) if ∂K is a circular polygon of a certain type. The proofs depend on a series of lemmas.

Lemma 1. *Let $B \subset \partial G$, $\lambda_1(\partial G \setminus B) = 0$, $\delta > 0$ and suppose that with each $z \in B$ there is associated an $r(z) > 0$ and $\theta(z) \in \Gamma$ such that*

$$\{z + t\theta;\ 0 < t < r(z),\, \theta \in \Gamma,\, |\theta - \theta(z)| < \delta\} \subset G\,.$$

If $z \mapsto r(z)$ is λ_1-measurable and $\int_B r^{-a}(z)\, d\lambda_1(z) < \infty$ for some $a \in [0,\infty[$, then $\lambda_1(\partial G) < \infty$.

Proof. Fix $R > 0$ large enough to have $K \subset B_R(0)$ and put $\Omega = G \cap B_R(0)$, so that $\partial\Omega = \partial G \cup \{\zeta;\ |\zeta| = R\}$. Assumptions of our lemma guarantee that with each $z \in C \equiv B \cup \{\zeta;\ |\zeta| = R\}$ it is possible to associate a circular sector $\{z + t\theta;\ 0 < t < r(z),\, \theta \in \Gamma,\, |\theta - \theta(z)| < \delta_0\} \subset \Omega$, where $0 < \delta_0 \le \delta$, $z \mapsto r(z)$ is λ_1-measurable on C and $\int_C r^{-a}(z)\, d\lambda_1(z) < \infty$. Put $C_1 = \{z \in C;\ r(z) \ge 1\}$, $C_2 = C \setminus C_1$. Clearly,

$$\lambda_1(C_2) \le \int_{C_2} r^{-a}(z)\, d\lambda_1(z) \le \int_C r^{-a}(z)\, d\lambda_1(z) < \infty\,,$$

so that it is sufficient to verify that $\lambda_1(C_1) < \infty$. Let $\mathscr{S}$ by the system of all circular sectors of the form

$$S(z,\theta_z,\delta_0) \equiv \{z+t\theta;\, 0<t<1, \theta\in\Gamma, |\theta-\theta_z|<\delta_0\}$$

with $z \in C_1$, $\theta_z \in \Gamma$ such that $S(z,\theta_z,\delta_0) \subset \Omega$. Let $S = \cup\mathscr{S}$, which is an open bounded set. If $S_1, \dots, S_k$ are mutually different components of S, then each of them must contain a sector isometric with $S(0,1,\delta_0)$, whence

$$k\lambda_2(S(0,1,\delta_0)) \le \sum_{j=1}^{k} \lambda_2(S_j) \le \lambda_2(S), \quad k \le \lambda_2(S)/\lambda_2(S(0,1,\delta_0))\,.$$

We see that S has only finitely many components $S_1, \dots, S_k$. We shall show that each S_j has the cone property in the following sense: There is an $r>0$ such that with each $z \in \partial S_j$ it is possible to associate a $\theta_z \in \Gamma$ with

$$B_r(z) \cap S(z,\theta_z,r) \subset S_j\,. \tag{17}$$

Let $z \in \partial S_j$, $\mathscr{S}_j = \{D \in \mathscr{S};\, D \subset S_j\}$. There is a sequence $x_n \in S_j$ with $\lim_{n\to\infty} x_n = z$. Since $S_j = \cup\mathscr{S}_j$, for each n there is a $D_n \in \mathscr{S}_j$ with $x_n \in D_n$. Denote by z_n the vertex of D_n and by $\theta^n \equiv \theta_{z_n}$ the corresponding vector in Γ determining $D_n = S(z_n,\theta^n,\delta_0)$. Since $\{z_n\} \subset \partial\Omega$ which is compact, passing to subsequences, if necessary, we may achieve that $\lim_{n\to\infty} z_n = y \in \partial\Omega$ and $\lim_{n\to\infty}\theta^n = \tilde\theta \in \Gamma$ for suitable y and $\tilde\theta$. Writing $\tilde D = S(y,\tilde\theta,\delta_0)$ we observe that

$$\tilde D \subset \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} D_n \subset \bigcap_{k=1}^{\infty} \operatorname{cl} \bigcup_{n=k}^{\infty} D_n \subset \operatorname{cl}\tilde D\,,$$

so that $\tilde D \subset S_j \subset \Omega$, $\tilde D \in \mathscr{S}_j$. As $x_n \in D_n$ tend to z, we have $z \in \operatorname{cl}\tilde D$. Since $z \in \partial S_j$ while $\tilde D \subset S_j$, we see that $z \in \partial\tilde D$. It remains to realize that $\tilde D$ is isometric with $S(0,1,\delta_0)$, so that there is an $r>0$ (depending on δ_0 only) such that with each $\tilde z \in \partial\tilde D$ it is possible to associate a $\theta_{\tilde z} \in \Gamma$ with $S(\tilde z,\theta_{\tilde z},r)\cap B_r(\tilde z) \subset \tilde D$; this is in particular true for $\tilde z = z$, so that the cone property (17) of S_j has been verified. Now we recall the following result established in [**4**]:

If $\mathcal{U}$ is a bounded domain having the cone property, then there are open sets $\mathcal{U}_1,\dots,\mathcal{U}_p$ with $\cup_{i=1}^{p}\mathcal{U}_i = \mathcal{U}$ such that each $\mathcal{U}_i$ has locally lipschitzian boundary (and, in particular, $\lambda_1(\partial\mathcal{U}_i) < \infty$); consequently, $\lambda_1(\partial\mathcal{U}) \le \sum_{i=1}^{p}\lambda_1(\partial\mathcal{U}_i) < \infty$.

Applying this to $\mathcal{U} = S_j$ $(j=1,\dots,k)$ we get $\lambda_1(\partial S) \le \sum_{j=1}^{k}\lambda_1(\partial S_j) < \infty$. Since $C_1 \subset \partial S$, $\lambda_1(C_1) < \infty$ has been verified and the proof is complete. □

Lemma 2. *Denote by $\hat{\partial} K$ the set of all $y \in \mathbb{R}^2$, for which there exists $n^K(y) \in \Gamma$ (which is called the Federer exterior normal of K at y and is uniquely determined) such that*

$$\lim_{r \to 0^+} r^{-2} \lambda_2[B_r(y) \cap \{x \in K; \langle x-y, n^K(y) \rangle > 0\}] = \lim_{r \to 0^+} r^{-2} \lambda_2[B_r(y) \cap \{x \in G; \langle x-y, n^K(y) \rangle < 0\}] = 0 .$$

If $y \in \hat{\partial} K$, $z \in \partial K \setminus \{y\}$, $\zeta(y) \in \mathbb{R}^2$ and $|y - \zeta(y)| = r(y) > 0$, then the following implications hold:

(18)
$$B_{r(y)}(\zeta(y)) \subset K \implies -\langle \text{grad } h_z(y), n^K(y) \rangle = \frac{1}{4\pi r(y)} + \frac{r^2(y) - |z - \zeta(y)|^2}{4\pi r(y) |y-z|^2} \le \frac{1}{4\pi r(y)} ,$$

(19)
$$K \subset \text{cl}\, B_{r(y)}(\zeta(y)) \implies -\langle \text{grad } h_z(y), n^K(y) \rangle = \frac{1}{4\pi r(y)} + \frac{r^2(y) - |z - \zeta(y)|^2}{4\pi r(y) |y-z|^2} \ge \frac{1}{4\pi r(y)} ,$$

(20)
$$K \cap B_{r(y)}(\zeta(y)) = \emptyset \implies -\langle \text{grad } h_z(y), n^K(y) \rangle = -\frac{1}{4\pi r(y)} - \frac{r^2(y) - |z - \zeta(y)|^2}{4\pi r(y) |y-z|^2} \ge -\frac{1}{4\pi r(y)} .$$

Proof. If $y \in \hat{\partial} K$ and the assumptions from (18) or (19) are valid, then

$$n^K(y) = \frac{y - \zeta(y)}{r(y)} ,$$

while

$$\frac{y - \zeta(y)}{r(y)} = -n^K(y)$$

under the assumption occurring in (20). Since calculation yields

$$-\Big\langle \text{grad } h_z(y), \frac{y-\zeta(y)}{r(y)} \Big\rangle = \frac{1}{2\pi} \left\langle \frac{y-z}{|y-z|^2}, \frac{y-\zeta(y)}{r(y)} \right\rangle$$
$$= \frac{1}{2\pi r(y)} \cdot \frac{|y-\zeta(y)|^2 - \langle z - \zeta(y), y - \zeta(y) \rangle}{|y-z|^2}$$
$$= \frac{1}{2\pi r(y)} \cdot \frac{|y-\zeta(y)|^2 - 2\langle z - \zeta(y), y - \zeta(y) \rangle + |z-\zeta(y)|^2}{2|y-z|^2} + \frac{r^2(y) - |z-\zeta(y)|^2}{4\pi r(y)|y-z|^2}$$
$$= \frac{1}{4\pi r(y)} + \frac{r^2(y) - |z-\zeta(y)|^2}{4\pi r(y)|y-z|^2} .$$

It remains to note that $r^2(y)-|z-\zeta(y)|^2 \le 0$ under the assumptions occurring in (18), (20), while $r^2(y)-|z-\zeta(y)|^2 \ge 0$ under the assumption occurring in (19).□

Lemma 3. *If the assumptions of Theorem* 1 *are fulfilled, then*

$$V^K = \|T^K\| < \infty\,.$$

Proof. Lemma 1 shows that $\lambda_1(\partial K) < \infty$, so that K has finite perimeter $P(K)$ in the sense of 2.10 in [**8**] (see 4.5 in [**3**]). For $y \in \hat{\partial} K$ the vector $n^K(y) \in \Gamma$ has been defined in Lemma 2; we shall further put $n^K(y) = 0$ $(\in \mathbb{R}^2)$ for $y \in \mathbb{R}^2 \setminus \hat{\partial} K$. Then the vector-valued function $y \mapsto n^K(y)$ is defined on $\mathbb{R}^2$ and is Borel measurable (cf. Remark 2.14 in [**8**]), so that we may introduce

$$2\int_{\partial K} |\langle n^K(y), \operatorname{grad}\, h_z(y)\rangle|\, d\lambda_1(y) \equiv v^K(z)$$

(which agrees with the quantity occurring in (28) in [**8**] up to the multiplicative factor 2). Then a necessary and sufficient condition for extendability of $W_{\partial K}$ (defined so far on $\mathcal{C}^{(1)}(\partial K)$ only) to a bounded linear operator on $\mathcal{C}(\partial K)$ consists in finiteness of the quantity

$$V^K \equiv \sup\{v^K(z);\ z \in \partial K\}$$

which is then equal to the norm of the operator T^K defined by (10) (cf. §2 in [**8**], in particular 2.19–2.25; notice that our V^K coincides with $2V^G$ occurring in [**8**]). We should remark that the quantity $v^K(z)$ can be equivalently defined by various expressions, one of them being

$$v^K(z) = \frac{1}{\pi}\int_\Gamma n^K_\infty(\theta, z)\, d\lambda_1(\theta)\,,$$

where $n^K_\infty(\theta, z)$ is the number of so-called hits of the half-line

$$H_z(\theta) = \{z + t\theta;\ t > 0\}$$

on K in the sense of 1.7 in [**8**] (note that, according to 1.11 in [**8**], $\theta \mapsto n^K_\infty(\theta, z)$ is a Baire function of the variable $\theta \in \Gamma$). As pointed out by M. Chlebík [**6**], methods of geometric measure theory [**3**] permit to show that $n^K_\infty(\theta, z)$ coincides with the total number of points in $H_z(\theta) \cap \partial_e K$ for λ_1-a.e. $\theta \in \Gamma$, so that $v^K(z)$ has the same meaning as described in the introduction. Fix now an arbitrary $z \in \partial K$ and consider $\delta > 0$ such that

$$\lambda_1(\partial B_\delta(z) \cap \partial K) = 0 \tag{21}$$

(as $\lambda_1(\partial K) < \infty$, all but countable many values $\delta > 0$ enjoy this property). Under the conditions of Theorem 1, for λ_1-a.e. $y \in \hat{\partial} K$ either the assumption in (19) or that occurring in (20) is fulfilled; accordingly,

$$-\langle \text{grad } h_z(y), n^K(y) \rangle \geq -\frac{1}{4\pi r(y)}, \quad \lambda_1\text{-a.e. } y \in \hat{\partial} K\,. \tag{22}$$

Put $Q = K - B_\delta(z)$. Employing (21) we see that λ_1-a.e. $y \in \hat{\partial} Q \cap \partial B_\delta(z)$ belongs to $\hat{\partial} Q \cap \operatorname{int} K \subset \partial B_\delta(z) \cap \operatorname{int} K$, so that $n^Q(y) = \frac{z-y}{\delta}$ and

$$\langle \text{grad } h_z(y), n^Q(y) \rangle = \frac{1}{2\pi\delta}, \quad \lambda_1\text{-a.e. } y \in \hat{\partial} Q \cap \partial B_\delta(z)\,. \tag{23}$$

Noting that $n^Q(\cdot) = n^K(\cdot)$ on $\hat{\partial} Q \setminus \partial B_\delta(z) \subset \hat{\partial} K$ we get by (22), (23)

$$\begin{aligned}
\frac{1}{2} v^Q(z) &= \int_{\hat{\partial} Q} |\langle \text{grad } h_z(y), n^Q(y) \rangle| \, d\lambda_1(y) \\
&\leq \int_{\hat{\partial} Q \cap \partial B_\delta(z)} \left[\frac{1}{\pi\delta} - \langle \text{grad } h_z(y), n^Q(y) \rangle \right] d\lambda_1(y) \\
&\quad + \int_{\hat{\partial} Q \setminus \partial B_\delta(z)} \left[\frac{1}{4\pi r(y)} - \langle \text{grad } h_z(y), n^Q(y) \rangle \right] d\lambda_1(y) \\
&\quad + \int_{\hat{\partial} Q \setminus \partial B_\delta(z)} \frac{1}{4\pi r(y)} \, d\lambda_1(y) \\
&\leq - \int_{\hat{\partial} Q} \langle \text{grad } h_z(y), n^Q(y) \rangle \, d\lambda_1(y) + \frac{1}{\pi\delta} \cdot 2\pi\delta + 2 \int_{\partial K} \frac{1}{4\pi r(y)} \, d\lambda_1(y) \\
&= 2 + \frac{1}{2\pi} \int_{\partial K} \frac{1}{r(y)} \, d\lambda_1(y)\,,
\end{aligned}$$

where we have used the fact that $y \mapsto h_z(y)$ is harmonic in some neighbourhood of $\operatorname{cl} Q$, whence it follows by the divergence theorem for sets with finite perimeter (cf. p. 49 in [**8**]) that

$$\int_{\hat{\partial} Q} \langle \text{grad } h_z(y), n^Q(y) \rangle \, d\lambda_1(y) \rangle = 0\,.$$

Since $\partial K \setminus B_\delta(z) \subset \partial Q$ and $n^K(\cdot) = n^Q(\cdot)$ holds λ_1-a.e. on $\partial K \setminus B_\delta(z)$ by (21), we arrive at

$$\int_{\partial K \setminus B_{\delta(z)}} | \big\langle \text{grad } h_z(y), n^K(y) \big\rangle | \, d\lambda_1(y) \leq \frac{1}{2} v^Q(z) \leq 2 + \frac{1}{2\pi} \int_{\partial K} \frac{1}{r(y)} \, d\lambda_1(y)\,,$$

whence we get making $\delta \to 0^+$ (with δ obeying (21))

$$v^K(z) = 2 \int_{\partial K} |\langle \text{grad } h_z(y), n^K(y) \rangle| \, d\lambda_1(y) \leq 4 + \frac{1}{\pi} \int_{\partial K} \frac{1}{r(y)} \, d\lambda_1(y)\,.$$

Since $z \in \partial K$ has been arbitrarily chosen, we have

$$V^K \leq 4 + \pi^{-1} \int_{\partial K} r^{-1}(y) \, d\lambda_1(y) < \infty$$

and the proof is complete. $\square$

Lemma 4. *If the assumptions of Theorem 2 are fulfilled, then*

$$V^K = \|T^K\| < \infty .$$

Proof. Choose $R > 0$ large enough to have $K \subset B_R(0)$ and put $L = \operatorname{cl}[B_R(0) \setminus K]$. If K satisfies the assumptions of Theorem 2, then L satisfies the assumptions of Theorem 1 (where K is replaced by L) and Lemma 3 implies $V^L < \infty$. It remains to observe that $V^K \le V^L$. □

Lemma 5. *Let $V^K < \infty$. Then the density*

$$d_K(z) = \lim_{r \to 0+} \frac{\lambda_2[K \cap B_r(z)]}{\lambda_2[B_r(z)]}$$

is well defined for any $z \in \mathbb{R}^2$. Denoting by δ_z the Dirac unit point-mass concentrated at z define for any $z \in \partial K$ the signed Borel measure τ_z on ∂K by

$$d\tau_z(y) = [1 - 2d_K(z)]d\delta_z(y) - 2\langle n^K(y), \operatorname{grad} h_z(y)\rangle d\lambda_1(y) . \tag{24}$$

Then

$$T^K f(z) = \int_{\partial K} f\, d\tau_z, \quad z \in \partial K, \quad f \in \mathcal{C}(\partial K) . \tag{25}$$

Proof. See §3 in [**8**] (p. 73). □

Lemma 6. *Let $V^K < \infty$ and let D be a dense subset of ∂K. Let us agree to denote by $\|\nu\|$ the total variation of an arbitrary signed Borel measure ν on ∂K. Then*

$$\|T^K\|_0 = \frac{1}{2} \sup\{\|\tau_u - \tau_v\|;\ u, v \in D\} \tag{26}$$

and for each signed Borel measure μ on ∂K the following estimate holds

$$\|T^K\|_0 \le \sup\{\|\tau_z - \mu\|;\ z \in D\} . \tag{27}$$

Proof. If $f \in \mathcal{C}(\partial K)$, then we denote by $\|f\|_0 = \frac{1}{2}\operatorname{osc} f(\partial K)$ the norm in $\mathcal{C}(\partial K)/\text{Const}\,(\partial K)$ of the class containing f. Hence

$$\begin{aligned}
\|T^K\|_0 &= \sup\left\{\frac{1}{2}\operatorname{osc} T^K f(\partial K);\ f \in \mathcal{C}(\partial K),\ \|f\|_0 \le 1\right\} \\
&= \frac{1}{2}\sup\left\{\left|\int_{\partial K} f\, d\tau_u - \int_{\partial K} f\, d\tau_v\right|;\ u, v \in D,\ f \in D,\ f \in \mathcal{C}(\partial K),\ \|f\|_0 \le 1\right\} \\
&= \sup\left\{\left|\int_{\partial K} f\, d(\tau_u - \tau_v)\right|;\ u, v \in D,\ f \in \mathcal{C}(\partial K),\ \|f\|_0 \le \frac{1}{2}\right\} .
\end{aligned}$$

In view of (11) we have $\int_{\partial K} d(\tau_u - \tau_v) = 0$, so that the last expression transforms into

$$\|T^K\|_0 = \sup\left\{\left|\int_{\partial K} f\, d(\tau_u - \tau_v)\right|;\ u, v \in D,\ f \in \mathcal{C}(\partial K),\ \|f\| \le \frac{1}{2}\right\}$$
$$= \frac{1}{2}\sup\{\|\tau_u - \tau_v\|;\ u, v \in D\}$$

which is (26). Given $f \in \mathcal{C}(\partial K)$ we have for any $\gamma \in \mathbb{R}$:

$$\|T^K f\|_0 \le \|T^K f - \gamma 1_{\partial K}\| = \sup\left\{\left|\int_{\partial K} f\, d\tau_z - \gamma\right|;\ z \in D\right\}.$$

Choosing $\gamma = \int_{\partial K} f\, d\mu$ we arrive at

$$\|T^K f\|_0 \le \sup\left\{\left|\int_{\partial K} f\, d(\tau_z - \mu)\right|;\ z \in D\right\} \le \|f\| \sup\{\|\tau_z - \mu\|;\ z \in D\}.$$

In this inequality we replace f by $f - \alpha 1_{\partial K}$ for any $\alpha \in \mathbb{R}$. Since

$$\|T^K f\|_0 = \|T^K f - \alpha 1_{\partial K}\|_0$$

we get

$$\|T^K f\|_0 \le \|f - \alpha 1_{\partial K}\| \cdot \sup\{\|\tau_z - \mu\|;\ z \in D\}, \quad \alpha \in \mathbb{R},$$

so that

$$\|T^K f\|_0 \le \|f\|_0 \cdot \sup\{\|\tau_z - \mu\|;\ z \in D\}, \quad f \in \mathcal{C}(\partial K),$$

and (27) follows. □

We are in position to present proofs of Theorems 1, 2 stated above.

Proof of Theorem 1. We know from Lemma 3 that $V^K < \infty$. Define a signed Borel measure μ on ∂K putting for each Borel set $M \subset \partial K$

$$\mu(M) = \frac{1}{2\pi}\left(\int_{M\cap B_2} \frac{d\lambda_1(y)}{r(y)} - \int_{M\cap B_1} \frac{d\lambda_1}{r(y)}\right).$$

Fix $z \in \hat{\partial} K$, so that $d_K(z) = \frac{1}{2}$. Using (24), (19), (20) we get

$$\|\tau_z - \mu\| = \int_{B_1}\left[-2\left\langle \operatorname{grad} h_z(y), n^K(y)\right\rangle + \frac{1}{2\pi r(y)}\right] d\lambda_1(y)$$
$$+ \int_{B_2}\left[-2\left\langle \operatorname{grad} h_z(y), n^K(y)\right\rangle - \frac{1}{2\pi r(y)}\right] d\lambda_1(y)$$
$$= \int_{\partial K} d\tau_z(y) + \frac{1}{2\pi}\left(\int_{B_1} \frac{d\lambda_1(y)}{r(y)} - \int_{B_2} \frac{d\lambda_1(y)}{r(y)}\right)$$
$$= T^K 1_{\partial K}(z) + \frac{1}{2\pi}\left(\int_{B_1} \frac{d\lambda_1(y)}{r(y)} - \int_{B_2} \frac{d\lambda_1(y)}{r(y)}\right),$$

which in combination with (11), (27) completes the proof, because $\hat{\partial} K$ is dense in ∂K thanks to our assumption that K is massive at each point of ∂K (cf. [**8**], p. 54 and isoperimetric lemma on p. 50). □

Proof of Theorem 2. Lemma 4 shows that $V^K < \infty$. Fix again an arbitrary $z \in \hat{\partial} K$ and define now the signed measure μ on Borel sets $M \subset \partial K$ by

$$\mu(M) = \frac{1}{2\pi} \int_{M \cap B_0} \frac{d\lambda_1(y)}{r(y)} .$$

It follows from (24), (18) that

$$\begin{aligned} \|\mu - \tau_z\| &= \int_{B_0} \left[\frac{1}{2\pi r(y)} + 2 \left\langle \text{grad } h_z(y), n^K(y) \right\rangle \right] d\lambda_1(y) \\ &= \frac{1}{2\pi} \int_{B_0} \frac{d\lambda_1(y)}{r(y)} - T^K 1_{\partial K}(z) \end{aligned}$$

which together with (11), (27) proves (16), because $\hat{\partial} K$ is dense in ∂K as observed above. □

Notation. We now specialize to the case that K is bounded by a simple oriented circular polygon

$$\partial K = \bigcup_{m=1}^{n} C_m \cup \{z_m\} ,$$

where C_m is an open oriented circular arc situated on the boundary of a disk $B_{r_m}(\zeta_m)$ and z_m is the initial point of C_m; for $m < n$ the end-point of C_m coincides with z_{m+1}, the end-point of C_n is z_1. Further suppose that for $1 \le k < m \le n$ either $C_k \cap \partial B_{r_m}(\zeta_m) = \emptyset$ or else $C_k \subset \partial B_{r_m}(\zeta_m) \setminus C_m$. We put

$$\begin{aligned} \alpha_m &= \lambda_1(C_m)/r_m , & \mathcal{A}_0 &= \{m;\, B_{r_m}(\zeta_m) \subset K\} , \\ \mathcal{A}_1 &= \{m;\, B_{r_m}(\zeta_m) \cap K = \emptyset\} , & \mathcal{A}_2 &= \{m;\, K \subset \text{cl}\, B_{r_m}(\zeta_m)\} \end{aligned}$$

and adopt the following assumption:

$$\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 = \{1, \ldots, n\} .$$

Then we may state the following result.

Theorem 3. *Let i run over $\mathcal{A}_0$, j run over $\mathcal{A}_1$ and k run over $\mathcal{A}_2$. If $\mathcal{A}_0 = \emptyset$, then*

$$\|T^K\|_0 \le 1 + \frac{1}{2\pi} \left(\sum_j \alpha_j - \sum_k \alpha_k \right) , \tag{28}$$

where the sign of equality holds in case $n \leq 4$. If $\mathcal{A}_1 = \emptyset = \mathcal{A}_2$, then

$$\|T^K\| \leq \frac{1}{2\pi} \sum_{i=1}^{n} \alpha_i - 1\,, \tag{29}$$

where again the sign of equality holds provided $n \leq 4$; now the condition

$$\operatorname{int} K \setminus \bigcup_{i=1}^{n} B_{r_i}(\zeta_i) \equiv \bigcap_{i=1}^{n} [\operatorname{int} K \setminus B_{r_i}(\zeta_i)] \neq \emptyset \tag{30}$$

implies that

$$\frac{1}{2\pi} \sum_{i=1}^{n} \alpha_i - 1 \geq 1 \tag{31}$$

(*so that in case $n \leq 4$ the operator T^K cannot be contractive on $\mathcal{C}(\partial K)/_{\mathrm{Const}\,(\partial K)}$ in view of the equality in* (29)), *while the conditions*

$$\bigcap_{i=1}^{n} [\operatorname{int} K \setminus B_{r_i}(\zeta_i)] = \emptyset, \qquad \bigcap_{i=1}^{n} B_{r_i}(\zeta_i) \neq \emptyset \tag{32}$$

together imply the inequality

$$\frac{1}{2\pi} \sum_{i=1}^{n} \alpha_1 - 1 < 1 \tag{33}$$

(*guaranteeing contractivity of T^K on $\mathcal{C}(\partial K)/_{\mathrm{Const}\,(\partial K)}$*).

Corollary 1. *If $\mathcal{A}_0 = \emptyset = \mathcal{A}_1$, then* (28) *implies the inequality*

$$\|T^K\|_0 \leq 1 - \frac{1}{2\pi} \sum_{k=1}^{n} \alpha_k$$

guaranteeing contractivity of T^K on $\mathcal{C}(\partial K)/_{\mathrm{Const}\,(\partial K)}$. If $\mathcal{A}_0 = \emptyset = \mathcal{A}_2$ and $n \leq 4$ then the equality

$$\|T^K\|_0 = 1 + \frac{1}{2\pi} \sum_{k=1}^{n} \alpha_k$$

holds, so that T^K cannot be contractive on $\mathcal{C}(\partial K)/_{\mathrm{Const}\,(\partial K)}$.

The proof will depend on the following lemma.

Lemma 7. *Put for any* $m \in \{1, \dots, n\}$

$$\sigma_m = \begin{cases} 1, & \text{in case } K \cap B_{r_m}(\zeta_m) \neq \emptyset\,, \\ -1, & \text{in case } K \cap B_{r_m}(\zeta_m) = \emptyset\,. \end{cases}$$

If $z \in C_m$, *then*

$$-2 \int_{\partial K \setminus C_m} \langle \text{grad } h_z(y), n^K(y) \rangle \, d\lambda_1(y) = 1 - \frac{1}{2\pi} \sigma_m \alpha_m, \quad m \in \{1, \dots, n\}\,; \tag{34}$$

further we have

$$-2 \int_{\partial K \setminus C_1 \setminus C_n} \langle \text{grad } h_{z_1}(y), n^K(y) \rangle \, d\lambda_1(y) = 2d_K(z_1) - \frac{1}{2\pi} \sigma_1 \alpha_1 - \frac{1}{2\pi} \sigma_n \alpha_n\,, \tag{35}$$

$$-2 \int_{\partial K \setminus C_{m-1} \setminus C_m} \langle \text{grad } h_{z_m}(y), n^K(y) \rangle \, d\lambda_1(y) = 2d_K(z_m) - \frac{1}{2\pi} \sigma_{m-1} \alpha_{m-1} - \frac{1}{2\pi} \sigma_m \alpha_m \quad \text{for } 1 < m \leq n\,. \tag{36}$$

Proof. If $z \in C_m$, then (11), (25), (24) yield

$$-2 \int_{\partial K} \left\langle n^K(y), \text{grad } h_z(y) \right\rangle \, d\lambda_1(y) = \int_{\partial K} d\tau_z(y) + [2d_K(z) - 1] = 2d_K(z)\,. \tag{37}$$

From Lemma 2 we get for $y, z \in C_m$, $y \neq z$

$$- \left\langle \text{grad } h_z(y), n^K(y) \right\rangle = \frac{\sigma_m}{4\pi r_m}\,,$$

whence

$$-2 \int_{C_m} \left\langle n^K(y), \text{grad } h_z(y) \right\rangle \, d\lambda_1(y) = \frac{1}{2\pi} \sigma_m \alpha_m\,, \tag{38}$$

which together with (37) implies (34).

If $y \in C_1$, then Lemma 2 combined with $|z_1 - \zeta_1| = r_1$ yields again

$$- \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle = \frac{\sigma_1}{4\pi r_1}\,,$$

whence

$$-2\int_{C_1}\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\, d\lambda_1(y) = \frac{1}{2\pi}\sigma_1\alpha_1\,. \tag{39}$$

Similarly we get from Lemma 2 for $y \in C_n$ in view of $|z_1 - \zeta_n| = r_n$

$$-\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle = \frac{\sigma_n}{4\pi r_n}\,,$$

so that

$$-2\int_{C_n} -\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\, d\lambda_1(y) = \frac{1}{2\pi}\sigma_n\alpha_n\,. \tag{40}$$

Combining (37), (39), (40) we get (35). Similar reasoning proves (36). □

Proof of Theorem 3. Assuming $\mathcal{A}_0 = \emptyset$ put $B_1 = \cup C_j$ $(j \in \mathcal{A}_1)$, $B_2 = \cup C_k$ $(k \in \mathcal{A}_2)$, $B = B_1 \cup B_2$, $B(z) = B_{r_m}(\zeta_m)$ for $z \in C_m$ $(1 \le m \le n)$. Then $\partial K \setminus B = \{z_1, \dots, z_n\}$ and Theorem 1 implies

$$\|T^K\|_0 \le 1 + \frac{1}{2\pi}\left(\sum_j \lambda(C_j)/_{r_j} - \sum_k \lambda(C_k)/_{r_k}\right)$$

which is (28). Now we shall verify that the sign of equality holds in (28) provided $1 \le n \le 4$. This is clear when $n = 1$, because then $\mathcal{A}_3 = \emptyset$, $\alpha_1 = 2\pi$ and $0 \le \|T^K\|_0 \le 1 - \frac{1}{2\pi}\alpha_1 = 0$. Let now $n = 2$ and fix $u \in C_1$, $v \in C_2$. According to Lemma 2 we have for $y \in C_1$

$$-\left\langle \text{grad } h_u(y), n^K(y)\right\rangle = \frac{\sigma_1}{4\pi r_1}\,, \quad -\left\langle \text{grad } h_v(y), n^K(y)\right\rangle - \frac{\sigma_1}{4\pi r_1} \ge 0\,,$$

while for $y \in C_2$

$$-\left\langle \text{grad } h_v(y), n^K(y)\right\rangle = \frac{\sigma_2}{4\pi r_2}\,, \quad -\left\langle \text{grad } h_u(y), n^K(y)\right\rangle - \frac{\sigma_2}{4\pi r_2} \ge 0\,.$$

Hence we get by (24)

$$\begin{aligned}
\|\tau_u - \tau_v\| = &-\int_{C_1}\left[\frac{\sigma_1}{2\pi r_1} + 2\left\langle \text{grad } h_v(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\
&-\int_{C_2}\left[\frac{\sigma_2}{2\pi r_2} + 2\left\langle \text{grad } h_u(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\
= &-\frac{\sigma_1}{2\pi}\frac{\lambda_1(C_1)}{r_1} - 2\int_{\partial K\setminus C_2}\left\langle \text{grad } h_v(y), n^K(y)\right\rangle\, d\lambda_1(y) \\
&-2\int_{\partial K\setminus C_1}\left\langle \text{grad } h_u(y), n^K(y)\right\rangle\, d\lambda_1(y) - \frac{\sigma_2}{2\pi}\frac{\lambda_1(C_2)}{r_2}\,.
\end{aligned}$$

Using (34) we arrive at

$$\begin{aligned}\|\tau_u - \tau_v\| &= -\frac{\sigma_1}{2\pi}\alpha_1 + \big(1 - \frac{\sigma_2}{2\pi}\alpha_2\big) + \big(1 - \frac{\sigma_1}{2\pi}\alpha_1\big) - \frac{\sigma_2}{2\pi}\alpha_2 \\ &= 2\big(1 - \frac{1}{2\pi}\sigma_1\alpha_1 - \frac{1}{2\pi}\sigma_2\alpha_2\big)\,.\end{aligned}$$

Hence we get by (26)

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_u - \tau_v\| = 1 - \frac{1}{2\pi}(\sigma_1\alpha_1 + \sigma_2\alpha_2)$$

which is the inequality opposite to (28) for $n = 2$.

Next we shall consider the case $n = 3$. Observing that

$$\big\langle \text{grad } h_{z_1}(y), n^K(y)\big\rangle = \big\langle \text{grad } h_{z_3}(y), n^K(y)\big\rangle \quad \text{for } y \in C_3$$

by Lemma 2, we get from (24) and this lemma

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| &= |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\ &\qquad - \int_{C_1}\left[\frac{\sigma_1}{2\pi r_1} + 2\left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\ &\qquad - \int_{C_2}\left[\frac{\sigma_2}{2\pi r_2} + 2\left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\right] d\lambda_1(y) \\ &\geq [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] - \frac{1}{2\pi}\sigma_1\alpha_1 \\ &\qquad - 2\int_{\partial K\setminus C_2\setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y)\right\rangle\, d\lambda_1(y) - \frac{1}{2\pi}\sigma_2\alpha_2 \\ &\qquad - 2\int_{\partial K\setminus C_1\setminus C_3} \left\langle \text{grad } h_{z_1}(y), n^K(y)\right\rangle\, d\lambda_1(y)\,.\end{aligned}$$

Employing (36) and (35) we obtain

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| &\geq [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] - \frac{1}{2\pi}\sigma_1\alpha_1 + \Big[2d_K(z_3) - \frac{1}{2\pi}\sigma_2\alpha_2 \\ &\qquad - \frac{1}{2\pi}\sigma_3\alpha_3\Big] - \frac{1}{2\pi}\sigma_2\alpha_2 + \Big[2d_K(z_1) - \frac{1}{2\pi}\sigma_1\alpha_1 - \frac{1}{2\pi}\sigma_3\alpha_3\Big] \\ &= 2\left(1 - \sum_{m=1}^{3}\frac{1}{2\pi}\sigma_m\alpha_m\right)\,,\end{aligned}$$

whence it follows by (26) that

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_{z_1} - \tau_{z_3}\| \geq 1 - \frac{1}{2\pi}\sum_{m=1}^{3}\sigma_m\alpha_m$$

which gives the inequality opposite to (28) for $n = 3$.

Finally we shall treat the case $n = 4$. We obtain from (24) and Lemma 2

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| = |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\ - \sum_{m=2}^{3} \int_{C_m} \left[2 \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle + \frac{\sigma_m}{2\pi r_m} \right] d\lambda_1(y) \\ - \sum_{m\in\{1,4\}} \int_{C_m} \left[2 \left\langle \text{grad } h_{z_3}(y), n^K(y) \right\rangle + \frac{\sigma_m}{2\pi r_m} \right] d\lambda_1(y) \\ \geq [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] - \sum_{m=1}^{4} \frac{1}{2\pi} \sigma_m \alpha_m \\ - 2 \int_{\partial K \setminus C_1 \setminus C_4} \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle \, d\lambda_1(y) \\ - 2 \int_{\partial K \setminus C_2 \setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y) \right\rangle \, d\lambda_1(y) \,.\end{aligned}$$

Applying (35), (36) we finally get

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| \geq [1 - 2d_K(z_1)] + [1 - 2d_K(z_3)] \\ - \frac{1}{2\pi} \sum_{m=1}^{4} \sigma_m \alpha_m + \left[2d_K(z_1) - \frac{1}{2\pi} \sigma_1 \alpha_1 - \frac{1}{2\pi} \sigma_4 \alpha_4 \right] \\ + \left[2d_K(z_3) - \frac{1}{2\pi} \sigma_2 \alpha_2 - \frac{1}{2\pi} \sigma_3 \alpha_3 \right] = 2 \left(1 - \frac{1}{2\pi} \sum_{m=1}^{4} \sigma_m \alpha_m \right)\end{aligned}$$

which again yields the inequality

$$\|T^K\|_0 \geq 1 - \frac{1}{2\pi} \sum_{m=1}^{4} \sigma_m \alpha_m$$

opposite to (28) for $n = 4$.

The first part of Theorem 3 dealing with the inequality (28) concerning the case $\mathcal{A}_0 = \emptyset$ is completely proved. We now proceed to the case $\mathcal{A}_1 = \emptyset = \mathcal{A}_2$ and put $B_0 = \cup_{i=1}^{n} C_i$. Then $\partial K \setminus B_0 = \{z_1, \ldots, z_n\}$ and letting again $B(z) = B_{r_m}(\zeta_m)$ for $z \in C_m$ $(1 \leq m \leq n)$ we get from Theorem 2

$$\|T^K\|_0 \leq \frac{1}{2\pi} \sum_{i=1}^{n} \lambda(C_i) \big/ r_i - 1 \,,$$

which is the inequality (29). It remains to discuss the case $1 \leq n \leq 4$. If $n = 1$ then $\alpha_1 = 2\pi$ and $\|T^K\|_0 = 0$ as in the first part of the proof. If $n = 2$ we again choose $u \in C_1$, $v \in C_2$ and get by (24) and Lemma 2

$$\begin{aligned}\|\tau_u - \tau_v\| &= \int_{C_1} \left[2 \left\langle \text{grad } h_v(y), n^K(y) \right\rangle + \frac{1}{2\pi r_1}\right] d\lambda_1(y) \\ &\quad + \int_{C_2} \left[2 \left\langle \text{grad } h_u(y), n^K(y) \right\rangle + \frac{1}{2\pi r_2}\right] d\lambda_1(y) \\ &= \frac{1}{2\pi}(\alpha_1 + \alpha_2) + 2 \int_{\partial K \setminus C_2} \left\langle \text{grad } h_v(y), n^K(y) \right\rangle d\lambda_1(y) \\ &\quad + 2 \int_{\partial K \setminus C_1} \left\langle \text{grad } h_u(y), n^K(y) \right\rangle d\lambda_1(y) .\end{aligned}$$

Hence it follows by (34) that

$$\|\tau_u - \tau_v\| = \frac{1}{2\pi}(\alpha_1 + \alpha_2) - 1 + \frac{1}{2\pi}\alpha_1 - 1 + \frac{1}{2\pi}\alpha_2 = \frac{1}{\pi}(\alpha_1 + \alpha_2) - 2$$

which together with (26) implies

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_u - \tau_v\| = \frac{1}{2\pi}(\alpha_1 + \alpha_2) - 1 ,$$

so that equality holds in (29) for $n = 2$. If $n = 3$, then (24) and Lemma 2 imply

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| &= |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\ &\quad + \int_{C_1} \left[2 \left\langle \text{grad } h_{z_3}(y), n^K(y) \right\rangle + \frac{1}{2\pi r_1}\right] d\lambda_1(y) \\ &\quad + \int_{C_2} \left|2 \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle + \frac{1}{2\pi r_2}\right] d\lambda_1(y) \\ &\geq 2d_K(z_1)] + 2d_K(z_3) - 2 + \frac{1}{2\pi}\alpha_1 + \frac{1}{2\pi}\alpha_2 \\ &\quad + 2 \int_{\partial K \setminus C_2 \setminus C_3} \left\langle \text{grad } h_{z_3}(y), n^K(y) \right\rangle d\lambda_1(y) \\ &\quad + 2 \int_{\partial K \setminus C_1 \setminus C_3} \left\langle \text{grad } h_{z_1}(y), n^K(y) \right\rangle d\lambda_1(y) .\end{aligned}$$

Using (36), (35) we get

$$\begin{aligned}\|\tau_{z_1} - \tau_{z_3}\| &\geq 2d_K(z_1) + 2d_K(z_3) - 2 + \frac{1}{2\pi}(\alpha_1 + \alpha_2) \\ &\quad - 2d_K(z_3) + \frac{1}{2\pi}(\alpha_2 + \alpha_3) - 2d_K(z_1) + \frac{1}{2\pi}(\alpha_1 + \alpha_3) \\ &= \frac{1}{\pi}(\alpha_1 + \alpha_2 + \alpha_3) ,\end{aligned}$$

whence

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_{z_1} - \tau_{z_3}\| \geq \frac{1}{2\pi}\sum_{i=1}^{3}\alpha_i - 1$$

by (26), which shows that equality holds in (29) for $n = 3$. Finally, if $n = 4$ we obtain similarly from (24) and Lemma 2

$$\begin{aligned}
\|\tau_{z_1} - \tau_{z_3}\| = {} & |1 - 2d_K(z_1)| + |1 - 2d_K(z_3)| \\
& + \sum_{i=2}^{3}\int_{C_i}\left[2\left\langle \operatorname{grad} h_{z_1}(y), n^K(y)\right\rangle + \frac{1}{2\pi r_i}\right] d\lambda_1(y) \\
& + \sum_{i\in\{1,4\}}\int_{C_i}\left[2\left\langle \operatorname{grad} h_{z_3}(y), n^K(y)\right\rangle + \frac{1}{2\pi r_i}\right] d\lambda_1(y) \\
\geq {} & 2d_K(z_1) - 1 + 2d_K(z_3) - 1 + \sum_{i=1}^{4}\frac{1}{2\pi}\alpha_i \\
& + 2\int_{\partial K\setminus C_1\setminus C_4}\left\langle \operatorname{grad} h_{z_1}(y), n^K(y)\right\rangle d\lambda_1(y) \\
& + 2\int_{\partial K\setminus C_2\setminus C_3}\left\langle \operatorname{grad} h_{z_3}(y), n^K(y)\right\rangle d\lambda_1(y) \\
= {} & \frac{1}{\pi}\sum_{i=1}^{4}\alpha_i - 2 \quad \text{(see (35) and (36))}\,,
\end{aligned}$$

so that by (26) we have again

$$\|T^K\|_0 \geq \frac{1}{2}\|\tau_{z_1} - \tau_{z_3}\| \geq \frac{1}{2\pi}\sum_{i=1}^{4}\alpha_i - 1$$

which yields equality in (29) for $n = 4$.

Now we assume (30) together with $\mathcal{A}_0 = \{1, \ldots, n\}$ and choose $z_0 \in \operatorname{int} K \setminus \cup_{i=1}^{n} B_{r_i}(\zeta_i)$. Denote by $\triangle \arg[y - z_0;\, y \in C_i]$ the increment of the argument of $y - z_0$ as y describes the oriented arc C_i. Assuming, as we may, that the Jordan curve ∂K arising as the union of the oriented arcs $\operatorname{cl} C_1, \ldots, \operatorname{cl} C_n$ is positively oriented we get

$$\begin{aligned}
2\pi &= \sum_{i=1}^{n}\triangle \arg[y - z_0;\, y \in C_i] = \sum_{i=1}^{n}\int_{C_i}\frac{\left\langle n^K(y), y - z_0\right\rangle}{|y - z_0|^2}\, d\lambda_1(y) \\
&= -2\pi\sum_{i=1}^{n}\int_{C_i}\left\langle n^K(y), \operatorname{grad} h_{z_0}(y)\right\rangle d\lambda_1(y)\,.
\end{aligned}$$

We have seen in the proof of (18) in Lemma 2 that for $i \in \{1,\dots,n\}$ and any $z_0 \notin \partial K$

$$\Big(y \in C_i, B_{r_i}(\zeta_i) \subset K\Big) \implies -\big\langle \text{grad } h_{z_0}(y), n^K(y)\big\rangle = \frac{1}{4\pi r_i} + \frac{r_i^2 - |z_0-\zeta_i|^2}{4\pi r_i |y-z_0|^2}\,, \tag{41}$$

whence we get noting that $|z_0 - \zeta_i| \geq r_i$ for $i \in \{1,\dots,n\}$

$$2\pi \leq \frac{1}{2}\sum_{i=1}^n \int_{C_i} \frac{d\lambda_1(y)}{r_i} = \frac{1}{2}\sum_{i=1}^n \alpha_i$$

which proves (31).

Finally suppose that (32) holds together with $\mathcal{A}_0 = \{1,\dots,n\}$ and choose $z_0 \in \cap_{i=1}^n B_{r_i}(\zeta_i) \subset \operatorname{int} K$. Keeping the assumption that ∂K is positively oriented we obtain from (41) in view of $|z_0 - \zeta_i| < r_i$ $(1 \leq i \leq n)$ by the above reasoning

$$\begin{aligned}2\pi &= -2\pi \sum_{i=1}^n \int_{C_i} \big\langle n^K(y), \text{grad } h_{z_0}(y)\big\rangle \; d\lambda_1(y)\\ &> \frac{1}{2}\sum_{i=1}^n \int_{C_i} \frac{d\lambda_1(y)}{r_i} = \frac{1}{2}\sum_{i=1}^n \alpha_i\end{aligned}$$

which is (33). The proof of Theorem 3 is complete. □

Corollary 2. *If $n = 2$ in Theorem 3 then T^K is always contractive on $\mathcal{C}(\partial K)/\text{Const}\,(\partial K)$ if both C_1 and C_2 are convex w.r. to K (i.e. $\sigma_1 = 1 = \sigma_2$); if only C_1 is convex while C_2 is concave (i.e. $\sigma_1 = 1 = -\sigma_2$), then $\|T^K\|_0 < 1$ iff $\alpha_1 > \alpha_2$.*

Remark. If $\mathcal{A}_1 = \emptyset = \mathcal{A}_2$ and $\operatorname{int} K \subset \cup_{i=1}^n B_{r_i}(\zeta_i)$ then, as we have seen in Theorem 3,

$$\bigcap_{i=1}^n B_{r_i}(\zeta_i) \neq \emptyset \tag{42}$$

is sufficient for $\|T^K\|_0 < 1$; to see that (42) is not necessary consider $\alpha \in]0, \pi/2[$ and form the region

$$K = \text{cl}\, B_1(-2\cos\alpha) \cup \text{cl}\, B_1(0) \cup \text{cl}\, B_1(2\cos\alpha)$$

whose boundary consists of four circular arcs

$$\begin{aligned}
&C_1 = \{-2\cos\alpha + \exp i\theta; \alpha < \theta < 2\pi - \alpha\} &&(\text{so that } \alpha_1 = 2\pi - 2\alpha)\,,\\
&C_2 = \{\exp i\theta; -\pi + \alpha < \theta < -\alpha\} &&(\text{so that } \alpha_2 = \pi - 2\alpha)\,,\\
&C_3 = \{+2\cos\alpha + \exp i\theta; -\pi + \alpha < \theta < \pi - \alpha\} &&(\text{so that } \alpha_3 = 2\pi - 2\alpha)\,,\\
&C_4 = \{\exp i\theta; \alpha < \theta < \pi - \alpha\} &&(\text{so that } \alpha_4 = \pi - 2\alpha)\,,
\end{aligned}$$

and their end-points $z_1, \dots, z_4$. Elementary considerations show that (42) holds iff $\alpha > \pi/3$ while the equality occurring in (29) (Theorem 3) for $n = 4$ tells us that $\|T^K\|_0 < 1$ iff $\alpha > \pi/4$.

Comments. The estimate $\|T^K\|_0 < 1$ guarantees convergence of the Neumann series for the inverse of $I \pm T^K$ in the operator norm; it is not indispensable for the convergence of the Neumann series $\sum_{n=0}^{\infty}(-1)^n(T^K)^n g$ (corresponding to an individual $g \in \mathcal{C}(\partial K)$) to the solution f of the equation $(I + T^K)f = g$ in $\mathcal{C}(\partial K)$ (cf. [**20**], [**15**]). Nevertheless, evalation or estimates of $\|T^K\|_0$ are useful in connection with iterative techniques connected with the equations of the type (13), (14) (cf. [**7**], [**19**]). C. Neumann started investigation of the quantity $\|T^K\|_0$ (which he called the configuration constant of K) in order to get a proof for the existence of the solution of the Dirichlet problem for any continuous boundary condition g prescribed on the boundary of a convex region K ([**17**]); Dirichlet's principle used for this purpose previously by Riemann lost credit after Weierstrass' criticism concerning attaining minima in variational problems. C. Neumann's first proof dealing with the inequality $\|T^K\|_0 < 1$ for convex regions $K \subset \mathbb{R}^2$ different from triangles and quadrangles was only sketchy (as he himself admitted cf. [**18**], p. 759) and was followed by a detailed and correct proof in [**18**], §6 (which was known in his time – cf. [**5**]). This contribution was forgotten later and after Lebesgue's criticism [**12**] of Neumann's first proof (which apparently contained the same gap connected with attaining minima as Riemann's reasoning based on the Dirichlet principle) there remained a common belief that Neumann's proof of $\|T^K\|_0 < 1$ for general convex $K \subset \mathbb{R}^2$ different from triangles and quadrangles was insufficient (cf. [**16**], [**2**], chap. 8, p. 572); Neumann's original proof has been included in [**11**], characterization of convex bodies in higher dimensional spaces for which the operator of the arithmetical mean is contractive is presented in [**10**], where also historical comments are included. We refer the reader to [**13**] for the description of the role played by the Neumann operator in the development of the theory of integral equations.

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