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FREE BOUNDED DISTRIBUTIVE LATTICES OVER FINITE ORDERED SETS AND THEIR SKELETONS

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1. Introduction

R. Dedekind described in [**2**] the free bounded distributive lattice generated by three elements. Up to now the number of elements of a free distributive lattice is known only for a generating set up to eight elements. In [**9**], D. Wiedemann gives an algorithm to compute the number of elements of the free distributive lattice generated by eight elements. There are representations of free bounded distributive lattices and their r-skeletons as concept lattices [**14**] and as lattices of specific convex sets [**13**]. Both papers use the notion of the skeleton of a (finite) lattice to analyze free bounded distributive lattices generated by antichains. In this paper we will extend the use of concept lattices, skeletons and specific convex sets to analyse the structure of free bounded distributive lattices generated by finite ordered sets.

For the main results of this paper we use the methods of formal concept analysis. This approach was developed by R. Wille and others (see [**4**], [**10**], and [**3**]). In the second section we give the basic definitions and results of formal concept analysis and introduce the skeleton of a finite lattice. The third section contains representations of free bounded distributive lattices generated by an ordered set and their skeletons as concept lattices. The representation of the second skeleton of a free bounded distributive lattice as concept lattices leads to some problems. We give an example to exhibit these problems. In Section 4 we introduce specific convex sets which allow us to characterize covering elements of the skeleton of a free bounded distributive lattice. Using this characterization, we prove that two covering blocks (which in the distributive case are maximal Boolean intervals) of a free bounded distributive lattice intersect in a Boolean interval half the size of the smaller one. In the fifth section we investigate a connection between the coverings in a free bounded distributive lattice generated by a finite ordered set and the blocks of a free bounded distributive lattice generated by the same ordered set plus one more incomparable element.

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2. Preliminary Definitions and Remarks

We call $FCD(S, \leq)$ the **free completely distributive complete lattice** generated by the ordered set $(S, \leq)$ if every order-preserving map from $(S, \leq)$ into a completely distributive complete lattice L can be extended to an homomorphism from $FCD(S, \leq)$ to L. For a detailed definition see ([**5**, Definition 2, p. 39]).

The results described in the following sections are based on the theory of formal concept analysis. Therefore we recall some definitions and basic facts of formal concept analysis. Proofs are omitted and interested readers are refered to [**4**], [**10**] and [**3**].

A triple $\mathbf{K} := (G, M, I)$ is called a **context** if G and M are sets and I is a binary relation between G and M, i.e., $I \subseteq G \times M$. The elements of G and M are called **objects** and **attributes** and I is called the **incidence relation** between G and M. For $A \subseteq G$ and $B \subseteq M$ the derivations are defined by $A' := \{m \in M \mid gIm$ for all $g \in A\}$ and $B' := \{g \in G \mid gIM$ for all $m \in B\}$. Instead of A' or B' we sometimes write A^I and B^I and for $g \in G$ we abbreviate $\{g\}'$ by g', analogously, for $m \in M$, we abbreviate $\{m\}'$ by m'. A pair (A, B) with $A \subseteq G$ and $B \subseteq M$ is called a **concept** of $\mathbf{K}$ if $A' = B$ and $B' = A$. The sets A and B are called the **extent** and the **intent** of the concept (A, B). The set of all concepts of $\mathbf{K}$ is denoted by $\mathfrak{B}(\mathbf{K})$. It is ordered by $(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C (\Leftrightarrow B \supseteq D)$. The resulting ordered set is denoted by $\underline{\mathfrak{B}}(\mathbf{K}) := (\mathfrak{B}(\mathbf{K}), \leq)$, the set of its extents by $\mathfrak{U}(\mathbf{K})$ and the set of its intents by $\mathfrak{J}(\mathbf{K})$. Note that $\underline{\mathfrak{B}}(\mathbf{K}) \cong (\mathfrak{U}(\mathbf{K}), \subseteq) \cong (\mathfrak{J}(\mathbf{K}), \supseteq)$. The ordered set $\underline{\mathfrak{B}}(\mathbf{K})$ is indeed a complete lattice, called the **concept lattice** of $\mathbf{K}$, whose infima and suprema are described by the following **basic theorem** of formal concept analysis:

Theorem 2.1. *For $(A_t, B_t) \in \mathfrak{B}(\mathbf{K})$, with $t \in T$, we have*

$$\bigwedge_{t\in T} (A_t, B_t) = \Big(\bigcap_{t\in T} A_t, \Big(\bigcup_{t\in T} B_t\Big)''\Big) \quad \textit{and} \quad \bigvee_{t\in T} (A_t, B_t) = \Big(\Big(\bigcup_{t\in T} A_t\Big)'', \bigcap_{t\in T} B_t\Big).$$

For any index set T and $A_t \subseteq G$ $(B_t \subseteq M)$ for each $t \in T$, we have $(\cup_{t\in T} A_t)' = \cap_{t\in T} A_t' ((\cup_{t\in T} B_t)' = \cap_{t\in T} B_t')$. Let $\mathbf{K} := (G, M, I)$ be a context and let $H \subseteq G$ and $N \subseteq M$. Then $\mathbf{L} := (H, N, I \cap (H \times N))$ is called **subcontext** of $\mathbf{K}$.

Lemma 2.2 (Hilfssatz 29 of [**4**]). *The mapping $\iota\colon \underline{\mathfrak{B}}(\mathbf{L}) \to \underline{\mathfrak{B}}(\mathbf{K})$ with $\iota(A, B) := (A'', A')$ is an order embedding.*

In this paper a **diagram** representing a concept lattice is labelled by the elements of G. It is read as follows: Given an element of the lattice, the extent of this element consists of the objects labelled to the joint-irreducible elements in its principal ideal. That means in a diagramm of a concept lattice, the elements are described in terms of $(\mathfrak{U}(\mathbf{K}), \subseteq)$.

A common way to represent a context is via a crosstable where we label rows by objects, columns by attributes and make a cross on the intersection of row g and column m iff gIm.

Given a finite lattice L, one has $L \cong \underline{\mathfrak{B}}(J(L), M(L), \leq)$ where $J(L)$ denotes the set of all join-irreducible elements and $M(L)$ denotes the set of all meet-irreducible elements. For a finite lattice L we define $\mathbf{K}(L) := (J(L), M(L), \leq)$. For each $g \in G$, the concept $\gamma g := (g'', g')$ is called the **object concept** of g and, for each $m \in M$, the concept $\mu m := (m', m'')$ is called the **attribute concept** of m. We note that $\{\gamma g \mid g \in J(L)\}$, the set of all object concepts, is join-dense in $\underline{\mathfrak{B}}(\mathbf{K}(L))$ and dually $\{\mu m \mid m \in M(L)\}$, the set of all attribute concepts, is meet-dense in $\underline{\mathfrak{B}}(\mathbf{K}(L))$. A context is **purified** if, for $g, h \in G$, $g' = h'$ implies $g = h$ and, for $m, n \in M$, $m' = n'$ implies $m = n$. A purified context is called **reduced** if each object concept is completely join-irreducible and each attribute concept is completely meet-irreducible. We note that for a finite lattice L the context $\mathbf{K}(L)$ is reduced and unique up to isomorphism among the set of all reduced contexts $\mathbf{K}$ with $\underline{\mathfrak{B}}(\mathbf{K}) \cong L$ ([**4**, Hilfssatz 14, p. 26)]

Definition 2.3. For each context $\mathbf{K}$, $g \in G$, and $m \in M$, we define:

$$\begin{aligned}
&g \swarrow m :\Leftrightarrow (g, m) \notin I \quad \text{and} \quad (g' \subset h' \Rightarrow m \in h');\\
&g \nearrow m :\Leftrightarrow (g, m) \notin I \quad \text{and} \quad (m' \subset n' \Rightarrow g \in n');\\
&g \swarrow\!\!\!\nearrow m :\Leftrightarrow g \swarrow m \quad \text{and} \quad g \nearrow m\,.
\end{aligned}$$

We call the relationships down-, up- and double-arrows, respectively.

The arrows allow an easy characterization of (reduced) contexts of modular and distributive lattices. The proof of the following proposition can be found in [**4**].

Proposition 2.4. *Let L be a finite lattice.*

1. *If L is modular then all arrows in $\mathbf{K}(L)$ are double arrows.*
2. *L is distributive iff there is exactly one double arrow in each row and each column of $\mathbf{K}(L)$ and there are no other arrows.*

Now we introduce the notion of a skeleton of a finite lattice (see [**14**]). First we recall that a (**complete**) **tolerance relation** Θ of a lattice L is a binary relation on L which is reflexive, symmetric, and compatible with the meet and join operation, i.e., $x_t \Theta y_t$ for $t \in T$ implies $(\wedge_{t\in T} x_t)\Theta(\wedge_{t\in T} y_t)$ and $(\vee_{t\in T})\Theta(\vee_{t\in T} y_t)$. The **blocks** of Θ are the maximal intervals B of L satisfying $x\Theta y$ for all $x, y \in B$. The set L/Θ of all blocks of Θ becomes a lattice by defining

$$B_1 \leq B_2 :\Leftrightarrow \wedge B_1 \leq \wedge B_2 (\Leftrightarrow \vee B_1 \leq \vee B_2) \qquad \text{(see [\textbf{12}])}.$$

A (complete) tolerance relation is called **glued** if for every two of its blocks $B_1 < B_2$ there are blocks B_3 and B_4 with $B_1 \leq B_3 < B_4 \leq B_2$ and $B_3 \cap B_4 \neq \emptyset$. For a lattice L of finite length we denote with $\Sigma(L)$ the smallest tolerance relation which contains all covering pairs of elements in L.

Note that for a finite lattice L the smallest tolerance relation $\Sigma(L)$ always exists, because the meet of two tolerance relations is again a tolerance relation. $S(L) := L/\Sigma(L)$ is called the **skeleton** of L. This construction may be iterated as follows: $S_0(L) := L$ and $S_r(L) := S(S_{r-1}(L))$ for $r \in \mathbf{N}$. We call $S_r(L)$ the r-**skeleton** of L.

The following proposition is proved in [**7**, Lemma 1.4].

Proposition 2.5. *Let L be a finite lattice. Then $\Sigma(L)$ is the smallest glued tolerance relation of L.*

Definition 2.6. Let $\mathbf{K} := (G, M, I)$ be a finite context. $J \subseteq G \times M$ is called a **block relation** of $\mathbf{K}$ if

(i) $I \subseteq J$,
(ii) $(\forall X \subseteq G) : X^J$ is an intent of $\mathbf{K}$, and
(iii) $(\forall Y \subseteq M) : Y^J$ is an extent of $\mathbf{K}$.

In [**12**, Theorem 8], it has been shown that the lattice of all block relations of a context $\mathbf{K}$ with set inclusion as its order and the lattice of all complete tolerance relations of $\underline{\mathfrak{B}}(\mathbf{K})$ are isomorphic. For the next theorem let β denote this isomorphism.

Theorem 2.7 (Theorem 8 in [**12**]). *For a context (G, M, I), there is an isomorphism β from the lattice of all complete tolerance relations of $\underline{\mathfrak{B}}(G, M, I)$ onto the lattice of all block relations of (G, M, I) given by*

$$g\beta(\Theta)m :\Leftrightarrow \gamma g\Theta(\gamma g \vee \mu m)(\Leftrightarrow (\gamma g \vee \mu m)\Theta\mu m\,;$$

furthermore, $(A, B)\beta^{-1}(J)(C, D) \Leftrightarrow A \times D \cup B \times C \subseteq J$.

Theorem 2.8 (Theorem 10 in [**12**]). *Let (G, M, I) be a context such that $L := \underline{\mathfrak{B}}(G, M, I)$ has finite length. Then $J := \beta(\Sigma(L))$ is the smallest block relation of (G, M, I) containing all pairs (g, m) such that g' is maximal in $\{h' \mid h \in G$ and $(h, m) \notin I\}$ or m' is maximal in $\{n' \mid n \in M$ and $(g, n) \notin I\}$; especially, an isomorphism from $\underline{\mathfrak{B}}(G, M, J)$ onto the skeleton of L is given by $(H, N) \to \{(A, B) \in L \mid A \subseteq H$ and $B \subseteq N\}$.*

3. Free Distributive Lattices as Concept Lattices

Up to isomorphism, there is exactly one free completely distributive complete lattice, generated as a complete lattice by an ordered set $(S, \leq)$. We denote this lattice by $FCD(S, \leq)$. In this section we represent $FCD(S, \leq)$ as the concept lattice of a suitable context. Furthermore, we represent the 1-skeleton of $FCD(S, \leq)$ as a concept lattice if the generating ordered set $(S, \leq)$ is finite. In [**14**] one can find a representation of $FCD(S)$ and its r-skeletons in the case that S is an antichain. First we have to introduce some notions for describing the context of our representation.

Definition 3.1. For an ordered set $(S, \leq)$ we define the set of its order filters $G_S := \{[X)_S \mid X \subseteq S\}$, where $[X)_S := \{y \in S \mid (\exists\, x \in X) : y \geq x\}$, and the set of its order ideals $M_S := \{(Y]_S \mid Y \subseteq S\}$, where $(Y]_S := \{x \in S \mid (\exists\, y \in Y) : x \leq y\}$. Between G_S and M_S, the relation Δ is defined by $A \Delta B :\Leftrightarrow A \cap B \neq \emptyset$ and the relation Σ_r^S by $A \Sigma_r^S B :\Leftrightarrow |S \setminus (A \cup B)| \leq r - 1$ for $r \in \mathbf{N}$.

Obviously, if $(S, \leq)$ is an antichain then $G_S = \mathfrak{P}(S) = M_S$. If $(S, \leq)$ is linearly ordered then $(G_S, \subseteq)$ and $(M_S, \subseteq)$ are chains of length $|S|$. In general $(G_S, \subseteq)$ and $(M_S, \subseteq)$ are complete sublattices of $(\mathfrak{P}(S), \subseteq)$. The ordered sets $(G_S, \subseteq)$ and $(M_S, \subseteq)$ are dually isomorphic to each other; an isomorphism is given by mapping an element of one set to its complement. Now we can state the main theorem. As an exemption of our general rule the next theorem is valid for arbitrary ordered sets.

Theorem 3.2. *Let $(S, \leq)$ be an ordered set. Then $FCD(S, \leq)$ is isomorphic to $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$.*

Proof. The extents and intents of the context (G_S, M_S, Δ) are order filters of $G_S, \subseteq)$ and $(M_S, \subseteq)$, respectively. For each order filter $\mathscr{F}$ of $(G_S, \subseteq)$, the pair $(\mathscr{F}, \mathscr{F}^\sharp)$ with $\mathscr{F}^\sharp := \{Y \in M_S \mid S \setminus Y \notin \mathscr{F}\}$ is a concept of (G_S, M_S, Δ). This can be derived from the following equivalences for $Y \in M_S$:

$$(\exists\, X \in \mathscr{F}) : X \cap Y = \emptyset \Leftrightarrow (\exists\, X \in \mathscr{F}) : X \subseteq (S \setminus Y) \Leftrightarrow S \setminus Y \in \mathscr{F}\,.$$

Dually, for each order filter $\mathscr{F}$ of $(M_S, \subseteq)$, the pair $(\mathscr{F}^\sharp, \mathscr{F})$ is a concept of (G_S, M_S, Δ). We have $\mathscr{F}^\sharp = \mathscr{F}^\Delta$ and $\mathscr{F}^{\sharp\sharp} = \mathscr{F}$ for each order filter $\mathscr{F}$ of $(G_S, \subseteq)$ or $(M_S, \subseteq)$. Hence $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ consists of all pairs $(\mathscr{F}, \mathscr{F}^\Delta)$ for which $\mathscr{F}$ is an order filter on $(G_S, \subseteq)$.

Next we show that $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ is completely distributive: $\vee_{t\in T}(\mathscr{F}_t, \mathscr{F}_t^\Delta) = ((\cup_{t\in T}\mathscr{F}_t)^{\Delta\Delta}, \cap_{t\in T}\mathscr{F}_t^\Delta) = (\cup_{t\in T}\mathscr{F}_t, \cap_{t\in T}\mathscr{F}_t^\Delta)$ because $\cup_{t\in T}\mathscr{F}_t$ is an order filter in $(G_S, \subseteq)$. Thus, $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ is isomorphic to a complete sublattice of $(\mathfrak{P}(G_S), \subseteq)$.

Now we describe the desired isomorphism between $FCD(S, \leq)$ and $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$. For $p \in S$, let $E_p^S := \{X \in G_S \mid p \in X\}$. Then $(E_p^S)^\Delta = \{Y \in M_S \mid p \in Y\}$ and $(E_p^S, (E_p^S)^\Delta \in \underline{\mathfrak{B}}(G_S, M_S, \Delta)$. The pairs $(E_p^S, (E_p^S)^\Delta)$, $p \in S$, generate $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ because for $(\mathscr{F}, \mathscr{F}^\Delta) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta)$ we can prove $\mathscr{F} = \cup_{X\in\mathscr{F}} \cap_{p\in X} E_p^S$: Obviously we have $X \in \cap_{p\in X} E_p^S$ for all $X \in \mathscr{F}$; on the other hand, let $Y \in \cup_{X\in\mathscr{F}} \cap_{p\in X} E_p^S$. Then there exists some $X_0 \in \mathscr{F}$ such that $Y \in \cap_{p\in X_0} E_p^S$, i.e., for all $p \in X_0$ we have $Y \in E_p^S$. Therefore $X_0 \subseteq Y$ and hence $Y \in \mathscr{F}$ since $\mathscr{F}$ is an order filter. Let α be an order-preserving map from

$(\{(E_p^S, (E_p^S)^\Delta) \mid p \in S\}, \leq)$ into a completely distributive complete lattice L. We will prove that the mapping $\hat{\alpha}\colon \underline{\mathfrak{B}}(G_S, M_S, \Delta) \to L$ defined by

$$\hat{\alpha}(\mathscr{F}, \mathscr{F}^\Delta) := \bigvee_{X\in\mathscr{F}} \bigwedge_{p\in X} \alpha(E_p^S, (E_p^S)^\Delta)$$

is a complete homomorphism which extends to α.

First we note that $\hat{\alpha}_{|(\{(E_p^S,(E_p^S)^\Delta)|p\in S\},\leq)} = \alpha$ because α is order-preserving. Next we show that $\hat{\alpha}$ is $\vee$-preserving:

$$\begin{aligned}\hat{\alpha}\Big(\bigvee_{t\in T}(\mathscr{F}_t, \mathscr{F}_t^\Delta)\Big) &= \hat{\alpha}\Big(\bigcup_{t\in T}\mathscr{F}_t, \bigcap_{t\in T}\mathscr{F}_t^\Delta\Big) = \bigvee_{X\in\bigcup_{t\in T}\mathscr{F}_t} \bigwedge_{p\in X} \alpha(E_p^S, (E_p^S)^\Delta)\\ &= \bigvee_{t\in T}\bigvee_{X\in\mathscr{F}_t}\bigwedge_{p\in X} \alpha(E_p^S, (E_p^S)^\Delta) = \bigvee_{t\in T}\hat{\alpha}(\mathscr{F}_t, \mathscr{F}_t^\Delta)\,.\end{aligned}$$

Because of the complete distributivity of $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ we have

$$\begin{aligned}\hat{\alpha}(\mathscr{F}, \mathscr{F}^\Delta) &= \bigvee_{X\in\mathscr{F}}\bigwedge_{p\in X} \alpha(E_p^S, (E_p^S)^\Delta) = \bigwedge_{\sigma\in\Pi\mathscr{F}}\bigvee_{X\in\mathscr{F}} \alpha(E_{\sigma X}^S, (E_{\sigma X}^S)^\Delta)\\ &= \bigwedge_{Y\in\mathscr{F}^\Delta}\bigvee_{p\in Y} \alpha(E_p^S, (E_p^S)^\Delta)\,,\end{aligned}$$

and therefore, dually to the $\vee$-case, $\hat{\alpha}$ is $\wedge$-preserving:

$$\begin{aligned}\hat{\alpha}\Big(\bigwedge_{t\in T}(\mathscr{F}_t, \mathscr{F}_t^\Delta)\Big) &= \hat{\alpha}\Big(\bigcap_{t\in T}\mathscr{F}_t, \bigcup_{t\in T}\mathscr{F}_t^\Delta\Big) = \bigwedge_{Y\in\bigcup_{t\in T}\mathscr{F}_t^\Delta} \bigvee_{p\in Y} \alpha(E_p^S, (E_p^S)^\Delta)\\ &= \bigwedge_{t\in T}\bigwedge_{Y\in\mathscr{F}_t^\Delta}\bigvee_{p\in Y} \alpha(E_p^S, (E_p^S)^\Delta) = \bigwedge_{t\in T}\hat{\alpha}(\mathscr{F}_t, \mathscr{F}_t^\Delta)\,.\end{aligned}$$

It follows from the previous remarks that $(\{(E_p^S, (E_p^S)^\Delta) \mid p \in S\}, \leq)$ is order-isomorphic to $(S, \leq)$. Thus, $FCD(S, \leq)$ has to be isomorphic to $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$. This finishes the proof. □

The following result can be used to represent the skeleton of a free distributive lattice over a finite ordered set as a concept lattice. A free bounded distributive lattice generated by a finite ordered set $(S, \leq)$ is denoted by $FBD(S, \leq)$. If S is a finite antichain with $|S| = n$, we write $FBD(n)$ instead of $FBD(S)$. From Theorem 3.2 we know that $FBD(S, \leq)$ is finite because $|\mathfrak{P}(G_S)|$ is finite and an upper bound for $|\underline{\mathfrak{B}}(G_S, M_S, \Delta)|$.

Proposition 3.3. *Let L be a finite modular lattice. Then $S_1(L)$ is isomorphic to $\underline{\mathfrak{B}}(J(L), M(L), \leq \cup \swarrow\!\!\!\nearrow)$.*

Proof. Let $I :=\leq$ and $J :=\leq \cup \swarrow\!\!\!\nearrow$. We show that J is a block relation of $(J(L), M(L), \leq)$. Clearly $I \subseteq J$. Suppose, for a $g \in J(L)$, there is some $p \in M(L)$ with $p \in \cap\{h^I \mid g^J \subseteq h^I,\ h \in J(L)\} \setminus g^J$. Then there exists a $g_p \in J(L)$ with $g^I \subset g_p^I$ and $g_p \swarrow p$. Using Proposition 2.4 we obtain $g_p \swarrow\!\!\!\nearrow p$. For $m \in M(L)$ with $g \swarrow\!\!\!\nearrow m$ we get $m \in g_p^I$ and hence $g^J \subseteq g_p^I$ which yields a contradiction. This proves $g^J = \cap\{h^I \mid g^J \subseteq h^I \text{ with } h \in J(L)\}$ for $g \in J(L)$. Dually we obtain $m^J = \cap\{n^I \mid m^J \subseteq n^I \text{ with } n \in M(L)\}$ for $m \in M(L)$. Therefore J is a blockrelation of $(J(L), M(L), I)$. By Theorem 2.8, J is the block relation corresponding to $\Sigma(L)$, the tolerance relation of the skeleton, i.e., $\underline{\mathfrak{B}}(J(L), M(L), J) \cong S_1(L)$. □

Proposition 3.4. *For a finite ordered set* $(S, \leq)$*, the skeleton* $S_1(FBD(S, \leq))$ *is isomorphic to* $\underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_1^S)$.

Proof. From Theorem 3.2 we know that $FBD(S, \leq) \cong \underline{\mathfrak{B}}(G_S, M_S, \Delta)$. To apply Proposition 3.3 we have to show that

$$(G_S, M_S, \Delta) \cong (J(\underline{\mathfrak{B}}(G_S, M_S, \Delta)), M(\underline{\mathfrak{B}}(G_S, M_S, \Delta)), \leq)\,.$$

It suffices to show that (G_S, M_S, Δ) is a reduced context. Suppose there exists some $g \in G_S$ with $g^\Delta = \cap\{h^\Delta \mid g^\Delta \subset h^\Delta \text{ with } h \in G_S\}$. Obviously $g \cap (S \setminus g) = \emptyset$ and hence there exists an $h \in G_S$ with $g^\Delta \subset h^\Delta$ and $h \cap (S \setminus g) = \emptyset$. Therefore we can conclude $h = g$ which contradicts the assumed equation. Hence (G_S, M_S, Δ) is object-reduced. Analogously, we obtain that (G_S, M_S, Δ) is attribute-reduced. Thus (G_S, M_S, Δ) is a reduced context. By Proposition 3.3, we get $S_1(FBD(S, \leq)) \cong \underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \swarrow\!\!\!\nearrow)$. It only remains to show that $\Delta \cup \Sigma_1^S = \Delta \cup \swarrow\!\!\!\nearrow$. This follows from the equivalences:

$$(X, Y) \in \Delta \cup \Sigma_1^S \Leftrightarrow X \cap Y \neq \emptyset \ \text{ or } \ X \,\dot\cup\, Y = S \Leftrightarrow X \cap Y \neq \emptyset \ \text{ or } \ (X, Y) \in \swarrow\!\!\!\nearrow\,,$$

because if $X \,\dot\cup\, Y = S$ then every proper superset of X has nonempty intersection with Y and every proper superset of Y has nonempty intersection with X. □

For a finite antichain S it was shown in [**14**] that

$$S_r(FBD(S)) \cong \underline{\mathfrak{B}}(\mathfrak{P}(S), \mathfrak{P}(S), \Delta \cup \Sigma_r^S)\,.$$

This does not hold for ordered sets in general if $r \geq 2$. The following example illustrates this fact: Let $\underline{n}$ denote a chain with n elements and let $\underline{n} + \underline{m}$ denote two (disjoint) chains. Then for $(S, \leq) \cong \underline{1} + \underline{2}$ where $S = \{1, 2, 3\}$ and $1 < 2$, the second skeleton $S_2(FBD(S, \leq))$ is a four-element chain because obviously the skeleton of a five-element chain is a four-element chain (see Figure 1). On the other hand the incidence relation $\Sigma \cup \Sigma_2^S$ of the context of the second skeleton contains all incidences and double-arrows of the context of Table 1 plus the pair $(2, 1)$. Therefore the concept lattice $\underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_2^S)$ is isomorphic to a three-element

$\emptyset$

2

3

12, 23

123

Figure 1. The lattice $S_1(FBD(\underline{1}+\underline{2}))$.

chain. In the following pictures the elements of G_S and M_S are abbreviated in the context and in the concept lattice, e.g., for the object $\{2,3\}$ in the context of $S_1(FBD(\underline{1}+\underline{2}))$ we write 23.

	$\emptyset$	1	3	12	13	123
$\emptyset$				$\swarrow\nearrow$	$\swarrow\nearrow$	$\times$
2		$\circ$	$\swarrow\nearrow$	$\times$	$\times$	$\times$
3		$\swarrow\nearrow$	$\times$	$\times$	$\times$	$\times$
12	$\swarrow\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$
23	$\swarrow\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$
123	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Table 1. The context of $S_1(FBD(\underline{1}+\underline{2}))$.

From Theorem 2.8 we know that, given a lattice L of finite length, the block relation of the skeleton of L contains all arrows of $\mathbf{K}(L)$. For nonmodular lattices this containment may be proper. Such an L can be constructed as follows. Every finite lattice is the skeleton of some finite distributive lattice (Satz 7.2 in [**6**]). Given a finite lattice L, define an ordered set $P := L \times \{0,1\}$ and $(x,1) > (y,0)$: $\Leftrightarrow x \not\geq y$. Then $\underline{\mathfrak{B}}(P,P,\not\geq)$ is distributive and $S_1(\underline{\mathfrak{B}}(P,P,\not\geq))$ is isomorphic to L (see [**11**] or [**6**]).

4 5

2 3

Figure 2. L.

In the following example we start with the lattice L displayed in Figure 2. As we will see below, it suffices to define $P := \{2,3,4,5\} \times \{0,1\}$. The context $(P,P,\not\geq)$ is given in Table 2 where (x,y) is abbreviated by xy. The distributive concept lattice $\underline{\mathfrak{B}}(P,P,\not\geq)$ and its skeleton $S_1(\underline{\mathfrak{B}}(P,P,\not\geq))$ are given in Figure 3 and Figure 4, respectively. Therefore $S_2(\underline{\mathfrak{B}}(P,P,\not\geq))$ consist of only one element.

21

31

51 41

20 50

40 30

Figure 3. $\underline{\mathfrak{B}}(P, P, \not\geq)$.

41 51

21 31

20, 30, 40, 50

Figure 4. $S_1(\underline{\mathfrak{B}}(P, P, \not\geq)$.

If we start with the context $(P, P, \not\geq)$ of Table 2 and add all (double) arrows to the given incidence we get a context whose concept lattice is isomorphic to L. If we again add all arrows of the new context (which are marked with index 2 in Table 2) to the incidence relation we get a context whose concept lattice is isomorphic to a four-element Boolean lattice. This shows that the underlying context of the second skeleton of a finite distributive lattice cannot be obtained just by adding twice the arrows to the given incidence relation of the underlying context of the finite distributive lattice.

$\not\geq$	20	30	40	50	21	31	41	51
20	$\swarrow\!\!\!\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
30	$\times$	$\swarrow\!\!\!\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
40	$\times$	$\times$	$\swarrow\!\!\!\nearrow$	$\times$	$\times$	$\times$	$\times$	$\times$
50	$\times$	$\times$	$\times$	$\swarrow\!\!\!\nearrow$	$\times$	$\times$	$\times$	$\times$
21	$\times$	$\swarrow_2$	$\swarrow\!\!\!\nearrow_2$		$\swarrow\!\!\!\nearrow$	$\times$	$\times$	$\times$
31	$\swarrow_2$	$\times$		$\swarrow\!\!\!\nearrow_2$	$\times$	$\swarrow\!\!\!\nearrow$	$\times$	$\times$
41	$\times$	$\swarrow\!\!\!\nearrow_2$	$\times$	$\nearrow_2$	$\times$	$\times$	$\swarrow\!\!\!\nearrow$	$\times$
51	$\swarrow\!\!\!\nearrow_2$	$\times$	$\nearrow_2$	$\times$	$\times$	$\times$	$\times$	$\swarrow\!\!\!\nearrow$

Table 2. $(P, P, \not\geq)$.

4. The Covering Relation in r-Skeletons

The representations of $FBD(S)$, where S is an antichain, suggest to assume that two blocks, which cover each other, intersect in a Boolean interval which is half the size of the smaller one of the two blocks. We prove this conjuncture in this section for all finite ordered sets $(S, \leq)$. The proof is stated in terms of antichains of the ordered set $(M_r^S, \leq_r)$. We will use lower case letters for elements of M_S and upper case letters for the elements of M_r^S. In the sequel let $(S, \leq)$ always be a finite ordered set.

Definition 4.1. 1. For $a, b \in M_S$ and a natural number $r \leq |S|$, we define $a \subseteq_r b$: $\Leftrightarrow a \subseteq b$ and $|b \backslash a| \geq r$.

2. Let $C \subseteq M_S$. We call C **convex**, if a, $b \in C$, $x \in M_S$ with $a \subseteq x \subseteq b$ implies $x \in C$. We call C an r-**family** of M_S, if a, $b \in C$ with $a \subseteq b$ implies $|b \backslash a| \leq r - 1$.

3. For $r \leq |S|$, we define the ordered set $(M_r^S, \leq)$ by $M_r^S := \{C \subseteq M_S \mid C$ is a maximal convex r-family of $(M_S, \subseteq)\}$. For C_1, $C_2 \in M_r^S$ we define $C_1 \leq C_2$: $\Leftrightarrow$ for every $a \in C_1$, there exists $b \in C_2$ such that $a \subseteq b$.

If S is a finite antichain with $|S| = n$, we write M_r^n instead of M_r^S.

Lemma 4.2. $\underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_r^S) \cong \underline{\mathfrak{B}}(M_S, M_S, \not\supseteq_r)$.

Proof. Let α be the map from $\underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_r^S)$ to $\underline{\mathfrak{B}}(M_S, M_S, \not\supseteq_r)$ with $\alpha(A, B) := (A_1, B)$, where $A_1 := \{S \setminus a \mid a \in A\}$. Then we have

$$\begin{aligned}
&(A, B) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_r^S) \\
&\qquad \Leftrightarrow (\forall\, a \in A)(\forall\, b \in B)(a \cap b \neq \emptyset \quad \text{or} \quad |S \setminus (a \cup b)| \leq r - 1) \\
&\qquad \Leftrightarrow (\forall\, a \in A)(\forall\, b \in B)(S \setminus a \not\supseteq b \quad \text{or} \quad |(S \setminus a) \setminus b)| \leq r - 1) \\
&\qquad \Leftrightarrow (\forall\, a \in A)(\forall\, b \in B) S \setminus a \not\supseteq_r b \\
&\qquad \Leftrightarrow (A_1, B) \in \underline{\mathfrak{B}}(M_S, M_S, \not\supseteq_r)\,.
\end{aligned}$$

Since both concept lattices have the same intents, α is an isomorphism. □

Proposition 4.3 (Proposition 2 of [**13**]). *Let S be an ordered set of finite length and let k be a positive integer. Then $(A, B) \mapsto A \cap B$ describes an isomorphism from $\underline{\mathfrak{B}}(P, P, \not\supseteq_k)$ onto the lattice of all maximal convex k-families of P.*

Proposition 4.4. $\underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_r^S) \cong (M_r^S, \leq)$.

Proof. $\underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_r^S) \cong \underline{\mathfrak{B}}(M_S, M_S, \not\supseteq_r) \cong (M_r^S, \leq)$. The first isomorphism follows from Lemma 4.2, the second from Proposition 4.3 with $(M_S, \subseteq)$ as the given ordered set. The isomorphism between the two lattices of the proposition is given by $\iota(A, B) := A_1 \cap B$, where $A_1 := \{S \setminus a \mid a \in A\}$. □

For $r = 1$, M_1^S denotes the set of all maximal antichains of $(M_S, \subseteq)$. Note that for $C \in M_r^S$ not every maximal chain of C need to have the length $r - 1$. In the following example (Figure 5) let S be the five-element antichain and $r = 3$. For $C :=$

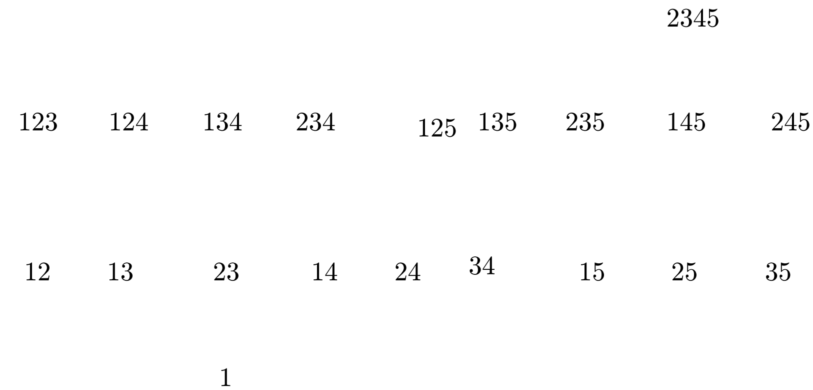

Figure 5. The maximal chain-length in $C \in M_3^S$.

$\{1, 12, 13, 23, 14, 24, 34, 15, 25, 35, 45, 123, 124, 134, 234, 125, 135, 235, 145, 245, 345, 2345\} \in M_3^5$, there is a maximal chain $\{23, 123\}$ of length one in C.

In Proposition 4.6 we show that every element of $C \in M_r^S$ is contained in a chain of length $r-1$. From Proposition 4.6 we can conclude that $r \geq 3$ is necessary for an example as given above. Before proving the main theorem of this chapter we need some properties of the ordered set $(M_r^S, \leq)$. For $C \in M_r^S$ we denote with $\mathrm{Max}\,(C)$ the set of all maximal elements of C and with $\mathrm{Min}\,(C)$ the set of all minimal elements of C.

Lemma 4.5. 1. *Let $C \in M_r^S$. Then for each $d \in M_S \setminus C$ there exists either $d_1 \in \mathrm{Min}\,(C)$ with $d_1 \subseteq_r d$ or $d_2 \in \mathrm{Max}\,(C)$ with $d \subseteq_r d_2$ or both.*

2. *Let $C_1, C_2 \in M_r^S$, $C_1 \leq C_2$. Then for any $b \in C_2$ there exists an $a \in C_1$ such that $a \subseteq b$.*

3. *For every $C \in M_r^S$ the sets $\mathrm{Max}\,(C)$ and $\mathrm{Min}\,(C)$ form maximal antichains in $(M_S, \subseteq)$.*

4. *Let $C_1, C_2 \in M_r^S$. Then the following inequalities are equivalent:*
(a) $C_1 \leq C_2$,
(b) $\mathrm{Max}\,(C_1) \leq \mathrm{Max}\,(C_2)$,
(c) $\mathrm{Min}\,(C_1) \leq \mathrm{Min}\,(C_2)$.

5. *For $C \in M_{r+1}^S$ let $D := C \setminus \mathrm{Max}\,(C)$ and $E := C \setminus \mathrm{Min}\,(C)$. Then $D \in M_r^S$ and $E \in M_r^S$.*

Proof. 1. Let $d \in M_S \setminus C$. Defining $X := \{x \in M_S \mid \exists c_1, c_2 \in C \cup \{d\} : c_1 \subseteq x \subseteq c_2\}$ we have that X is a convex subset of $(M_S, \subseteq)$. Since $C \in M_r^S$ we have $X \notin M_r^S$. Therefore X cannot be an r-family, i.e., there exist two elements $e_1, e_2 \in X$ with $e_1 \subseteq_r e_2$ and $|e_2 \setminus e_1|$ maximal. Since we cannot have $c_1, c_2 \in C$ with $c_1 \subseteq e_1 \subseteq_r e_2 \subseteq c_2$ this implies $d \in \{e_1, e_2\}$. If $d = e_1$ we have $e_2 \in \mathrm{Max}\,(C)$, if $d = e_2$ we have $e_1 \in \mathrm{Min}\,(C)$.

2. Let $b \in C_2 \setminus C_1$. Then by 1. there exists $d_1 \in \mathrm{Min}\,(C_1)$ with $d_1 \subseteq_r b$ or $d_2 \in \mathrm{Max}\,(C_1)$ such that $b \subseteq_r d_2$. In the first case we are done, in the second, because of $C_1 \leq C_2$, there exists $b_1 \in C_2$ such that $d_2 \subseteq b_1$. But $b \subseteq_r d_2 \subseteq b_1$ contradicts $C_2 \in M_r^S$, so the second case does not occur.

3. $\mathrm{Max}\,(C)$ and $\mathrm{Min}\,(C)$ are antichains. We prove that $\mathrm{Max}\,(C)$ is a maximal antichain. Let $d \in M_S \setminus C$. Then by 1. there exists $a_1 \in \mathrm{Min}\,(C)$ with $a_1 \subseteq_r d$ or $a_2 \in \mathrm{Max}\,(C)$ with $d \subseteq_r a_2$. In the second case $\mathrm{Max}\,(C) \cup \{d\}$ is not an antichain. For the first case assume $|a_1|$ minimal with $a_1 \subseteq_r d$ and choose $g \in C$ with $a_1 \subseteq g \subset d$, g maximal in C below d. If $g \notin \mathrm{Max}\,(C)$, we have $|g \setminus a_1| < r-1$. Let $h \in M_S$ with $g \subset h \subseteq d$ and $|h \setminus g| = 1$. Then $|h \setminus a_1| \leq r-1$. For all $a \in \mathrm{Min}\,(C)$ with $a \subseteq_r d$ we have $|a| \geq |a_1|$, hence $C \cup \{h\}$ is an r-family, which can be extended to a convex r-family because of the minimality of $|a_1|$ under d. This contradicts $C \in M_r^S$. Therefore $g \in \mathrm{Max}\,(C)$ and $\mathrm{Max}\,(C) \cup \{d\}$ is not an antichain.

The proof for $\mathrm{Min}\,(C)$ follows dually.

4. (a) $\Rightarrow$ (b): For $a \in \mathrm{Max}\,(C_1)$ there exists $b \in C_2$ such that $a \subseteq b$. We can choose $b \in \mathrm{Max}\,(C_2)$.
(b) $\Rightarrow$ (a): For $a \in C_1$ there exist $a_1 \in \mathrm{Max}\,(C_1)$ and $b_1 \in \mathrm{Max}\,(C_2)$ with $a \subseteq a_1 \subseteq b_1$. Hence $C_1 \leq C_2$.
Because of 2. the equivalence of (a) and (c) follows dually.

5. The proof of the last part is left to the reader. □

Proposition 4.6. *Let $c \in M_r^S$, $r \in \mathbf{N}$. Then for every $b \in \mathrm{Max}\,(C)$ there exists an $a \in \mathrm{Min}\,(C)$ such that $|b \setminus a| = r-1$. Dually for every $a \in \mathrm{Min}\,(C)$ there exists $b \in \mathrm{Max}\,(C)$ such that $|b \setminus a| = r-1$.*

Proof. We prove the first claim by induction on r. The statement is true for $r = 1$. Assume that the statement holds for k with $1 \leq k < |S|$. We will show that the statement is true for $k+1$. Let $C \in M_{k+1}^S$. From Lemma 4.5(5) we have $D := C \setminus \mathrm{Max}\,(C) \in M_k^S$. Let $b \in \mathrm{Max}\,(C)$. By Lemma 4.5(1) and since $\mathrm{Max}\,(D) \leq \mathrm{Max}\,(C)$ there exists $d_1 \in \mathrm{Min}\,(D)$ with $d_1 \subseteq_k b$. Because of $d_1 \in C$, we get $|b \setminus d_1| = k$, otherwise we get a contradiction C begin a $(k+1)$-family. This proves the case $k+1$.

The proof of the second statement is similar. □

The following main theorem of this chapter gives us a good characterization of the covering relation in $(M_r^S, \leq)$.

Theorem 4.7 (Characterization of the covering relation in $(M_r^S, \leq)$). *Let C_1, $C_2 \in M_r^S$, $C_1 < C_2$, $1 \leq r \leq |S|$. Then $C_1 \prec C_2$ iff*
(1) *Every element of $C_1 \setminus C_2$ has the same cardinality u.*
(2) *Every element of $C_2 \setminus C_1$ has the cardinality $u + r$.*
(3) *Each element of $C_1 \setminus C_2$ is contained in every element of $C_2 \setminus C_1$.*

Proof. First let us assume that C_1 and C_2 have the properties described in (1), (2) and (3). Suppose there exists a $C \in M_r^S$ with $C_1 < C < C_2$.

Because of $C_1 < C$, there exists $a_1 \in C_1 \setminus C$ and $b_1 \in C \setminus C_1$ with $a_1 \subseteq_r b_1$. Because of $C < C_2$, there exists $b_2 \in C \setminus C_2$ and $c_2 \in C_2 \setminus C$ with $b_2 \subseteq_r c_2$. Because of $C < C_2$, there exists $c_1 \in C_2$ with $a_1 \subseteq_r b_1 \subseteq c_1$. It follows $a_1 \in C_1 \setminus C_2$ and $c_1 \in C_2 \setminus C_1$. By the assumptions we get $b_1 = c_1$, hence $b_1 \in (C \cap C_2) \setminus C_1$.

Because of $C_1 < C$, there exists $a_2 \in C_1$ with $a_2 \subseteq b_2 \subseteq_r c_2$. It follows $a_2 \in C_1 \setminus C_2$ and $c_2 \in C_2 \setminus C_1$. By the assumptions we get $a_2 = b_2$, hence $b_2 \in (C \cap C_1) \setminus C_2$.

From $b_2 \in C_1 \setminus C_2$, for all $c \in C_2 \setminus C_1$, we get $b_2 \subseteq_r c$, in particular for $b_1 \in C_2 \setminus C_1$ we get $b_2 \subseteq_r b_1$. But $b_1, b_2 \in C$, so we have a contradiction because C is an r-family.

This finishes one direction of the proof.

Now we assume that C_1 is the lower cover of C_2. Because of $C_1 < C_2$ we have $C_1 \setminus C_2 \neq \emptyset$ and $C_2 \setminus C_1 \neq \emptyset$. With the following definitions we describe C_2, so that we can derive the properties (1), (2) and (3). Let $g \in C_1 \setminus C_2$ with minimal cardinality, $u := |g|$. Because of $C_1 < C_2$ there exists an $h \in C_2 \setminus C_1$ with $g \subseteq_r h$. We define

$$\begin{aligned} M_u &:= \{a \in C_1 \setminus C_2 \mid |a| = u\}, \\ G &:= \{b \in C_2 \setminus C_1 \mid \exists\, a \in M_u : a \subseteq_r b\}, \quad \text{and} \\ E &:= \{e \in M_S \mid (\exists\, a \in M_u)(\exists\, b \in G) : a \subset e \subseteq b \quad \text{and} \quad |e \setminus a| = r\}. \end{aligned}$$

For $e \in E$ we define $N_e := \{a \in M_u \mid a \subseteq_r e\}$. Let $c \in E$ such that N_c has minimal cardinality. Then we define $D := \{d \in E \mid ((d]_r \setminus (d]_{r+1}) \cap M_u = N_c\}$ and $T := D \cup (C_1 \setminus N_c)$. We note that, for all $e \in E$, the set N_e is nonempty and that $c \in D$.

h

G

D c E

M_u

g N_c

Figure 6. Visualization of the defined sets.

The drawing given in Figure 6 illustrates the sets defined above. The horizontal lines denote the sets G, E, M_u, D, and N_c where $D \subseteq E$ and $N_c \subseteq M_u$. Vertical lines between points denote a subset-relation between the corresponding elements. The cross lines between N_c and D means that every element of N_c is a subset of every element of D.

The main part of the proof will be to show that $N_c = C_1 \setminus C_2$ and that $D = C_2 \setminus C_1$, i.e., that $T = C_2$. In the next three steps we show that $T \in M_r^S$.

Step 1: T is an r-family.

Suppose there exist $d, e \in T$ with $d \subseteq_r e$. Since C_1 is an r-family, we have $d \in D$ or $e \in D$. In the first case we get $e \in C_1$. There exists $a \in N_c \subseteq M_u$ with $a \subseteq_r d \subseteq_r e$ in contrary to the assumption that C_1 is an r-family. In the second case we have $d \in C_1$. There exists $b \in G$ with $e \subseteq b$ which implies $d \subseteq_r b$ and hence we have $d \in C_1 \setminus C_2$. The cardinality of e is $u + r$ and hence $|d| \leq u = |g|$. Since g has minimal cardinality in $C_1 \setminus C_2$, we get $|d| = u$. But then we have $d \in M_u \cap ((e]_r \setminus (e]_{r+1})$ and, since $e \in D$, $d \in N_c$ which contradicts $d \in T$. Therefore T is an r-family.

Step 2: T is a convex subset of $(M_S, \subseteq)$.

Suppose T is not convex. Then there exist elements $d \in \mathrm{Min}\,(T)$, $e \in \mathrm{Max}\,(T)$, and $f \in M_S \setminus T$ with $d \subset f \subset e$. Since C_1 is convex and all elements of N_c are minimal in $C_1 \setminus C_2$, we have $f \notin N_c$ and hence $f \notin C_1$. Therefore, it follows either $d \in D$ or $e \in D$. In the first case we get $e \in C_1$. There exists $a \in N_c$ with $a \subseteq_r d \subset e$ contradicting that C_1 is an r-family.

In the second case we have $d \in C_1$. Since $f \notin C_1$, by Lemma 4.5, there exists $g_1 \in \mathrm{Min}\,(C_1)$ with $g_1 \subseteq_r f$ or $g_2 \in \mathrm{Max}\,(C_1)$ with $f \subseteq_r g_2$. The second possibility yields, because of $d \subset f \subseteq_r g_2$, a contradiction for C_1 being an r-family. The first possibility yields $g_1 \subseteq_r f \subset e$, i.e., $g_1 \subseteq_{r+1} e$. This implies $|g_1| < u$. It exists $b \in G$ with $e \subseteq b$ and hence $g_1 \in C_1 \setminus C_2$, contradicting that u is the minimal absolute value in $C_1 \setminus C_2$.

Step 3: T is a maximal convex r-family.

Suppose there would exist a $C \in M_r^S$ with $T \subset C$. Let $p \in C \setminus T$. If $p \in N_c$ then from $p \subseteq_r c$ and $c \in D \subseteq T \subset C$ we have a contradiction to C begin an r-family. Therefore $p \notin C_1$ and by Lemma 4.5 there exists $e_1 \in \mathrm{Max}\,(C_1)$ with $p \subseteq_r e_1$ or $e_2 \in \mathrm{Min}\,(C_1)$ with $e_2 \subseteq_r p$. We need e_1 (respectively e_2) $\in N_c$, and hence, since $p \subseteq_r e_1 \subseteq_r c \in T$, the first case cannot occur. In the second case we first need the proof for $r = 1$.

Assume there exists an $e_3 \in C_1 \setminus \{e_2\}$ with $e_3 \subseteq_1 p$. It follows $e_3 \in N_c$ and hence $c = e_2 \cup e_3 \subseteq p$. For $c = p$ we get a contradiction to $p \notin T$, for $c \subset p$ we get a contradiction to C being an antichain.

Now, suppose $e_3 \nsubseteq p$ for all $e_3 \in C_1 \setminus \{e_2\}$. For all $d \in M_S$ with $e_2 \subset d \subseteq p$ and $|d \setminus e_2| = 1$, since $C_2 \in M_1^S$, we have $d \in C_2$ or there exists $b \in C_2 \setminus C_1$ with $d \subseteq_1 b$. The other possibility that there exists a $b \in C_2$ with $b \subseteq_1 d$ cannot occur; because $C_1 < C_2$ yields an $e \in C_1$ with $e \subseteq b \subseteq_1 d \subseteq p$ and since $e_3 \nsubseteq p$ for all $e_3 \in C_1 \setminus \{e_2\}$, we get $e = e_2$ and therefore $b = e_2$, c contradiction to the assumption that $e_2 \in N_c \subseteq C_1 \setminus C_2$. From $e_2 \in N_c \subseteq M_u$ follows $d \in E$. We get $N_d = \{a \in M_u \mid a \subseteq_1 d\} = \{e_2\} \subseteq N_c$ where the second equality follows from $e_3 \not\subseteq p$ for all $e_3 \in C_1 \setminus \{e_2\}$. N_c has minimal cardinality, hence $N_d = N_c = \{e_2\}$. Therefore we have $d \in D$ and for $d = p$ we get a contradiction to $p \notin T$, for $d \subset p$ we get a contradiction to C being an antichain. Hence we have shown the maximality of T in the case $r = 1$.

Now let $r > 1$. Let $f \in M_S$ with $e_2 \subset f \subset p$ and $|f| < u + r$. Such an f exists because $|e_2| = u$ and hence $|p| \geq u + r$. We claim $f \in C_1$, since otherwise there would exist $g_1 \in \mathrm{Min}\,(C_1)$ with $g_1 \subseteq_r f$ or $g_2 \in \mathrm{Max}\,(C_1)$ with $f \subseteq_r g_2$; in the second case we get, because of $e_2 \subset f \subseteq_r g_2$, a contradiction to the assumption that C_1 is an r-family. In the first case we get $g_1 \subseteq_{r+1} p$. C is supposed to be an r-family, hence we have $g_1 \in N_c$ which means $|g_1| = u$. This contradicts $g_1 \subseteq_r f$ since $|f| < u + r$.

Since $e_2 \in N_c$ we get for $f \in M_S$ with $e_2 \subset f \subset p$ and $|f| = u + 1$ that $f \in C_1 \setminus N_c$, i.e., $f \in T$. From $f \subseteq_{r-1} p$ it follows that $|p| \geq u + r$. C is an r-family, thus $|p| = u + r$. Altogether in the second case it follows: Supposing $C \in M_r^S$ with $T \subset C$ and $p \in C \setminus T$ there exists an $e_2 \in N_c$ with $e_2 \subseteq_r p$ and $|p| = u + r$.

Suppose $p \not\subseteq b$ for all $b \in C_2$. Then $p \notin C_2$ and, by Lemma 4.5, there exists $g_1 \in \mathrm{Min}\,(C_2)$ with $g_1 \subseteq_r p$. We have $g_1 \in C_1$, since otherwise Lemma 4.5 implies the existence of an $f_1 \in \mathrm{Min}\,(C_1)$ with $f_1 \subseteq_r g_1$ or an $f_2 \in \mathrm{Max}\,(C_1)$ with $g_1 \subseteq_r f_2$. In the first case it follows from $f_1 \subseteq_r g_1 \subseteq_r p$ that $f_1 \in C_1 \setminus C_2$ and because of $|p| = u + r$ we have $|f_1| < u$ contradicting that u is the minimal cardinality in $C_1 \setminus C_2$. In the second case, since $C_1 < C_2$, we have an $f_3 \in C_1$ with $f_3 \subseteq g_1 \subseteq_r f_2$; this contradicts that C_1 is an r-family. Therefore we get $g_1 \in C_1$ and hence $g_1 \in C_1 \cap C_2$. It follows $g_1 \notin N_c$, which yields $g_1 \in T$. Since $g_1 \subseteq_r p$, this contradicts that C is an r-family.

Hence there exists $b \in C_2$ with $p \subseteq b$. $e_2 \subseteq_r p \subseteq b$ and $e_2 \in N_c \subseteq M_u$ implies $b \in G$; therefore $p \in E$. We have $N_p \subseteq N_c$, since otherwise there would be $d \in N_p \setminus N_c$ with $d \in C_1 \setminus N_c \subseteq T$, which contradicts that C is an r-family. The set N_c has minimal cardinality beneath the sets N_e, $e \in E$, hence $N_p = N_c$.

$((p]_r \setminus (p]_{r+1}) \cap M_u = \{a \in M_u \mid a \subseteq_r p \text{ and } |p \setminus a| = r\} = \{a \in M_u \mid a \subseteq_r p\} = N_p = N_c$. Hence we get $p \in D$ contrary to the assumption $p \in M_S \setminus T$. So we have proven that $T \in M_r^S$.

Step 4: $T \leq C_2$.

Because of $C_1 < C_2$, there is nothing to show for $a \in C_1 \setminus N_c$. For $a \in D$ we know that $a \in E$ and that there exists $b \in G \subseteq C_2$ with $a \subseteq b$.

Step 5: $N_c = C_1 \setminus C_2$ and $D = C_1 \setminus C_1$.

From $C_1 < T \leq C_2$ and $C_1 \prec C_2$ it follows that $C_2 = T$. Because of $C_1 \setminus C_2 = N_c$, we have $|a| = u$ for all $a \in C_1 \setminus C_2$. Because of $C_2 \setminus C_1 = D \subseteq E$, we have $|b| = u + r$ for all $b \in C_2 \setminus C_1$. Let $a \in N_c$ and $b \in D$. Then we know

$a \in (b]_r \setminus (b]_{r+1}$, in particular $a \subseteq b$. Hence from the construction of N_c and D we get that each element of $C_1 \setminus C_2$ is contained in every element of $C_2 \setminus C_1$. Thus finishes the proof of 4.7. $\square$

For the following we have to apply the special case $r = 1$ of the last theorem. The characterization of two covering elements becomes easier in that case.

Corollary 4.8. *Let $C_1, C_2 \in M_1^S$, $C_1 < C_2$. Then $C_1 \prec C_2$ implies $|C_2 \setminus C_1| = 1$ or $|C_1 \setminus C_1| = 1$.*

Proof. For the proof we use the three conditions (1), (2) and (3) of Theorem 4.7 which are equivalent to $C_1 \prec C_2$. Suppose $|C_1 \setminus C_2| \neq 1$. Since $C_1, C_2 \in M_1^S$ we have $|C_1 \setminus C_2| > 1$. From (3) we get for any $a_1, a_2 \in C_1 \setminus C_2$ with $a_1 \neq a_2$, and $b_1, b_2 \in C_2 \setminus C_1$ that $a_1 \cup a_2 \subseteq b_1$ and $a_1 \cup a_2 \subseteq b_2$. Because of (1) and (2), we have $|a_1| = |a_2| = u$ and $|b_1| = |b_2| = u + 1$. We get $b_1 = a_1 \cup a_2 = b_2$ and hence $|C_2 \setminus C_1| = 1$.

Dually, for $|C_2 \setminus C_1| \neq 1$ we can conclude $|C_1 \setminus C_2| = 1$. $\square$

The "or" of the Corollary above is not excluding. We give an example where for $r = 1$ and S the five-element antichain the sets $C_1 \setminus C_2$ and $C_2 \setminus C_1$ both consist of one element: Defining $C_1 := \{13, 23, 14, 125, 245, 345\}$ and $C_2 := \{23, 14, 125, 135, 245, 345\}$ we have $C_1, C_2 \in M_1^5$, $C_1 \prec C_2$, and $|C_1 \setminus C_2| = 1 = |C_2 \setminus C_1|$. On the other hand in the case $r > 1$ the sets $C_1 \setminus C_2$ and $C_2 \setminus C_1$ both may contain more than one elements. As an illustration we give the following example where $(M_2^{17}, \leq)$ is the given ordered set.

Example. For $0 \leq i \leq |S|$ let $\mathfrak{P}_i(S) := \{a \in \mathfrak{P}(S) \mid |a| = i\}$.
M=$\{\{1,4,6,7\}, \{2,4,8,9\}, \{3,4,10,11\}, \{1,5,12,13\}, \{2,5,14,15\}, \{3,5,16,17\}\}$,
$C_1 := M \cup \mathfrak{P}_3(S) \cup \mathfrak{P}_2(S) \setminus \{a \in \mathfrak{P}_2(S) \mid \exists\, m \in M : a \subseteq m\}$,
$C_2 := \{\, \{1,2,3,4\}, \{1,2,3,5\}\,\} \cup C_1 \setminus \{\, \{1,2\}, \{1,3\}, \{2,3\}\,\}$.
We have $C_1, C_2 \in M_2^{17}$, $C_1 \prec C_2$, $|C_1 \setminus C_2| = 3$, and $|C_2 \setminus C_1| = 2$.

By Lemma 4.5 we know that $\operatorname{Max}(C)$ is a maximal antichain for every $C \in M_r^S$. The following example shows that there are maximal antichains which are not the set of the maximal elements of some element from M_r^S.

Example. Define $S := \{1,2,3,4\}$. Then we have $C_1 := \{34, 123, 124\} \in M_1^4$. For $D_2 := \{4, 12, 13, 23, 14, 24, 34, 123\}$ and $E_2 := \{12, 13, 23, 14, 24, 34, 123, 124, 134, 234\}$ we have $D_2, E_2 \in M_2^4$, $D_2 \prec E_2$, but $\operatorname{Max}(D_2) < C_1 < \operatorname{Max}(E_2)$. For $D_3 := \{\emptyset, 1, 2, 3, 4, 12, 13, 23, 14, 24, 34\}$ and $E_3 := \{1, 2, 3, 4, 12, 13, 23, 14, 24, 34, 123, 124, 134, 234\}$ we have $D_3, E_3 \in M_3^4$, $D_3 \prec E_3$, but $\operatorname{Max}(D_3) < C_1 < \operatorname{Max}(E_3)$. Hence for $C_1 \in M_1^4$ no $A \in M_r^4$, $r > 1$, exists with $C_1 = \operatorname{Max}(A)$.

Now we come back to the main goal of this chapter. In Proposition 4.10, we describe explicitly the connection between the elements of $(M_1^S, \leq)$ and the blocks of $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$. From this we get immediately the intersection of two covering blocks. First we show that for a finite distributive lattice the blocks are the maximal Boolean intervals. Generalized for modular lattices of finite length, Christian Herrmann has shown this result in [**6**].

Lemma 4.9. *The blocks of* $\Sigma(FBD(S,\leq))$ *are the maximal Boolean intervals of* $FBD(S,\leq)$.

Proof. By the definition of $\Sigma(L)$ we know that each covering pair of $FBD(S,\leq)$ is in the relation $\Sigma(FBD(S,\leq))$. So in particular the pairs consisting of the 0-element and an atom of a maximal Boolean interval are in $(FBD(S,\leq))$. Since $\Sigma(FBD(S,\leq))$ is compatible with join and meet we get that every maximal Boolean interval of $FBD(S,\leq)$ is contained in a block of $\Sigma(FBD(S,\leq))$. From Theorem 3.1, p. 377 in [**1**], we know for a modular lattice L that $\Phi := \{(x,y) \in L^2 \mid [x\wedge y, x\vee y] \text{ is complemented}\}$ is a tolerance relation on L. In a distributive lattice L a complemented interval of L is Boolean, hence we get $\Phi = \{(x,y) \in L^2 \mid [x\wedge y, x\vee y] \text{ is Boolean}\}$. Obviously Φ is glued, because it contains all covering pairs of L. Since $\Sigma(FBD(S,\leq))$ is the smallest glued tolerance relation and $\Phi \subseteq \Sigma(FBD(S,\leq))$, we have $\Phi = \Sigma(FBD(S,\leq))$. $\square$

From Proposition 4.4 we know that $\underline{\mathfrak{B}}(G_S, M_S, \Delta\cup\Sigma_1^S) \cong (M_1^S,\leq)$, i.e., for a concept $(A,B) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta\cup\Sigma_1^S)$ the set $A_1 \cap B$ is a maximal antichain in $(M_S,\subseteq)$. For the next proposition for $(A,B) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta\cup\Sigma_1^S)$ we define $\operatorname{Min}(A) := \{a_1,\ldots,a_l\}$ where $a_i \in G_S$ for $1 \leq i \leq l \in \mathbf{N}$. Then the set $\{S\setminus a_1,\ldots,S\setminus a_l\}$ is a maximal antichain in $(M_S,\subseteq)$.

Proposition 4.10. *Let* $(A, B) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta \cup \Sigma_1^S)$ *and define* $A_2 := ([a_1)_{G_S} \setminus \{a_1\}) \cup \cdots \cup ([a_l)_{G_S} \setminus \{a_l\})$. *Then we have*

1. $A_3 \in \mathfrak{U}(G_S, M_S, \Delta)$ *for all* A_3 *with* $A_2 \subseteq A_3 \subseteq A_1$.
2. $[(A_2, A_2^\Delta), (A_1, A_1^\Delta)]$ *is a maximal Boolean interval of* $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ *of cardinality* 2^l.

Proof. 1. A_3 is an order filter of $(G_S, \subseteq)$. From the proof of Theorem 3.2 we know that each order filter of $(G_S, \subseteq)$ is an extent of $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$.

2. If $[(A_2, A_2^\Delta), (A_1, A_1^\Delta)]$ would not be maximal, then a maximal Boolean interval containing the given one would have more than l atoms and more than l coatoms. Then either (A_1, A_1^Δ) would have more than l lower covers or (A_2, A_2^Δ) would have more than l upper covers or both. We check the lower covers of (A_1, A_1^Δ). Let (C_1, D_1) be an element of $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ with $(C_1, D_1) \prec (A_1, A_1^\Delta)$. Then there exists an $i \in \{1, \ldots, l\}$ with $a_i \notin C_1$. Hence $(C_1, D_1) \in [(A_2, A_2^\Delta), (A_1, A_1^\Delta)]$. To check the upper covers of (A_2, A_2^Δ) the proof follows dually.

This proves the maximality; the rest follows from 1. □

From Theorem 3.2 and Lemma 4.9 we get that the blocks of $\Sigma(\underline{\mathfrak{B}}(G_S, M_S, \Delta))$ are the maximal Boolean intervals of $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$. In the following proposition we use this fact to describe the intersection of two covering blocks.

Proposition 4.11. *Let* B_1, B_2 *be blocks of* $\Sigma(\underline{\mathfrak{B}}(G_S, M_S, \Delta))$ *with* $B_1 \prec B_2$. *Then* $2|B_1 \cap B_2| = \min\{|B_1|, |B_2|\}$.

Proof. Let A_1 and A_2 denote the extents of the maximal elements of the blocks B_1 and B_2. We define $\mathrm{Min}\,(A_1) := \{a_1, \ldots, a_l\}$ and $\mathrm{Min}\,(A_2) := \{b_1, \ldots, b_m\}$ where $a_i, b_j \in G_S$ for $1 \leq i \leq l$ and $1 \leq j \leq m$. Then $A_1 = [a_1)_{G_S} \cup \cdots \cup [a_l)_{G_S}$ and $A_2 = [b_1)_{G_S} \cup \cdots \cup [b_m)_{G_S}$. From Proposition 4.4 we know that $C_1, C_2 \in M_1^S$ defining $C_1 := \{(S \setminus a_1), \ldots, (S \setminus a_l)\}$ and $C_2 := \{(S \setminus b_1), \ldots, (S \setminus b_m)\}$. Since $B_1 \prec B_2$, from Lemma 4.9 and Proposition 4.3 we get that $C_1 \prec C_2$. From Corollary 4.8 we know that $|C_1 \setminus C_2| = 1$ or $|C_2 \setminus C_1| = 1$. Therefore we have $|C_1 \cap C_2| = \min\{|C_1|, |C_2|\} - 1$. Hence it follows for the blocks that $|B_1 \cap B_2| = \min\{|B_1|, |B_2|\} \cdot 2^{-1}$. This proves the statement of the proposition. □

Proposition 4.11 shows that the intersection of two covering blocks in $FBD(S, \leq)$ is a Boolean lattice half the size of the smaller one of the two blocks.

5. Embedding of $FBD(S, \leq)$ into the 1-Skeleton of $FBD(S \cup \{w\}, \leq)$

Again we consider the free distributive lattice generated by an antichain. One observes that for every element of $FBD(n)$ there exists a maximal Boolean interval (block) of $FBD(n+1)$ such that the number of upper and lower covers of the element in $FBD(n)$ is the number of atoms of that block. R. Wille has shown in Proposition 5 and Corollary 8 in [**14**] that $FBD(n)$ is a 0-1-sublattice of $S_1(FBD(n+1))$. In this section we give an embedding of $FBD(S,\leq)$ into $S_1(FBD(S^1,\leq))$, where S^1 is the set S plus one element noncomparable to all elements of S. This will be proved in Proposition 5.2. We define $X' := X^{\Delta\cup\Sigma_1^{S^1}}$ for $X \in G_{S^1}\cup M_{S^1}$ and $X'^S := X^\Delta$ for $X \in G_S \cup M_S$.

Theorem 5.1. *Let ι be the map from $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ into $\underline{\mathfrak{B}}(G_{S^1}, M_{S^1}, \Delta\cup\Sigma_1^{S^1})$ defined by $\iota(A,B) := (A'', A')$ for $(A,B) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta)$. Then ι is an embedding.*

Proof. Let w denote the element of $S^1 \setminus S$. Clearly $G_S \subseteq G_{S^1}$ and $M_S \subseteq M_{S^1}$. Since $w \notin X \cup Y$, for all $X \in G_S$ and for all $Y \in M_S$ we get $\Delta = (\Delta\cup\Sigma_1^{S^1}) \cap (G_S \times M_S)$. Therefore by Lemma 2.2 ι is an order-embedding.
ι **is $\vee$-preserving:** Let $(A,B),(C,D) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta)$. Since ι is order preserving, we have that $\iota(A,B)\vee\iota(C,D) \leq \iota((A,B)\vee(C,D))$. It remains to show that $(B\cap D)'^S \subseteq (B'^S \cup D'^S)$.

In contrary we suppose that there exists an $y \in (B\cap D)'^S \setminus (B'^S\cup D'^S)$. Then there exists $b \in B$ with $(y,b) \notin \Delta$ and $d \in D$ with $(y,d) \notin \Delta$. B and D are orderfilters, so $b\cup d \in B\cap D$. Then $y\cap(b\cup d) = \emptyset$ yields a contradiction to the last supposition.
ι **is $\wedge$-preserving:** Let $(A,B),(C,D) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta)$. Since ι is order preserving, we have that $\iota(A,B)\wedge\iota(C,D) \geq \iota((A,B)\wedge(C,D))$. It remains to show that $(A''\cap C'') \subseteq (A\cap C)''$.

In contrary we suppose $(A\cap C)'' \subset (A''\cap C'')$. $(A\cap C)''$ and $(A''\cap C'')$ are orderfilters in $(G_{S^1},\subseteq)$, hence there exists $y \in \mathrm{Min}((A\cap C)'')$ and $p \in y$ such that $x := y\setminus\{p\} \in (A''\cap C'') = (A'\cup C')'$. We define $I := \Delta\cup\Sigma_1^{S^1}$. Because of $(x,(S^1\setminus y)) \notin I$ we have $(S^1\setminus y) \notin (A'\cup C')$. Hence there exists $a \in A$ with $(a,(S^1\setminus y)) \notin I$ and $c \in C$ with $(c,(S^1\setminus y)) \notin I$. Therefore we have $a \subset y$ and $c \subset y$, hence $(a\cup c) \subseteq y$ with $(a\cup c) \in (A\cap C) \subseteq (A\cap C)''$. Since y is minimal in $(A\cap C)''$ it follows that $y = a\cap c$; thus, $w \notin y$ and $w \notin x$. Therefore we have $(x,(S^1\setminus(x\cup\{w\})) \notin I$ and hence we get $S^1\setminus(x\cup\{w\}) \notin (A'\cup C')$. As above there exists $a_1 \in A$ with $(a_1,(S^1\setminus(x\cup\{w\})) \notin I$ and $c_1 \in C$ with $(c_1,(S^1\setminus(x\cup\{w\})) \notin I$. Hence $a_1 \subset (x\cup\{w\})$ and $c_1 \subset (x\cup\{w\})$. From $a_1 \in A$ and $c_1 \in C$ we get $a_1 \subseteq x$ and $c_1 \subseteq x$. Since A and C are orderfilters we have $(a_1\cup c_1) \in (A\cap C) \subseteq (A\cap C)''$ and $(A-1\cup c_1) \subseteq x \subset y$ which contradicts the minimality of y in $(A\cap C)''$. Therefore ι is an embedding. □

Proposition 5.2. *The number of coverings of an element $(A,B) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta)$ is equal to the number of elements of $\alpha(\iota(A,B))$, where ι is the embedding of Theorem 5.1 from $\underline{\mathfrak{B}}(G_S, M_S, \Delta)$ into $\underline{\mathfrak{B}}(G_{S^1}, M_{S^1}, \Delta\cup\Sigma_1^{S^1})$ and α is the isomorphism of Proposition 4.4 from $\underline{\mathfrak{B}}(G_{S^1}, M_{S^1}, \Delta\cup\Sigma_1^{S^1})$ onto $(M_1^{S^1},\leq)$.*

Proof. Let w denote the element of $S^1 \setminus S$. From Theorem 3.2 we know that $(\{(F, F^\Delta) \mid F \text{ orderfilter in } (G_S, \subseteq)\}, \leq) \cong \underline{\mathfrak{B}}(G_S, M_S, \Delta)$. Therefore we get

$$\begin{aligned}
&|\{(C,D) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta) | (C,D) \prec (A,B) \quad \text{or} \quad (C,D) \succ (A,B)\}| \\
&= |\{(C,D) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta) | (C,D) \prec (A,B)\}| \\
&\quad + |\{(C,D) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta) | (C,D) \succ (A,B)\}| \\
&= |\operatorname{Min}(A)| + |\operatorname{Max}(G_S \setminus A)| .
\end{aligned}$$

On the other hand we have $|\alpha(\iota(A,B))| = |\alpha(A'', A')| = |A_1'' \cap A'|$ where $A_1'' := \{(S^1 \setminus a) \mid a \in A''\}$ and $(A_1'' \cap A') \in M_1^{S^1}$. From Lemma 4.5 we know $\operatorname{Min}(A') \in M_1^{S^1}$ and hence we have $(A_1'' \cap A') = \operatorname{Min}(A')$.

Next we prove $\operatorname{Min}(A') = \{(S^1 \setminus a) \mid a \in \operatorname{Min}(A)\} \,\dot\cup\, \operatorname{Min}(B)$. We abbreviate $\{(S^1 \setminus a) \mid a \in \operatorname{Min}(A)\}$ by $\operatorname{Max}(A_1)$.

$\underline{\subseteq:}$ First not that the two sets on the right are disjoint and that they form an antichain, because for $b \in \operatorname{Min}(B)$ and $(y \cup w) \in \operatorname{Max}(A_1)$, $b \subset (y \cup w)$ leads to $b \subseteq y$. But $(S \setminus y) \cap b = \emptyset$ contradicts $(S \setminus y) \in A$. Now suppose there exists $x \in (\operatorname{Min}(A') \setminus \operatorname{Min}(B))$. It follows $w \in x$. Let $z := (x \setminus w)$. Then there exists an $a \in \operatorname{Min}(A)$ with $a \cap z = \emptyset$, since $x \in \operatorname{Min}(A')$ and $a \cup z \subset S^1$. Therefore we have $a \cap x = \emptyset$, and $x \in A'$ leads to $a \cup x = S^1$, hence $x = S^1 \setminus a$.

$\underline{\supseteq:}$ Let $b \in \operatorname{Min}(B)$. Then we have $a \,\Delta\, b$ for all $a \in A$ and hence $\operatorname{Min}(B) \subseteq A'$. For every lower cover $b_1 \prec_{G_S} b$ there exists an $a_1 \in \operatorname{Min}(A)$ such that $a_1 \cap b_1 = \emptyset$, thus $b_1 \notin A'$. This implies $\operatorname{Min}(B) \subseteq \operatorname{Min}(A')$. On the other hand we have $x \in \operatorname{Min}(A')$ for $x \in \{(S^1 \setminus a) \mid a \in \operatorname{Min}(A)\}$. This proves $\operatorname{Min}(A') = \{(S^1 \setminus a) \mid a \in \operatorname{Min}(A)\} \,\dot\cup\, \operatorname{Min}(B)$.

Because of $B = A^\Delta = \{Y \in M_S \mid (S \setminus Y) \notin A\} = \{Y \in M_S \mid (S \setminus Y) \in (G_S \setminus A)\}$, we have $\operatorname{Min}(B) = \{Y \in M_S \mid (S \setminus Y) \in \operatorname{Max}(G_S \setminus A)\}$. We conclude $|\alpha(\iota(A,B))| = |\operatorname{Min}(A')| = |\{(S^1 \setminus a) \mid a \in \operatorname{Min}(A)\}| + |\operatorname{Min}(B)| = |\operatorname{Min}(A)| + |\operatorname{Max}(G_S \setminus A)| = |\{(C,D) \in \underline{\mathfrak{B}}(G_S, M_S, \Delta) | (C,D) \prec (A,B) \text{ or } (C,D) \succ (A,B)\}|$. □

For every ordered set $(S, \leq)$, the $FBD(S, \leq)$ can be embedded into $FBD(n)$ where $n = |S|$. The same holds for the 1-skeletons. But the embedding of Theorem 5.1 does not work for r-skeletons with $r \geq 2$. As an example we take $FBD(S, \leq)$ and $FBD(S)$, where S is the three-element antichain and $(S, \leq) \cong \underline{1} + \underline{2}$. In the following figures the sublattices of $FBD(3)$ and $S_1(FBD(3))$ consisting of the filled circles are isomorphic to the lattices $FBD(\underline{1} + \underline{2})$ and $S_1(FBD(\underline{1} + \underline{2}))$, respectively. For the second skeletons there is no embedding from $S_2(FBD(\underline{1} + \underline{2}))$ into $S_2(FBD(3))$. Note that the incidence relation of the underlying context of the 2-skeleton of $FBD(S, \leq)$ is a proper subset of $\Delta \cup \Sigma_2^S$ (see Table 1).

0

0

2

3 1 2 3

12 23

123 12 13 23

123

Figure 7. $FBD(\underline{1}+\underline{2})$. **Figure 8.** $FBD(3)$.

0 0

2

3 1 2 3

$12, 23$ $12, 13, 23$

123 123

Figure 9. $S_1(FBD(\underline{1}+\underline{2}))$. **Figure 10.** $S_1(FBD(3))$.

0

2 0

3 $1, 2, 3$

$12, 23, 123$ $12, 13, 23, 123$

Figure 11. $S_2(FBD(\underline{1}+\underline{2}))$. **Figure 12.** $S_2(FBD(3))$.

From [**8**] we know that $S_1(FBD(6))$ is not rank-ordered. $S_1(FBD(6))$ contains more than 10^3 elements. For ordered sets we have much smaller examples, e.g., $S_1(FBD(S, \leq))$ where $(S, \leq) \cong \underline{2}+\underline{1}+\underline{1}$ is not rank-ordered but contains only 15 elements.

Outlook

To construct $FBD(S, \leq)$ from its r-skeletons does not seen to be too promising. For such a construction the blockrelations $\Delta \cup \Sigma_r^S$ seem to be more useful. For an antichain S, in [**13**] R. Wille constructed $FBD(S)$, by taking the concept lattices of parts of the context $(\mathfrak{P}(S), \mathfrak{P}(S), \Delta)$ and gluing them together properly by using specific mappings. A similar idea may work for $FBDs$ over finite ordered sets.

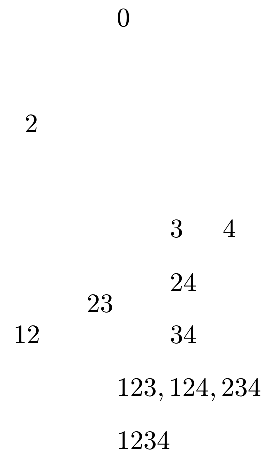

Figure 13. $S_1(FBD(\underline{1}+\underline{1}+\underline{2}))$.

One can start with the factor-lattice of the r-th level and replace each point by the corresponding block to receive the $(r-1)$-th level. It could be fruitful to look for such a level construction of $FBD(S,\leq)$. This paper does not solve the equation of constructing $FBD(S,\leq)$. At least the characterization of the covering of the blocks in $FBD(S,\leq)$ clears up the local structure of $FBD(S,\leq)$.

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