Acta Math. Univ. Comenianae
Vol. LXV, 2(1996), pp. 247–279


# GENERALIZED DIFFERENCE POSETS AND ORTHOALGEBRAS

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Abstract. A difference on a poset $(P, \leq)$ is a partial binary operation $\ominus$ on $P$ such that $b \ominus a$ is defined if and only if $a \leq b$ subject to conditions $a \leq b \implies b \ominus (b \ominus a) = a$ and $a \leq b \leq c \implies (c \ominus a) \ominus (c \ominus b) = b \ominus a$. A difference poset (DP) is a bounded poset with a difference. A generalized difference poset (GDP) is a poset with a difference having a smallest element and the property $b \ominus a = c \ominus a \implies b = c$. We prove that every GDP is an order ideal of a suitable DP, thus extending previous similar results of Janowitz for generalized orthomodular lattices and of Mayet-Ippolito for (weak) generalized orthomodular posets. Various results and examples concerning posets with a difference are included.

## 0. Introduction

A difference (operation) on a partially ordered set (poset) $P$ is a partial binary operation $\ominus$ on $P$ such that $b \ominus a$ is defined if and only if $a \leq b$ satisfying some conditions. For example, $b \wedge a'$ is such an operation in an orthomodular poset. A difference poset (DP) is a bounded poset equipped with a difference operation. For example, every orthoalgebra (which is a natural generalization of an orthomodular lattice or poset) is a difference poset.

An introduction to difference posets is in [**K, Ch**]. A basic theory of orthoalgebras can be found in [**F, G, R**]. An orthoalgebra (OA) is defined as a partial binary algebra with a sum (operation) $\oplus$ on a set with two special elements. An exact relationship between difference posets and orthoalgebras was pointed out in [**N, P**]. A description of an orthoalgebra in terms of a difference operation on a poset is given there. A description of an orthomodular poset, resp. a difference poset in terms of a sum operation on a set is given in [**B, M**], resp. [**P**] and [**F, B**]. Yet more general approach is used in [**K, R**] when considering a difference operation on an arbitrary set with a special element.

We define a generalized difference poset (GDP) as a poset with a smallest element and with a difference operation subject to an additional condition in

Received December 16, 1996.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 06A06, 08A55, 06C15.
*Key words and phrases.* Difference (operation) on a poset, orthomodular poset, orthoalgebra, difference poset, order ideal, orthogonality relation, sum (operation) on a set, orthomodular group.

such a way that every order ideal of a difference poset is a generalized difference poset. We prove that every generalized difference poset $P$ is an order ideal of a difference poset $\hat{P}$. Similar results were obtained in [**J1**] and [**M-I**] for generalized orthomodular lattices (see also [**B**], [**K**]) and (weak) generalized orthomodular posets, respectively.

After a definition of a generalized difference poset, a generalized orthoalgebra (GOA) is then defined in a natural way and also by means of a difference operation. A simpler structure with sum operation has been described in [**Kr**]. A weak generalized orthomodular poset (WGOMP) is also characterized in terms of a difference operation. It is shown that if $P$ is a generalized difference poset then $\hat{P}$ is an orthoalgebra if and only if $P$ is a generalized orthoalgebra. Moreover, $\hat{P}$ is an orthomodular poset if and only if $P$ is a weak generalized orthomodular poset, and the construction of $\hat{P}$ coincides with that in [**M-I**].

We conclude our paper with a series of examples of GDPs. In particular, we study abelian groups with a special partial order introduced in [**Ch**] under the name "orthomodular groups". Actually, we study a generalization of orthomodular groups. We prove that an orthomodular group is always a generalized orthomodular poset (GOMP) (in [**Ch**], there is proved that an orthomodular group is a WGOMP).

As a further generalization of an orthomodular group we study subsets of abelian groups with a special partial order. We prove for example, that sets of idempotents (projections) in rings ($*$-rings) satisfying special conditions form WGOMPs.

In another concrete example motivated by a theory of triple systems (alternative and Jordan triples), we introduce a "triple group" as an abelian group endowed with a ternary operation, and prove that the set of all tripotents in it forms a WGOMP. As a special case, the set of all tripotents in a JBW*-triple ([**Ba**]) forms a GOMP. Using triple groups, known partial orders on idempotents (projections) in rings ($*$-rings) are extended to tripotents and it is shown that they remain to form WGOMPs.

For general theory of orthomodular lattices and orthomodular posets we refer to [**B**], [**K**], [**P, P**].

## 1. Posets with a Difference

**Definition 1.1.** ([**K, Ch**]) Let $(P, \leq)$ be a partially ordered set (poset). A partial binary operation $\ominus$ on $P$ such that $b \ominus a$ is defined if and only if $a \leq b$ is called a *difference* on $(P, \leq)$ if the following conditions are satisfied for all $a, b, c \in P$:

(D1) If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.
(D2) If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

**Proposition 1.2.** *Let $(P, \le, \ominus)$ be a poset with a difference and let $a, b, c, d \in P$. The following assertions are true:*

(i) *If $a \le b \le c$ then $b \ominus a \le c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*
(ii) *If $b \le c$ and $a \le c \ominus b$ then $a \le c$ and $b \le c \ominus a$, and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.*
(iii) *If $a \le b \le c$ then $b \ominus a \le c$ and $a \le c \ominus (b \ominus a)$, and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.*
(iv) *If $a, b \le c$ and $c \ominus a = c \ominus b$ then $a = b$.*
(v) *If $d \le a, b \le c$ and $c \ominus a = b \ominus d$ then $c \ominus b = a \ominus d$.*
(vi) *If $a \le b, c \le d$ and $b \ominus a = c \ominus a$ then $b = c$.*

*Proof.* Conditions (i)–(v) are proved in [**K, Ch**]. To prove (vi) let $a \le b, c \le d$ and $b \ominus a = c \ominus a$. Then $(d \ominus a) \ominus (d \ominus b) = b \ominus a = c \ominus a = (d \ominus a) \ominus (d \ominus c)$, hence $d \ominus b = d \ominus c$ and thus $b = c$. □

**Remark 1.3.** We show that it can be introduced another system of axioms not using the order relation, this means that a poset with a difference operation can be characterized as a set with a "difference" operation. Namely, let us observe that according to Proposition 1.2(ii), every difference operation $\ominus$ on a poset $(P, \le)$ has the following properties ($a, b, c \in P$):

(d1) If $b \ominus a$ is defined then $b \ominus (b \ominus a)$ is defined and $b \ominus (b \ominus a) = a$.
(d2) If $(a \ominus b) \ominus c$ is defined then $(a \ominus c) \ominus b$ is defined and $(a \ominus b) \ominus c = (a \ominus c) \ominus b$.
(d3) $a \ominus b$ and $b \ominus a$ are defined if and only if $a = b$.

And conversely, as shown in the following result, every nonempty set $P$ equipped with a partial binary operation $\ominus$ satisfying conditions (d1), (d2) and (d3), can be endowed with a partial order $\le$ (having but one meaning) in such a way that $\ominus$ becomes a difference operation on the poset $(P, \le)$.

**Proposition 1.4.** *If $\ominus$ is a partial binary operation on a nonempty set $P$ having properties* (d1), (d2) and (d3) *and if $\le$ is a binary relation on $P$ given by $a \le b$ if and only if $b \ominus a$ is defined, then $\le$ is a partial order on $P$ and $\ominus$ is a difference on the poset $(P, \le)$.*

*Proof.* According to (d3), $\le$ is reflexive and antisymmetric. To prove transitivity let $a, b, c \in P$ be such that $a \le b$ and $b \le c$. Using (d1) and (d2) we obtain

$$b \ominus a = (c \ominus (c \ominus b)) \ominus a = (c \ominus a) \ominus (c \ominus b),$$

hence $c \ominus a$ is defined, which means that $a \le c$. Thus $\le$ is a partial order on $P$.

Condition (D1) follows from (d1) and condition (D2) is clear from the proof of transitivity of $\le$. Thus $\ominus$ is a difference on the poset $(P, \le)$. □

**Lemma 1.5.** *Let $(P, \le, \ominus)$ be a poset with a difference. If $a, b \in P$ and $a \le b$, then $a \ominus a = b \ominus b$.*

*Proof.* Follows directly from Proposition 1.2(iii) if we put $c = b$. □

**Proposition 1.6.** *Every poset $P$ with a difference can be written as a disjoint union of posets with a difference, each of which contains a smallest element.*

*Proof.* Let $R$ be a binary relation on $P$ defined by $aRb$ iff $a$ and $b$ are comparable, i.e., $a \le b$ or $b \le a$. Clearly, $R$ is reflexive and symmetric relation. Let $\tilde{R}$ denote the transitive closure of $R$, that is, $a\tilde{R}b$ iff there are $c_1, \dots, c_n$ in $P$ with $a = c_1$, $b = c_n$ and $c_i R c_{i+1}$ for $i = 1, \dots, n-1$. Every equivalence class with respect to $\tilde{R}$ is a poset with a difference, and Lemma 1.5 implies that each of them contains its smallest element. □

**Lemma 1.7.** *Let $(P, \le, \ominus)$ be a poset with a difference. If $0$ is the smallest element in $P$ then for all $a \in P$, $a \ominus a = 0$ and $a \ominus 0 = a$. Moreover, for all $a, b \in P$ with $a \le b$ we have $b \ominus a = 0$ iff $a = b$ and $b \ominus a = b$ iff $a = 0$. If $1$ is the greatest element in $P$ then $1 \ominus 1$ is the smallest element in $P$.*

*Proof.* If 0 is the smallest element in $P$ and if $a \in P$ then by Lemma 1.5, $a \ominus a = 0$ and then $a \ominus 0 = a \ominus (a \ominus a) = a$. If $a, b \in P$, $a \le b$ and $b \ominus a = 0$ then $b = b \ominus 0 = b \ominus (b \ominus a) = a$. If $a, b \in P$, $a \le b$ and $b \ominus a = b$ then $0 = b \ominus b = b \ominus (b \ominus a) = a$.

If 1 is the greatest element in $P$, then by Lemma 1.5, $1 \ominus 1$ is the smallest element in $P$. □

**Proposition 1.8.** *Let $(P, \le)$ be a poset with the smallest element $0$ and let $\ominus$ be a partial binary operation on $P$ such that $b \ominus a$ is defined iff $a \le b$. Then $\ominus$ is a difference on $(P, \le)$ if and only if the following two conditions are satisfied for all $a, b, c \in P$:*

(i) *$a \ominus 0 = a$.*
(ii) *If $a \le b \le c$ then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

*Proof.* Assume that (i) and (ii) are satisfied. If $a, b \in P$ with $a \le b$ then from $0 \le a \le b$ it follows $b \ominus a \le b \ominus 0 = b$ and $b \ominus (b \ominus a) = a \ominus 0 = a$ which proves (D1).

The converse assertion is clear. □

Let us note that in [**D, P**] there is the following characterization of a poset with a difference having a smallest element, not using the order relation.

**Proposition 1.8*.** *Let $P$ be a set with a special element $0$ endowed with a partial binary operation $\ominus$. Let $\le$ be a binary relation on $P$ given by $a \le b$ if and only if $b \ominus a$ is defined. Then $\le$ is a partial order on $P$ with the smallest element $0$ and $\ominus$ is a difference on $(P, \le)$ if and only if the following conditions are satisfied for all $a, b, c \in P$:*

(01) *$a \ominus 0$ is defined, and $a \ominus 0 = a$;*
(02) *$a \ominus a$ is defined;*
(03) *If $b \ominus a$ and $c \ominus b$ are defined, then $c \ominus a$ and $(c \ominus a) \ominus (c \ominus b)$ are defined, and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$;*

(04) *If* $0 \ominus a$ *is defined, then* $a = 0$.

Let us observe that every poset with a difference having a smallest element is an RI-set in the sense of [**K, R**]. The converse is not true.

The following notion was introduced in [**K, Ch**] (see also [**N, P**]).

**Definition 1.9.** Let $(P, \le, \ominus)$ be a poset with a difference and let 0 and 1 be the smallest and greatest elements in $P$, respectively. The structure $(P, \le, 0, 1, \ominus)$ is called a *difference poset* (D-*poset, DP*).

Let us note that every interval $[0, a]$ of $(P, \le, 0, \ominus)$, a poset with a difference having a smallest element 0, is a difference poset $([0, a], \le, 0, a, \ominus)$, where $\le$ and $\ominus$ are inherited from $P$.

Let us observe that the following condition (a strengthening of (vi) in Proposition 1.2) need not be satisfied in a poset with a difference $(P, \le, \ominus)$, $(a, b, c \in P)$:

(C) If $a \le b, c$ and $b \ominus a = c \ominus a$, then $b = c$.

A simplest such an example is on Fig. 1, where $b \ominus a = c \ominus a = a$, $x \ominus x = 0$ and $x \ominus 0 = x$ for all $x$ (by Lemma 1.7, $\ominus$ is a unique difference on the poset).

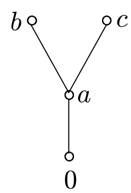


**Fig. 1**

By Proposition 1.2(vi), condition (C) is however satisfied in every difference poset $(P, \le, 0, 1, \ominus)$. In order to obtain a generalization $(P, \le, 0, \ominus)$ of a difference poset embeddable in a difference poset it appears that it is necessary to include condition (C) in a new definition (see the next paragraph).

**Remark 1.10.** Let $(P, \le, \ominus)$ be a poset with a difference satisfying condition (C). This means that for every $a, b \in P$ there is at most one $c \in P$ such that $a = c \ominus b$. Thus property (C) enables us to define a sum operation on $P$, that is, a partial binary operation $\oplus$ on $P$ given by $(a, b, c \in P)$:

(S) $a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $a = c \ominus b$.

The sum operation $\oplus$ on $P$ has as properties $(a, b, c \in P)$:

(S1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity).

(S2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).

(S3) If $a \oplus b$ and $a \oplus c$ are defined and $a \oplus b = a \oplus c$, then $b = c$ (cancellativity).

(S4) For every $a \in P$ there is $b \in P$ such that $a \oplus b$ is defined and $a \oplus b = a$ (zeros existence).

(S5) If $a \oplus a$ and $(a \oplus a) \oplus b$ are defined and if $(a \oplus a) \oplus b = a$, then $a \oplus a = a$ (zeros absorption).

To show conditions (S1)–(S5) we use properties (d1)–(d3) from Remark 1.3.

Condition (S1) follows by (d1). If $a, b, c \in P$ and $a \oplus b = c$ then $a = c \ominus b$, hence $c \ominus a = c \ominus (c \ominus b) = b$, which means that $b \oplus a = c$.

If $a, b, c, d, e \in P$, $a \oplus b = d$ and $d \oplus c = e$, then $b = d \ominus a$ and $d = e \ominus c$, hence by (d2), $b = (e \ominus c) \ominus a = (e \ominus a) \ominus c$. This means that $b \oplus c = e \ominus a$ and then $a \oplus (b \oplus c) = e$, which proves condition (S2).

If $a, b, c, d \in P$ with $a \oplus b = a \oplus c = d$ then $b = d \ominus a = c$ which shows condition (S3).

By (d3) and (d1), for every $a \in P$, $a \oplus (a \ominus a) = a$, which proves condition (S4).

To show (S5), let $a, b \in P$ be such that $(a \oplus a) \oplus b = a$. Then $a \oplus a = a \ominus b$, hence $a = (a \ominus b) \ominus a$. By (d1), $a \ominus (a \ominus b) = b$ which by (d3) gives $a \ominus b = a$. Therefore $a \oplus a = a$.

Conversely, if $(P, \oplus)$ is a partial binary algebra having at least property (S3), then a partial binary operation $\ominus$ on $P$ is enabled by the cancellativity (S3): for every $a, b \in P$ there is at most one $c \in P$ such that $a \oplus c = b$. $\ominus$ is then given by the following condition ($a, b, c \in P$):

(D) $b \ominus a$ is defined and $b \ominus a = c$ if and only if $a \oplus c$ is defined and $a \oplus c = b$.

Let $(P, \leq)$ be a poset and let $\ominus$ be a difference operation on $(P, \leq)$. The partial binary operation $\ominus$ on $P$ will be called a *cancellative difference* on $(P, \leq)$ if condition (C) is satisfied. The following result shows that there is a one to one correspondence between posets with a cancellative difference $\ominus$ and partial binary algebras with a sum operation $\oplus$ satisfying (S1)–(S5).

**Theorem 1.11.** *If* $(P, \leq, \ominus)$ *is a poset with a cancellative difference and if a partial binary operation* $\oplus$ *on* $P$ *is defined by* (S)*, then conditions* (S1)–(S5) *and* (D) *are satisfied. Conversely, if* $(P, \oplus)$ *is a partial binary algebra having properties* (S1)–(S5) *and if a partial binary operation* $\ominus$ *on* $P$ *is defined by* (D)*, then conditions* (d1)–(d3)*,* (C) *and* (S) *are satisfied, that is,* $P$ *becomes a poset with a cancellative difference.*

*Proof.* From $\ominus$ to $\oplus$. Conditions (S1)–(S5) are proved in Remark 1.10. If $a, b, c \in P$ then by (d1), $b \ominus a$ is defined and $b \ominus a = c$ if and only if $b \ominus c$ is defined and $b \ominus c = a$, which proves condition (D).

From $\oplus$ to $\ominus$. (d1) follows from (S1) and (d2) follows from (S1) and (S2). (S3) implies (C). By (S4), for every $a \in P$, $a \ominus a$ is defined. To finish the proof of (d3), let $a, b, c, d \in P$ be such that $a \ominus b = c$ and $b \ominus a = d$. This means that $b \oplus c = a$ and $a \oplus d = b$, hence $a = (a \oplus d) \oplus c$ and hence $a \oplus d = ((a \oplus d) \oplus c) \oplus d$. From this, using (S1), (S2) and (S3) we obtain $d = (d \oplus d) \oplus c$, which by (S5) gives $d \oplus d = d$.

Hence $d = d \oplus c$ and thus $a = a \oplus (d \oplus c) = a \oplus d = b$. By Proposition 1.4, $P$ is a poset with a cancellative difference. It remains to prove (S) to see that the correspondence is one to one. But this is clear from (S1). □

**Remark 1.12.** Let $(P, \leq, \ominus)$ be a poset with a cancellative difference. Consider the sum operation $\oplus$ on $P$ given by condition (S) (see Remark 1.10). Two elements $a, b \in P$ are said to be orthogonal (in notation $a \perp b$) if $a \oplus b$ is defined, i.e. if there is $c \in P$ with $a = c \ominus b$, or equivalently, if there is a unique $c \in P$ such that $a = c \ominus b$. The binary relation $\perp$ on $P$ is symmetric, i.e. for all $a, b \in P$, $a \perp b$ implies $b \perp a$ (since $\oplus$ is commutative). $\perp$ is called the *orthogonality relation* of $(P, \leq, \ominus)$.

If $a, b, c \in P$ with $a \leq c \ominus b$ then by (d1) and (d2) it follows $a = (c \ominus b) \ominus ((c \ominus b) \ominus a) = (c \ominus ((c \ominus b) \ominus a)) \ominus b$, which means that $a \perp b$. Moreover, $a \oplus b = c \ominus ((c \ominus b) \ominus a)$. Since $a \oplus b \leq c$, we get by (d1), $c \ominus (a \oplus b) = (c \ominus b) \ominus a$. If $a, b, c \in P$ with $a \oplus b \leq c$ then by (S) and Proposition 1.2(i), from $b \leq a \oplus b \leq c$ it follows $a = (a \oplus b) \ominus b \leq c \ominus b$. So, we have shown the following properties ($a, b, c \in P$):

(i) $a \perp b$ if and only if $a \leq d \ominus b$ for some $d \in P$.
(i)* $a \leq b$ and $b \perp c$ implies $a \perp c$.
(ii) If $a \leq c \ominus b$ then $a \oplus b = c \ominus ((c \ominus b) \ominus a)$.
(iii) If $a \leq c \ominus b$ then $(c \ominus b) \ominus a = c \ominus (a \oplus b)$.
(iii)* If $a \oplus b \leq c$ then $c \ominus (a \oplus b) = (c \ominus b) \ominus a$.

Let us note that a kind of orthomodularity holds in $P$ ($a, b \in P$):

(iv) If $a \leq b$ then $b = a \oplus (b \ominus a)$.

In particular, $a = a \oplus (a \ominus a)$ for every $a \in P$. Hence, if $P$ has a smallest element 0, then for all $a \in P$:

(v) $a \oplus 0 = a$.

If $a, b \in P$ and $a \oplus b = 0$ then $a = 0 \ominus b \leq 0$ which with $0 \leq a, b$ gives $a = b = 0$. Thus the following condition is satisfied for all $a, b \in P$:

(vi) $a \oplus b = 0 \implies a = b = 0$.

Another consequence of the "orthomodular law" is as follows ($a, b \in P$):

(vii) $a \leq b$ if and only if $a \oplus c = b$ for some $c \in P$.
(vii)* $a \leq b$ if and only if $a \oplus c = b$ for a unique $c \in P$.

Let us observe that if $(P, \oplus, 0)$ is a partial binary algebra with a special element 0 then in the presence of conditions (S1)–(S3), conditions (v) and (vi) imply conditions (S4) and (S5). Thus we have the following consequence of Theorem 1.11.

**Corollary 1.13.** *There is a one to one correspondence analogous to that in Theorem* 1.11*, between posets with a cancellative difference having a smallest element* 0 *and partial binary algebras with a sum operation and with a special element* 0 *having properties* (S1)–(S3) *and (v), (vi).*

**Remark 1.14.** Let $(P, \leq, 0, 1, \ominus)$ be a difference poset. Then for all $a, b \in P$ it is true:

$$\begin{aligned}&a \oplus b \text{ is defined iff } a \leq 1 \ominus b \text{ and}\\ &a \oplus b = 1 \ominus ((1 \ominus b) \ominus a),\\ &a \perp b \text{ iff } a \leq 1 \ominus b \text{ (equivalently, } \ b \leq 1 \ominus a).\end{aligned}$$

Consider a unary operation $'$ on $P$ given by: $a' = 1 \ominus a,\ a \in P$. Then $a'' = a$ and $a \leq b$ implies $b' \leq a'$ for all $a, b \in P$. A smallest example showing that $P$ need not be orthocomplemented is a three element chain $0 < a < 1$ (by Lemma 1.7, there is a unique way to make it into a difference poset) with $a = 1 \ominus a$, $x \ominus x = 0$ and $x \ominus 0 = x$ for all $x$.

A more precise relationship between D-posets and orthocomplemented posets is as follows. If $(P, \leq, 0, 1, \ominus)$ is a D-poset and $'$ is a unary operation on $P$ given by $a' = 1 \ominus a, a \in P$, then $(P, \leq, 0, 1, ')$ is an orthocomplemented poset if and only if for all $a \in P$, $a \leq 1 \ominus a$ implies $a = 0$. The latter condition is considered below in a connection with orthoalgebras (see Proposition 3.2). On the other hand, to characterize those orthocomplemented posets $(P, \leq, 0, 1, ')$ for which there can be defined a (unique) difference $\ominus$ on $P$ such that $a' = 1 \ominus a$ for all $a \in P$, is not so easy.

## 2. Generalized Difference Posets

From the preceding paragraph we have enough results about posets with a difference to introduce the following definition.

**Definition 2.1.** Let $(P, \leq, \ominus)$ be a poset with a cancellative difference containing a smallest element 0. The system $(P, \leq, 0, \ominus)$ is called a generalized difference poset (GDP).

In what follows we shall use also abbreviations as generalized D-poset, generalized DP and GD-poset.

Let us observe that every order ideal $J$ of a difference poset $(P, \leq, 0, 1, \ominus)$ is a generalized D-poset $(J, \leq, 0, \ominus)$ where $\leq, 0$ and $\ominus$ are inherited from $P$.

The set $R^+$ of all nonnegative real numbers with the usual difference of numbers is an example of a generalized difference poset. More generally, the positive cone $G^+$ of any partially ordered abelian group $(G, +, 0, \leq)$ with the usual difference of group elements is a generalized difference poset.

**Remark 2.2.** Let $(P, \leq, 0, \ominus)$ be a GDP. According to Lemma 1.7 and Proposition 1.2(ii), $P$ is an abelian RI-poset in the sense of [**K, R**]. Conversely, by Propositions 1.2(i) and 1.3(ii) of [**K, R**], every abelian RI-poset is a GD-poset.

**Remark 2.3.** Following the previous version of [**F, B**], call a system $(P, \oplus, 0)$, where $0 \in P$ and $\oplus$ is a partial binary operation on $P$ a cone if conditions

(S1)–(S3) from Remark 1.10 and conditions (v), (vi) from Remark 1.12 are satisfied. According to Corollary 1.13, generalized difference posets are in a one to one correspondence with cones. Cones arose as a convenient generalization of so called effect algebras (see the next paragraph for a definition) which are in an analogous one to one correspondence with difference posets (see [**F, B**]).

Our aim is to show that every generalized difference poset is an order ideal (with special properties) of a difference poset. Similar results have been already obtained for particular structures: generalized orthomodular lattices (which are order ideals of orthomodular lattices) [**J1**] and (weak) generalized orthomodular posets (which are order ideals of orthomodular posets) [**M-I**]. Another related result was obtained for so called relatively orthomodular lattices (which are dual ideals (with special properties) of generalized orthomodular lattices) [**H**].

Let $(P, \leq 0, \ominus)$ be a generalized difference poset. Let $P^\sharp$ be a set disjoint from $P$ with the same cardinality. Consider a bijection $a \mapsto a^\sharp$ from $P$ onto $P^\sharp$ and let us denote $P \cup P^\sharp = \hat{P}$. Define a partial binary operation $\ominus^*$ on $\hat{P}$ by the following rules $(a, b \in P)$:

(i) $b \ominus^* a$ is defined iff $b \ominus a$ is defined and $b \ominus^* a = b \ominus a$.
(ii) $b^\sharp \ominus^* a$ is defined iff $a \oplus b$ is defined and $b^\sharp \ominus^* a = (a \oplus b)^\sharp$.
(iii) $b^\sharp \ominus^* a^\sharp$ is defined iff $a \ominus b$ is defined and $b^\sharp \ominus^* a^\sharp = a \ominus b$.

Define a binary relation $\leq^*$ on $\hat{P}$ as follows: $x \leq^* y$ if and only if $y \ominus^* x$ is defined.

We are to show that the system $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ is a difference poset. With respect to Proposition 1.4 and Lemma 1.7, it suffices to prove that $\ominus^*$ has properties (d1), (d2) and (d3) from Remark 1.3, that $0^\sharp \ominus^* x$ is defined for every $x \in \hat{P}$ and that $0^\sharp \ominus^* 0^\sharp = 0$. This is done in the next theorem.

**Theorem 2.4.** $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ *is a difference poset.*

*Proof.* $0^\sharp \ominus^* 0^\sharp = 0$ since $0 \ominus 0 = 0$. $0^\sharp \ominus^* x$ is defined for every $x \in \hat{P}$ since $a \oplus 0$ and $a \ominus 0$ are defined for all $a \in P$. $x \ominus^* x$ is defined for every $x \in \hat{P}$ since $a \ominus a$ is defined for every $a \in P$. If $x, y \in \hat{P}$ and if $x \ominus^* y$ and $y \ominus^* x$ are defined then clearly $x = y$.

It remains to prove conditions (d1) and (d2). In the rest of the proof we recall results from Remark 1.10 and Remark 1.12.

1) If $a, b \in P$, $a \ominus b$ is defined, $x = a^\sharp$ and $y = b^\sharp$, then by (iv), $x = (b \oplus (a \ominus b))^\sharp = y \ominus^* (a \ominus b) = y \ominus^* (y \ominus^* x)$.

If $a, b \in P$, $a \oplus b$ is defined and $x = b^\sharp$ then by (S), $a = (a \oplus b) \ominus b = x \ominus^* (a \oplus b)^\sharp = x \ominus^* (x \ominus^* a)$.

2) If $a, b, c \in P$, $c \oplus (b \oplus a)$ is defined and $x = a^\sharp$, then by (S1) and (S2), $(x \ominus^* b) \ominus^* c = (b \oplus a)^\sharp \ominus^* c = (c \oplus (b \oplus a))^\sharp = (b \oplus (c \oplus a))^\sharp = (c \oplus a)^\sharp \ominus^* b = (x \ominus^* c) \ominus^* b$.

If $a, b, c \in P$, $c \ominus (b \oplus a)$ is defined, $x = a^\sharp$ and $y = c^\sharp$, then by (iii)*, $(x \ominus^* b) \ominus^* y = (b \oplus a)^\sharp \ominus^* y = c \ominus (b \oplus a) = (c \ominus a) \ominus b = (x \ominus^* y) \ominus^* b$.

If $a,b,c \in P$, $(b \ominus a) \ominus c$ is defined, $x = a^\sharp$ and $y = b^\sharp$, then by (iii), $(x \ominus^* y) \ominus^* c = (b \ominus a) \ominus c = b \ominus (c \oplus a) = (c \oplus a)^\sharp \ominus^* y = (x \ominus^* c) \ominus^* y$. □

Consider a unary operation $'^*$ on $\hat{P}$ given by: $x'^* = 0^\sharp \ominus^* x$, $x \in \hat{P}$. That is, if $a \in P$ then $a'^* = a^\sharp$ and $(a^\sharp)'^* = a$. Let us note that $(\hat{P}, \leq^*)$ contains $P$ as an order ideal with the property: if $a,b \in P$ and $a \perp^* b$ then $a \oplus^* b \in P$. (The sum operation $\oplus^*$ on $\hat{P}$ and the orthogonality relation $\perp^*$ on $\hat{P}$ are defined in $(\hat{P}, \leq^*, \ominus^*)$ as in Remark 1.10 and Remark 1.12.) Indeed, if $a,b \in P$ with $a \perp^* b$, then $a \leq^* b'^* = b^\sharp$ and $a \oplus^* b = 0^\sharp \ominus^* (b^\sharp \ominus^* a) = 0^\sharp \ominus^* (a \oplus b)^\sharp = (a \oplus b) \ominus 0 = a \oplus b \in P$ (see Remark 1.14). Moreover, $P$ is an order ideal of $\hat{P}$ which has the following property: for every $x \in \hat{P}$, either $x \in P$ or $x'^* \in P$.

As we have already observed, every order ideal $J$ of a D-poset $D$ is a GDP. The orthogonality relation $\perp_J$ in $J$ coincides with the orthogonality relation $\perp$ in $D$ if and only if $J$ is closed under the sum $\oplus$ of D. Indeed, it is clear that $\perp_J$ is always contained in $\perp$. Now, if $J$ is closed under $\oplus$ and if $a,b \in J$ are such that $a \perp b$, then $a \oplus b \in J$ and $a = (a \oplus b) \ominus b$, i.e. $a \perp_J b$. Conversely, if $a \perp_J b$ whenever $a,b \in J$ and $a \perp b$, then $a \leq c \ominus b$ for some $c \in J$, hence $a \oplus b \leq c$ (see Remark 1.12(iii)) and thus $a \oplus b \in J$.

**Proposition 2.5.** *Let $D$ be a D-poset and let $P$ be a proper order ideal in $D$ closed under $\oplus$ and such that for every $a \in D$, $a \in P$ or $1 \ominus a \in P$. Denote $a^\sharp = 1 \ominus a$, $a \in P$. Then the D-poset $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ coincides with $(D, \leq, 0, 1, \ominus)$.*

*Proof.* Since $P$ is proper, for every $a \in D$ we have either $a \in P$ or $1 \ominus a \in P$. It is now clear that $P^\sharp = D \setminus P$. Let $'$ be the following unary operation on $D$: $a' = 1 \ominus a$, $a \in D$. By Remark 1.14, for all $a,b \in D$, $a \oplus b$ is defined if and only if $a \leq b'$ and moreover, $a \oplus b = (b' \ominus a)'$. Hence from $a' \leq b'$, which is equivalent to $b \leq a$, it follows $b' \ominus a' = (a' \oplus b)' = (b \oplus a')' = a \ominus b$. According to the definition of $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ the proof is now clear. □

## 3. Generalized Orthoalgebras and (Weak) Generalized Orthomodular Posets

As observed in [**K, Ch**], orthoalgebras (see [**F, G, R**], [**G**], [**R1**]) and orthomodular posets (see [**B**], [**K**], [**P, P**]) are special examples of difference posets.

**Definition 3.1.** An *orthoalgebra (OA) is a set $A$ containing two special elements* $0, 1$ *and equipped with a partial binary operation $\oplus$ satisfying for all $a,b,c \in A$ the following conditions:*

(OA1) *If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity).*

(OA2) *If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity).*

(OA3) *For every $a \in A$ there is a unique $b \in A$ such that $a \oplus b$ is defined and $a \oplus b = 1$ (orthocomplementation).*

(OA4) *If $a \oplus a$ is defined, then $a = 0$ (consistency).*

*For an element $a \in A$, the unique element $b \in A$ satisfying condition (OA3) is denoted by $a'$ and is called the orthocomplement* of $a$.

Let us note that (OA1) is (S1) and (OA2) is (S2), where (S1) and (S2) are conditions from Remark 1.10.

Since every orthoalgebra $A$ satisfies cancellativity (S3) from Remark 1.10 (see [**F, G, R**]), a partial binary operation $\ominus$ on $A$ can be defined by ($a, b, c \in A$): $b \ominus a$ is defined and $b \ominus a = c$ if and only if $a \oplus c$ is defined and $a \oplus c = b$. $A$ then becomes a difference poset (cf. [**K, Ch**]). A partial order $\leq$ on $A$ is defined by (vii) or (vii)* in Remark 1.12. We note that in $A$, $a \oplus b$ exists if and only if $a \leq b'$, and if $a \leq b$ then $b \ominus a = (a \oplus b')'$.

In [**F, B**] an *effect algebra* is defined as a set $A$ containing two special elements $0, 1$ and endowed with a partial binary operation $\oplus$ satisfying conditions (OA1), (OA2), (OA3) and the following relaxation of (OA4):

(EA) If $1 \oplus a$ is defined, then $a = 0$.

It is shown in [**F, B**] that effect algebras and difference posets are the same things (the same result, independently, has been obtained in [**P**]).

**Proposition 3.2.** ([**N, P**]) *Let $(P, \leq, 0, 1, \ominus)$ be a difference poset. Then $(P, \oplus, 0, 1)$, where $\oplus$ is as in Remark 1.14, is an orthoalgebra if and only if the following condition is satisfied for all $a \in P$:*

(DOA)* *If $a \leq 1 \ominus a$, then $a = 0$.*

Let us note that in a difference poset $P$, condition (DOA)* is equivalent to the following condition ($a, b \in P$):

(DOA) If $a = b \ominus a$, then $a = 0$.

Or equivalently, $a \leq b \ominus a$ implies $a = 0$ ($a, b \in P$).

**Definition 3.3.** A generalized orthoalgebra (generalized OA, GOA) is a set $A$ containing a special element 0 and endowed with a partial binary operation $\oplus$ satisfying conditions (OA1), (OA2), (OA4) and (S3) from Remark 1.10 and (v) from Remark 1.12.

Let us note that a generalized OA is just a cone (see Remark 2.3) satisfying condition (OA4). To see this, it suffices to observe that every GOA satisfies condition (vi) from Remark 1.12. Indeed, if $a \oplus b = 0$, then $b = b \oplus 0 = 0 \oplus b = (a \oplus b) \oplus b = a \oplus (b \oplus b)$, hence $b = 0$ and also $a = 0$.

**Proposition 3.4.** *Let $(P, \leq, 0, \ominus)$ be a generalized D-poset. Then $(P, \oplus, 0)$, where $\oplus$ is given by condition (S) in Remark* 1.10*, is a generalized OA if and only if condition (DOA) is satisfied for all $a, b \in P$.*

*Proof.* See Remark 2.3, where it is observed that a generalized D-poset is a cone. $\square$

**Theorem 3.5.** *Let $(P, \leq, 0, \ominus)$ be a generalized D-poset and let $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ be the D-poset constructed in Theorem* 2.4*. Then $\hat{P}$ satisfies condition (DOA)* if and only if $P$ satisfies condition (DOA). This means that $\hat{P}$ is an orthoalgebra if and only if $P$ is a generalized orthoalgebra.*

*Proof.* If $a \in P$ then $0^\sharp \ominus^* a = a^\sharp$, hence $a \leq^* a^\sharp$ if and only if $a \oplus a$ is defined, which means that $a = b \ominus a$ for some $b \in P$.

If $a \in P$ then $0^\sharp \ominus^* a^\sharp = a$, but $a^\sharp \leq^* a$ is impossible. $\square$

In a partially ordered set $(P, \leq)$ we write $a \vee b$ for $\sup\{a, b\}$ and $a \wedge b$ for $\inf\{a, b\}$, if they exist for $a, b \in P$.

**Definition 3.6.** A partially ordered set $(P, \leq)$ with 0 and 1 as a least and a greatest element, respectively, endowed with a unary operation $' : P \to P$ is called an orthomodular poset (OMP) if the following conditions are satisfied:

(OMP1) $a'' = a$,

(OMP2) $a \leq b$ implies $b' \leq a'$,

(OMP3) $a \vee a' = 1$,

(OMP4) $a \leq b'$ implies $a \vee b$ exists,

(OMP5) $a \leq b$ implies $b = a \vee (a' \wedge b)$.

If $a \in P$, the element $a'$ is called the *orthocomplement* of $a$. Condition (OMP5) is the *orthomodular law*. Elements $a, b$ in $P$ are said to be *orthogonal* (in notation $a \perp b$) if $a \leq b'$.

Any OMP may be regarded as an OA by defining $a \oplus b = a \vee b$ precisely in case $a \leq b'$. The following proposition shows the relation between orthoalgebras and orthomodular posets.

**Proposition 3.7.** *([**F, G, R**]) The following conditions are equivalent for an orthoalgebra A:*

(i) *$A$ is an OMP.*
(ii) *If $a \oplus b$ is defined, then $a \vee b$ exists.*
(iii) *If $a \oplus b$ is defined, then $a \vee b$ exists and $a \oplus b = a \vee b$.*
(iv) *If $a \oplus b$, $a \oplus c$ and $b \oplus c$ exist, then $(a \oplus b) \oplus c$ exists.*

Any OMP may be regarded as a DP by defining $b \ominus a = b \wedge a'$ precisely when $a \leq b$. A relation between difference posets and orthomodular posets is as follows (cf. [**N, P**]).

**Proposition 3.8.** *A difference poset $P$ is an orthomodular poset if and only if (DOA)* and the following condition are satisfied for all $a, b \in P$: $a \leq 1 \ominus b$ implies $a \vee b$ exists.*

An orthomodular poset $P$ becomes an *orthomodular lattice (OML)* if the supremum $a \vee b$ (equivalently, the infimum $a \wedge b$) of any two elements $a, b \in P$ exists. The notion of a *generalized OML (GOML)* has been introduced by Janowitz [**J1**], and it has been shown that every GOML can be embedded into an OML as an orthomodular ideal (see also [**B**], [**K**]). A generalization of the latter result has been proved by Mayet-Ippolito [**M-I**] for a so called weak generalized OMP. (A generalization of the latter result in another direction is proved in [**H**] for a so called relatively OML.)

**Definition 3.9.** ([**M-I**]) Let $(P, \leq)$ be a poset with a smallest element 0, such that every interval $[0, a]$ of $P$ is equipped with a unary operation $x \mapsto x^{\sharp_a}$. $P$ is called a weak generalized orthomodular poset (WGOMP) if it satisfies the following conditions:

(G1) If $a \in P$ then $([0, a], \leq, 0, a, \sharp_a)$ is an OMP.
(G2) If $a \leq b \leq c$ then $a^{\sharp_b} = b \wedge a^{\sharp_c}$.

Elements $a, b \in P$ are said to be orthogonal (in notation $a \perp b$) if $a, b \leq c$ and $a \leq b^{\sharp_c}$ for some $c \in P$.

(G3) If $a \perp b$ then $a \vee b$ exists.
(G4) If $a \perp b$, $a \perp c$ and $b \perp c$ then $a \vee b \perp c$.

According to (G1) and (G2) every WGOMP can be regarded as a GDP by defining $b \ominus a = a^{\sharp_b}$ precisely when $a \leq b$. The following result shows a relation between generalized difference posets and weak generalized orthomodular posets.

**Theorem 3.10.** *Let $(P, \leq, 0, \ominus)$ be a generalized D-poset. For every $a \in P$ define $x^{\sharp_a} = a \ominus x$ ($x \in P$, $x \leq a$). Then $P$ is a weak generalized OMP if and only if the following conditions are satisfied:*

(W1) *If $a, b \in P$ and $a \perp b$ then $a \oplus b$ is the supremum of $a, b$.*
(W2) *If $a, b, c \in P$, $a \perp b$, $a \perp c$, and $b \perp c$, then $a \oplus b \perp c$.*

*Proof.* If $P$ is a WGOMP and $a, b \in P$ are such that $a \perp b$, then we get $(a \vee b) \ominus b = (a \vee b) \wedge ((a \oplus b) \ominus b) = (a \vee b) \wedge a = a$, hence $a \vee b = a \oplus b$. From this observation it is clear that (W1) and (W2) are satisfied.

Conversely, let (W1) and (W2) be satisfied. We have to check properties (G1)–(G4) of the preceding definition. (G3) and (G4) are clear. To prove (G1) consider the D-poset $[0, a]$ ($a \in P$). With respect to Proposition 3.7, for $[0, a]$ to be an OMP it suffices to show that $[0, a]$ is an orthoalgebra, i.e. that (DOA)* is satisfied (see Proposition 3.2). So, if $b \in P$, $b \leq a$ and $b \leq a \ominus b$, then $b \perp b$, hence by (W1), $b \oplus b = b$, i.e. $b = b \ominus b = 0$.

To prove (G2) let $a, b, c \in P$ be such that $a \leq b \leq c$. Since $[0, c]$ is an OMP we get $c \ominus (b \ominus a) = c \ominus ((c \ominus a) \ominus (c \ominus b)) = a \oplus (c \ominus b) = a \vee b^{\sharp_c} = (b \wedge a^{\sharp_c})^{\sharp_c}$, hence $b \ominus a = b \wedge a^{\sharp_c}$ which proves (G2). □

Another characterization of weak generalized orthomodular posets among posets with a difference having a smallest element is the following one which uses the difference operation only.

**Corollary 3.11.** *If $(P, \leq, 0, \ominus)$ is a poset with a difference having a smallest element* 0 *and if we define $x^{\sharp_a} = a \ominus x$ $(a, x \in P,\ x \leq a)$ then $P$ is a WGOMP if and only if (DOA) and the following condition are satisfied:*

(W) *If $a, b, c, d \in P$ and $b \ominus (c \ominus a) = d \ominus a$, then $a \vee b$ exists.*

*Proof.* If $P$ is a WGOMP then $P$ is a GDP and we can use Theorem 3.10. If $a, b \in P$ with $a = b \ominus a$ then, since $[0, b]$ is an OMP, $0 = a \wedge a^{\sharp_b} = a$. This shows (DOA). To prove (W) let $a, b, c, d \in P$ be such that $b \ominus (c \ominus a) = d \ominus a$. Then $a \perp c \ominus a$, $a \perp d \ominus a$ and $c \ominus a \perp d \ominus a$, hence by (W2), $a \perp (c \ominus a) \oplus (d \ominus a) = b$ and thus by (W1), $a \vee b$ exists.

Conversely, let (DOA) and (W) be satisfied. First we show that $P$ is then a GDP and thus Theorem 3.10 can be used.

Let us observe that (DOA) is equivalent to the following condition: $a \leq b \ominus a$ implies $a = 0$ $(a, b \in P)$. Indeed, if $a, b \in P$ are such that $a \leq b \ominus a$ and if we denote $c = (b \ominus a) \ominus a$, then $a = (b \ominus a) \ominus c = (b \ominus c) \ominus a$, hence by (DOA), $a = 0$. Using this we prove that $\ominus$ is a cancellative difference on $(P, \leq)$. So, let $a, b, c \in P$ be such that $a \leq b, c$ and $b \ominus a = c \ominus a$. Then $(b \ominus a) \ominus (c \ominus a) = a \ominus a$, hence by (W), $a \vee (b \ominus a)$ exists. Denote $d = a \vee (b \ominus a)$. We have $a \leq d \leq b$ which implies $b \ominus d \leq b \ominus a \leq d = b \ominus (b \ominus d)$ and thus $b \ominus d = 0$ which means that $b = d$. We obtain $a \vee (b \ominus a) = b$ and similarly $a \vee (c \ominus a) = c$. Therefore $b = c$.

We show that $P$ has properties (W1) and (W2) which, according to Theorem 3.10, means that $P$ is a WGOMP.

1) Let $c = a \oplus b$ where $a, b \in P$ with $a \perp b$. Then from $a \ominus (b \ominus b) = c \ominus b$ by (W) it follows that $a \vee b$ exists. Denote $d = a \vee b$. $b \leq d \leq c$ implies $c \ominus d \leq c \ominus b = a \leq d = c \ominus (c \ominus d)$, hence by (DOA), $c \ominus d = 0$ and thus $c = d$. This means that $a \oplus b = a \vee b$. (W1) is proved.

2) To prove (W2) let $a, b, c \in P$ be such that $a \perp b$, $a \perp c$ and $b \perp c$. Hence $(a \oplus b) \ominus ((a \oplus c) \ominus c) = (b \oplus c) \ominus c$ from which by (W) it follows that $(a \oplus b) \vee c$ exists. Denote $d = (a \oplus b) \vee c$. Using (W1) we get $a \oplus c, b \oplus c \leq d$, hence $a, b \leq d \ominus c$, and hence $a \oplus b \leq d \ominus c$ which means that $a \oplus b \perp c$. □

**Lemma 3.12.** *Let $(P, \leq, 0, 1, ')$ be an OMP and let $J$ be an order ideal of $P$ such that for all $a, b \in J$ with $a \leq b'$ also $a \vee b \in J$. Equip every interval $[0, a]$ of $J$ with a unary operation $\sharp_a$ given by $x \mapsto x^{\sharp_a} = a \wedge x'$. Then $J$ is a WGOMP.*

*Proof.* Conditions (G1)–(G4) of a WGOMP are clear from the following observation: for all $a, b \in J$, $a \leq b'$ if and only if $a \leq b^{\sharp_c}$ for some $c \in J$ with $a, b \leq c$. $\square$

**Theorem 3.13.** *Let* $(P, \leq, 0, \ominus)$ *be a generalized D-poset and let* $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ *be the D-poset constructed in Theorem* 2.4. *Let* $'^*$ *be a unary operation on* $\hat{P}$ *given by* $x'^* = 0^\sharp \ominus^* x$ *(this means that if* $a \in P$*, then* $a'^* = a^\sharp$ *and* $(a^\sharp)'^* = a$*). Then* $\hat{P}$ *is an OMP if and only if* $P$ *is a WGOMP.*

*Proof.* If $P$ is a WGOMP then $P$ satisfies (DOA), hence by Theorem 3.5, $\hat{P}$ satisfies (DOA)*. Owing to Proposition 3.8, it suffices to show that for all $x, y \in \hat{P}$, $x \vee^* y$ exists whenever $x \leq^* y'^*$. So, let $a, b \in P$. Since $a \leq^* (b^\sharp)'^*$ is equivalent to $b^\sharp \leq^* a'^*$, we have to check the following two possibilities.

1) If $a \leq^* (b^\sharp)'^*$, i.e. $a \leq b$, then clearly $a, b^\sharp \leq^* (b \ominus a)^\sharp$. If $a, b^\sharp \leq^* c^\sharp$ for some $c \in P$ then $a \perp c$ and $c \leq b$, hence by (W1), $a \oplus c \leq b$ and thus $c \leq b \ominus a$ which means that $(b \ominus a)^\sharp \leq^* c^\sharp$. Therefore $a \vee^* b^\sharp = (b \ominus a)^\sharp$.

2) If $a \leq^* b'^*$, i.e. $a \leq^* b^\sharp$, then $a \perp b$ and by (W1), $a \oplus b = a \vee b$. Clearly, $a, b \leq^* a \oplus b$. If $a, b \leq^* c^\sharp$ for some $c \in P$, then $a \perp c$ and $b \perp c$, hence by (W2), $a \oplus b \perp c$, i.e. $a \oplus b \leq^* c\sharp$. Therefore $a \vee^* b = a \vee b$.

Conversely, let $\hat{P}$ be an OMP. Since $\hat{P}$ satisfies (DOA)*, $P$ satisfies (DOA) by Theorem 3.5.

Let $a, b \in P$ with $a \leq^* b'^*$. This means that $a \leq^* b^\sharp$, i.e. $a \oplus b$ is defined, and $a \vee^* b$ exists. From $a, b \leq a \oplus b$ it follows $a, b \leq^* a \oplus b$, hence $a \vee^* b \leq^* a \oplus b$ which with $a \oplus b \in P$ gives $a \vee^* b \in P$ and thus $a \vee^* b = a \vee b$. Denote $c = a \vee b$ and $d = a \oplus b$. $b \leq c \leq d$ implies $d \ominus c \leq d \ominus b = a \leq c$, hence $d \ominus c \leq d \ominus (d \ominus c)$ which by (DOA) gives $d \ominus c = 0$, i.e. $d = c$. We have $a \oplus b = a \vee b$.

Since, by the preceding considerations, $P$, as an order ideal of $\hat{P}$, satisfies the condition of Lemma 3.12, it suffices now to show that for all $a, b \in P$ with $a \leq b$ it holds $b \ominus a = b \wedge^* a^\sharp$. $b \ominus a$ is a lower bound of $b$ and $a^\sharp$ in $(\hat{P}, \leq^*)$ because $b \ominus a \leq b$ and $(b \ominus a) \oplus a = b$. If $c \leq^* b$, $a^\sharp$ for some $c \in P$ then $c \leq b$ and $a \oplus c$ exists, hence $a \oplus c = a \vee c \leq b$ which implies $c \leq b \ominus a$, i.e. $c \leq^* b \ominus a$. Therefore $b \ominus a$ is the greatest lower bound of $b$ and $a^\sharp$. $\square$

By the preceding considerations, the embedding of a WGOMP $P$ into an orthomodular poset $\hat{P}$ from Theorem 3.13 coincides with that of [**M-I**] and, as observed there, this embedding preserves the infimum but not generally the supremum whenever they exist in $P$. If $a, b \in P$ with $a \perp b$, then, by (G4), the supremum of $a$ and $b$ in $P$ is also the supremum of $a$ and $b$ in $\hat{P}$.

According to [**M-I**], for a WGOMP $P$ the following conditions are equivalent:

(i) The embedding of $P$ into an OMP $\hat{P}$ preserves all existing suprema of two elements.

(ii) If $a, b, c \in P$ with $a \perp c$ and $b \perp c$ and if $a \vee b$ exists in $P$ then $a \vee b \perp c$.

(There is no difficulty to see that a similar statement is true for a GDP $P$ and a corresponding D-poset $\hat{P}$.)

A *generalized orthomodular poset (GOMP)* is defined in [**M-I**] as a poset $(P, \leq)$ with a smallest element 0 such that every interval $[0, a]$ of $P$ is equipped with a unary operation $x \mapsto x^{\sharp_a}$, satisfying the axioms (G1), (G2), (G3) and

(G4)' If $a, b, c \in P$ are such that $a \perp c$, $b \perp c$ and if $a \vee b$ exists, then $a \vee b \perp c$.

Since (G3) and (G4)' imply (G4), every GOMP is a WGOMP (see Definition 3.9). Theorem 2 in [**M-I**] and Theorem 3.13 imply the following.

**Theorem 3.14.** *Let* $(P, \leq, 0, \ominus)$, $(\hat{P}, \leq^*, 0, 0^\sharp, \ominus^*)$ *and* $'^*$ *be as in Theorem* 3.13. *Then* $\hat{P}$ *is an OMP for which the supremum of* $a, b \in P$ *in* $P$, *if it exists, is also the supremum of* $a, b$ *in* $\hat{P}$ *if and only if* $P$ *is a GOMP.*

## 4. Examples

There is an abundance of various examples of difference structures mentioned in this paper.

**Example 4.1.** The set $P = R^+$ of all nonnegative real numbers with the natural ordering and with the usual difference of numbers is a GDP. Since for every $a, b \in R^+$ also $a + b \in R^+$, we have $a \perp b$. Therefore, in $\hat{P} = P \cup P^\sharp$ every element in $P^\sharp$ is greater than any element in $P$.

More general, the positive cone $P = G^+$ of any partially ordered abelian group $(G, +, 0, \leq)$ with the usual difference of group elements is a GDP. For every $a, b \in G^+$ also $a + b \in G^+$, hence $a \perp b$. Thus, in $\hat{P} = P \cup P^\sharp$ every element in $P^\sharp$ is greater than any element in $P$.

Since every partially ordered vector space $V$ is at the same time a partially ordered abelian group, the ordering cone $P = V^+$ with a naturally defined difference is a GDP. The set of all positive operators on a complex Hilbert space, positive operators in a von Neumann algebra, positive elements in a $C^*$-algebra or a Jordan algebra are such examples of generalized difference posets.

**Example 4.2.** Let $X$ be a nonempty set and let $\mathcal{F} \subseteq [0, 1]^X$ satisfy

(i) $1 \in \mathcal{F}$,

(ii) $f, g \in \mathcal{F}$ and $f \leq g$ implies $g - f \in \mathcal{F}$,

where $\leq$ and $-$ are componentwise partial order and difference of real functions, respectively. Then $\mathcal{F}$ with the partial binary operation $\ominus$ given by: $g \ominus f$ is defined if and only if $f \leq g$ and $g \ominus f = g - f$, is a D-poset.

$\mathcal{F}$ is an orthoalgebra if and only if

(iii) $0 \neq f \in \mathcal{F}$ implies $2f \notin \mathcal{F}$.

$\mathcal{F}$ is an OMP if and only if

(iv) $f, g, h \in \mathcal{F}$, $f + g \leq 1$, $f + h \leq 1$, $g + h \leq 1$ imply $f + g + h \in \mathcal{F}$.

As concerns the latter example, see [**M, T**], [**B, M**] and [**P**].

**Example 4.3.** Let $X$ be a nonempty set and let $\mathcal{F} \subseteq (R^+)^X$ satisfy

(i) $0 \in \mathcal{F}$,
(ii) $f, g \in \mathcal{F}$ and $f \leq g$ implies $g - f \in \mathcal{F}$,

where $\leq$ and $-$ are as in Example 4.2. Then $\mathcal{F}$ with $\ominus$ as in Example 4.2 is a GDP. Observe that for $f, g \in \mathcal{F}$, $f \perp g$ if and only if $f + g \in \mathcal{F}$, where $+$ is pointwise sum of real functions.

$\mathcal{F}$ is a GOA if and only if

(iii) $0 \neq f \in \mathcal{F}$ implies $2f \notin \mathcal{F}$.

$\mathcal{F}$ is a WGOMP if and only if

(iv) $f, g, h, f + g \in \mathcal{F}$ and $f, g \leq h$ imply $f + g \leq h$,
(v) $f, g, h, f + g, f + h, g + h \in \mathcal{F}$ implies $f + g + h \in \mathcal{F}$.

As a concrete example we present the following set of real functions which is a GOA but which is not a WGOMP. Let $\mathcal{F} = \{0, f, g, h, f+g, h-f, h-g\} \subseteq (R^+)^{R^+}$ be a set of seven different functions $R^+ \to R^+$ described in Fig. 2. All functions on Fig. 2 are linear on the intervals $[0, 3]$, $[3, 6]$ and $[6, \infty]$, and $f(0) = 1$, $g(0) = 4$, $h(0) = 7$. $\mathcal{F}$ under pointwise partial order of real functions forms a poset which is on Fig. 3. Conditions (i)–(iii) and (v) are satisfied, but (iv) is not.

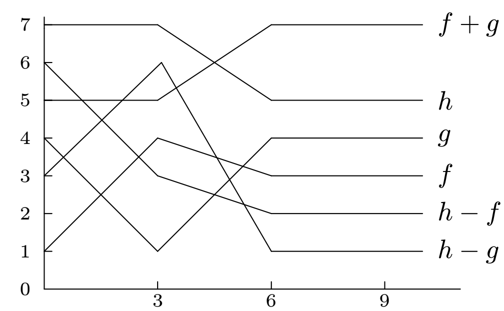


**Fig. 2**

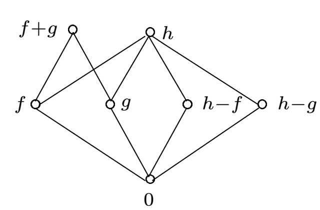


**Fig. 3**

**Example 4.4.** Let $X$ be a nonempty set and let $\mu : R^+ \to R^+$ be a strictly increasing continuous function such that $\mu(0) = 0$. Define a partial binary operation $\ominus$ on $(R^+)^X$ as follows: if $f, g \in (R^+)^X$ then $g \ominus f$ is defined if and only if $f \leq g$ and $(g \ominus f)(x) = \mu^{-1}(\mu(g(x)) - \mu(f(x)))$ for all $x \in X$, where $\leq$ is a pointwise partial order of real functions. Then $(R^+)^X$ is a GDP (cf. [**K, Ch**]).

**Example 4.5.** A very important class of D-posets can be obtained taking into account that every interval $[0, a]$, $a \geq 0$, in a partially ordered abelian group is a D-poset (cf [**B, F**]). Hence we have the following examples of D-posets: the set of all effects, that is, selfadjoint operators $A$ on a complex Hilbert space such that $0 \leq A \leq I$ (which plays an important role in quantum axiomatic [**B,L,M**]), the interval $[0, e]$ in an Archimedean order-unit space $(A, e)$ with the order unit $e$ [**Al**], the interval $[0, I]$ in a JB-algebra (see, e.g., [**H-O,S**] for the definition).

**Example 4.6.** Let $(G, +, 0)$ be an abelian group and let $\leq$ be a partial order on $G$ such that:

(i) If $a, b, c \in G$ and $a \leq b \leq c$ then $c - b \leq c - a$.

Define a partial binary operation $\ominus$ on $G$ by: if $a, b \in G$ then $b \ominus a$ is defined if and only if $a \leq b$ and let $b \ominus a = b - a$. Then the following three conditions are equivalent:

(1) $(G, \leq, 0, \ominus)$ is a GDP,
(2) 0 is a smallest element in $(G, \leq)$,
(3) if $a, b \in G$ and $a \leq b$ then $b - a \leq b$.

(1) $\Longrightarrow$ (2) This is clear.

(2) $\Longrightarrow$ (3) If $a, b \in G$ are such that $a \leq b$ then from $0 \leq a \leq b$ by (i) it follows that $b - a \leq b - 0 = b$.

(3) $\Longrightarrow$ (1) This follows from group properties.

Assume that $(G, \leq, 0, \ominus)$ is a GDP. Recall that for $a, b \in G$, $a \perp b$ if and only if $a = c \ominus b$ for some $c \in G$. From (3) and the fact that for all $a, b \in G$, $a = (a+b) - b$ and $b = (a + b) - a$ it follows the following:

(4) If $a, b \in G$ then $a \perp b$ if and only if $a \leq a + b$.

The sum $\oplus$ is then given by ($a, b \in G$): $a \oplus b$ is defined if and only if $a \perp b$ and $a \oplus b = a + b$. And $(G, \oplus, 0)$ is a GOA if and only if the following condition is satisfied:

(5) If $a \in G$ and $a \leq a + a$ then $a = 0$.

For every $a \in G$ define $x^{\sharp_a} = a \ominus x$ ($x \in G,\ x \leq a$). Then $G$ is a WGOMP if and only if the following two conditions are satisfied for all $a, b, c \in G$:

(ii) If $a \leq a + b$ then $a \vee b$ exists and $a \vee b = a + b$.
(iii) If $a \leq a + b,\ a + c$ and $b \leq b + c$ then $a \leq a + b + c$.

To prove this it suffices to observe that (2) is satisfied and then to use Theorem 3.10. Indeed, if $a \in G$ is arbitrary then from $a \leq a = a + 0$ it follows by (ii) that $a \vee 0$ exists and $a \vee 0 = a + 0 = a$ which means that $0 \leq a$.

Consequently, $G$ is a GOMP if and only if (ii) and the following condition are satisfied:

(iv) If $a \leq a+b,\ a+c$ and $b \vee c$ exists then $a \leq a+(b \vee c)$.

In [**Ch**] the notion of an *orthomodular group* is introduced as an abelian group $G$ equipped with a partial order $\leq$ satisfying the following conditions for all $a, b, c \in G$:

(OG1) $a \leq b \leq b+c$ implies $a \leq a+c$,

(OG2) $a \leq b \leq c$ implies $c-b \leq c-a$,

(OG3) $a \leq a+b$ implies $a \vee b$ exists and $a \vee b = a+b$,

(OG4) $a \leq a+b,\ a+c$ implies $a \leq a+b+c$.

We show that conditions (OG2)–(OG4) imply conditions (i), (ii) and (iv), hence every orthomodular group is a GOMP (in [**Ch**], it was proved that an orthomodular group is a WGOMP). So, (OG2) is (i) and (OG3) is (ii). Since (OG3) implies that 0 is a smallest element, $G$ is a GDP and hence (4) is satisfied. To prove (iv) let $a, b, c, d \in G$ be such that $a \leq a+b,\ a+c$ and $d = b \vee c$. From $b \leq b+a,\ b+(d-b)$ it follows by (OG4) that $b \leq a+b+(d-b) = a+d$. Similarly we get $c \leq a+d$ and thus $a+d$ is an upper bound of $b, c$, hence $d \leq a+d$, which implies $a \leq a+d$.

Let us note that from the preceding considerations it follows that, in the above definition of an orthomodular group, condition (OG1) can be omitted. Namely, (OG1) means that if $a, b, c \in G$ are such that $a \leq b$ and $b \perp c$, then $a \perp c$, which is true in every GDP (see condition (i)* in Remark 1.12).

Let $X$ be a nonempty set and let $S \subseteq 2^X$ be such that $\emptyset \in S$ and $a \Delta b \in S$ for all $a, b \in S$, where $a \Delta b = (a \cap b') \cup (a' \cap b)$. Then $(S, \Delta, \emptyset)$ is an abelian group and with respect to $\leq$ defined by set inclusion $S$ is an orthomodular group.

As shown in [**Ch**] there is an abundance of orthomodular groups:

(a) Let $A$ be an alternative ring with no nonzero nilpotent elements. Define a binary relation $\leq$ on $A$ by $a \leq b$ if and only if $ab = a^2$ [**M,J**], then $\leq$ is a partial order.

(b) Let $A$ be an associative $*$-ring with a proper involution [**Be**], that is, $a^*a = 0$ implies $a = 0$. Define a binary relation $\leq$ on $A$ by $a \leq b$ if and only if $aa^* = ba^*$ and $a^*a = a^*b$. Then $\leq$ is a partial order (called the $*$-order) [**D**] and $A$ with $\leq$ is a WGOMP [**M-I**]. In particular, a commutative ring $A$ without nonzero nilpotent elements, a Rickart $*$-ring $A$ [**Be**] (in [**M-I**], using results from [**J2**], it is shown that $A$ is a GOMP), a C*-algebra.

(c) Let $A$ be a Jordan algebra ([**T**], [**H-O,S**]) without nonzero nilpotent elements and satisfying the following condition:

$$[x, x, y] = 0 \text{ implies } [xy, x, y] = 0,$$

where $[a,b,c] = (ab)c - a(bc)$ is the associator of $a, b, c$. Define a binary relation $\leq$ on $A$ by $a \leq b$ if and only if $ab = a^2$ and $a^2b = a^3$ [**Ch**], then $\leq$ is a partial order.

(d) Let $A$ be a JB-algebra ([**H-O,S**], [**ML**]). Define a binary relation $\leq$ on $A$ by $a \leq b$ if and only if $a^2b = a^3$ [**Ch**], then $\leq$ is a partial order.

Since $A$ is always an abelian group, (a), (b), (c) and (d) with the order relations defined above are examples of orthomodular groups and thus examples of GOMPs.

We present yet an example of an abelian group which is not an orthomodular group (even which is not a WGOMP) but which is a GOA. Consider the abelian group $(Z_7, +, 0)$ of integers modulo 7 partially ordered as in Fig. 4.

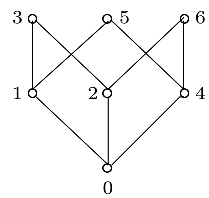


**Fig. 4**

Then conditions (i), (2), (5) and (ii) are satisfied, hence $Z_7$ is a GDP which is a GOA (let us note that, in general, (ii) need not be satisfied in a GOA). $Z_7$ is not a WGOMP since (iii) is not satisfied: we have $1 \leq 1+2$, $1+4$ and $2 \leq 2+4$, but $1 \not\leq 1+2+4$. Thus $Z_7$ is not a GOMP, and hence $Z_7$ is not an orthomodular group.

A simple example of an abelian group which is not an orthomodular group but which is a GOMP is the group $(Z_4, +, 0)$ of integers modulo 4 partially ordered as in Fig. 5. Conditions (i), (2), (ii) and (iv) are satisfied, hence $Z_4$ is a GOMP, but condition (OG4) is not satisfied, since $2 \leq 1+2$ but $2 \not\leq 1+1+2$.

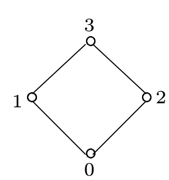


**Fig. 5**

There is also an example of an abelian group which is a WGOMP but which is not a GOMP (hence, which is not an orthomodular group). Consider the abelian group $(Z_9, +, 0)$ of integers modulo 9 partially ordered as in Fig. 6.

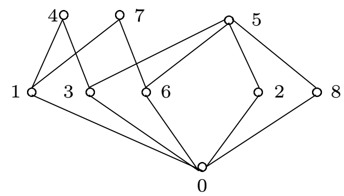


**Fig. 6**

Then conditions (i)–(iii) and (2) are satisfied (this means that $Z_9$ is a WGOMP) but condition (iv) is not satisfied, since $1 \le 1+3, 1+6$ and $3 \vee 6$ exists but $1 \not\le 1 + (3 \vee 6)$ (thus $Z_9$ is not a GOMP). Let us note that the poset in Fig. 6 considered as an abstract poset is an example of Roddy from [**M-I**].

**Example 4.7.** We present generalizations (modifications) of examples (a), (b) and (c) in Example 4.6.

(a) Let $R$ be a ring such that the following binary relation $\le$ on $R$ is a partial order:

$$a \le b \text{ if and only if } ab = ba = a^2.$$

Clearly, 0 is a smallest element in $(R, \le)$. Since $\le$ is antisymmetric, $a^2 = 0$ implies $a = 0$ for every $a \in R$. If $a, b, c \in R$ and $a \le b \le c$ then

$$\begin{aligned} &(c-b)(c-a) = c^2 - bc - ca + ba = c^2 - b^2, \text{ similarly} \\ &(c-a)(c-b) = c^2 - b^2, \\ &(c-b)^2 = c^2 - bc - cb + b^2 = c^2 - b^2, \end{aligned}$$

which means that $c - b \le c - a$. Hence $R$ is a GDP and by (4) in Example 4.6, for all $a, b \in R$, $a \perp b$ if and only if $a \le a + b$. It is easy to see that for all $a, b \in R$,

$$a \perp b \text{ if and only if } ab = ba = 0.$$

We prove that $(R, +, 0)$ is an orthomodular group. It remains to show conditions (OG3) and (OG4).

(OG3): If $a, b \in R$ and $a \le a + b$ then $b \le a + b$ (since $\perp$ is symmetric), hence $a + b$ is an upper bound of $a, b$. If $c \in R$ and $a, b \le c$ then

$$\begin{aligned} &(a+b)c = ac + bc = a^2 + b^2 = ca + cb = c(a+b), \\ &(a+b)^2 = a^2 + ba + ab + b^2 = a^2 + b^2, \end{aligned}$$

hence $a + b \le c$. Thus $a + b$ is the join of $a, b$ in $(R, \le)$.

(OG4): If $a, b, c \in R$ and $a \leq a+b, a+c$, then $a \perp b$ and $a \perp c$, hence $a(b+c) = ab + ac = 0 + 0 = 0$ and $(b+c)a = ba + ca = 0 + 0 = 0$, which means that $a \perp b + c$ and thus $a \leq a + b + c$.

For example, $\leq$ as defined above is a partial order on every member $R$ of the class of rings studied by Abian in [**Ab**], hence every $R$ is an orthomodular group. As shown by Hentzel in [**He**], Lemma 1, these rings generalize alternative rings without nonzero nilpotent elements. Let us note that the partial order $\leq$ on $R$ reduces to:

$$a \leq b \text{ if and only if } ab = a^2.$$

Cf. Example 4.6(a).

(b) Let $R$ be a $*$-ring (i.e., a ring $R$ with an involution $^*$) such that the following binary relation $\leq$ on $R$ is a partial order:

$$a \leq b \text{ if and only if } aa^* = ba^* \text{ and } a^*a = a^*b.$$

Observe that 0 is a smallest element in $(R, \leq)$ and that for all $a, b \in R$, $a \leq b$ implies $aa^* = ab^*$ and $a^*a = b^*a$. If $a, b, c \in R$ and $a \leq b \leq c$ then

$$((c-b) - (c-a))(c-b)^* = (a-b)(c^* - b^*) = ac^* - bc^* - ab^* + bb^* = 0,$$

and similarly we get $(c-b)^*((c-b) - (c-a)) = 0$, hence $c - b \leq c - a$. Thus $R$ is a GDP and by (4) in Example 4.6, for all $a, b \in R$, $a \perp b$ if and only if $a \leq a + b$. It is easy to show that for all $a, b \in R$,

$$a \perp b \text{ if and only if } a^*b = ba^* = 0.$$

To prove that $(R, +, 0)$ is an orthomodular group, it remains to show conditions (OG3) and (OG4).

(OG3): If $a, b \in R$ and $a \leq a + b$ then $b \leq a + b$. If $c \in R$ and $a, b \leq c$ then

$$\begin{aligned} &(a+b)(a+b)^* = aa^* + ba^* + ab^* + bb^* = aa^* + bb^*, \\ &c(a+b)^* = ca^* + cb^* = aa^* + bb^*, \end{aligned}$$

and similarly we get $(a+b)^*(a+b) = a^*a + b^*b = (a+b)^*c$, hence $a + b \leq c$. Thus $a + b$ is the join of $a, b$ in $(R, \leq)$.

(OG4): If $a, b, c \in R$ and $a \leq a+b, a+c$, then $a \perp b$, $a \perp c$, hence $a^*(b+c) = a^*b + a^*c = 0 + 0 = 0$ and $(b+c)a^* = ba^* + ca^* = 0 + 0 = 0$, which means that $a \perp b + c$ and thus $a \leq a + b + c$.

(c) Let $R$ be a ring in which for all $a \in R$, $a^2a = aa^2$. Consider the following binary relation $\leq$ on $R$:

$$a \leq b \text{ if and only if } ab = ba = a^2 \text{ and } a^2b = ba^2 = ab^2 = b^2a = a^3.$$

Then $\leq$ is reflexive and $0 \leq a$ for all $a \in R$. The relation $\leq$ is antisymmetric if and only if for all $a \in R$, $a^2 = 0$ implies $a = 0$. Assume that $\leq$ is a partial order on $R$. Using Example 4.6 we show that the abelian group $(R, +, 0)$ is a WGOMP, where for every $a \in R$ we define $x^{\sharp_a} = a - x$ $(x \in R, x \leq a)$. So, we prove that conditions (i)–(iii) of Example 4.6 are satisfied.

(i) If $a, b, c \in R$ and $a \leq b \leq c$ then $(c-a)^2 = c^2 - a^2$, $(c-b)^2 = c^2 - b^2$ and $(c-b)^3 = c^3 - b^3$. Then we get

$$\begin{aligned}
(c-b)(c-a) &= c^2 - bc - ca + ba = c^2 - b^2,\\
(c-b)^2(c-a) &= c^3 - b^2c - c^2a + b^2a = c^3 - b^3,\\
(c-b)(c-a)^2 &= c^3 - bc^2 - ca^2 + ba^2 = c^3 - b^3,
\end{aligned}$$

and similarly we get $(c-a)(c-b) = c^2 - b^2$ and $(c-a)(c-b)^2 = c^3 - b^3 = (c-a)^2(c-b)$, which means that $c - b \leq c - a$.

Hence $R$ is a GDP and by (4) in Example 4.6, for all $a, b \in R$, $a \perp b$ if and only if $a \leq a + b$. It is easy to see that for all $a, b \in R$,

$$a \perp b \text{ if and only if } ab = ba = a^2b = ba^2 = ab^2 = b^2a = 0.$$

(ii) We show that if $a, b \in R$ with $a \leq a + b$ then $a + b$ is the join of $a$ and $b$ in $(R, \leq)$. Since $\perp$ is symmetric, from $a \leq a + b$ it follows $b \leq a + b$, hence $a + b$ is an upper bound of $a$ and $b$. From $a \perp b$ we get $(a+b)^2 = a^2 + b^2$ and $(a+b)^3 = a^3 + b^3$. If $c \in R$ is an upper bound of $a$ and $b$ then we obtain

$$\begin{aligned}
(a+b)c &= ac + bc = a^2 + b^2 = (a+b)^2,\\
(a+b)^2c &= (a^2+b^2)c = a^2c + b^2c = a^3 + b^3 = (a+b)^3,\\
(a+b)c^2 &= ac^2 + bc^2 = a^3 + b^3 = (a+b)^3,
\end{aligned}$$

and similarly $c(a+b) = (a+b)^2$ and $c(a+b)^2 = (a+b)^3 = c^2(a+b)$, which means that $a + b \leq c$.

(iii) If $a, b, c \in R$, $a \leq a + b$, $a + c$ and $b \leq b + c$ then $a \leq a + b + c$ since $a(b+c) = ab+ac = 0$, $a^2(b+c) = a^2b+a^2c = 0$, $a(b+c)^2 = a(b^2+c^2) = ab^2+ac^2 = 0$ and similarly $0 = (b+c)a = (b+c)a^2 = (b+c)^2a$.

Let us note that if $R$ is zero commutative (i.e., if for all $x, y \in R$, $xy = 0$ implies $yx = 0$) or commutative then the partial order $\leq$ on $R$ as defined above reduces to:

$$a \leq b \text{ if and only if } ab = a^2 \text{ and } a^2b = ab^2 = a^3.$$

And the orthogonality relation $\perp$ on $R$ reduces to:

$$a \perp b \text{ if and only if } ab = a^2b = ab^2 = 0.$$

A Jordan ring is a commutative ring $R$ satisfying $(xy)x^2 = x(yx^2)$ for all $x, y \in R$, this is to say $[x, y, x^2] = 0$. Let $R$ be a Jordan ring satisfying the condition $2x = 0$ implies $x = 0$ for all $x \in R$, without nonzero nilpotent elements and satisfying the condition in Example 4.6(c). In [**G,M**] it is proved that the binary relation $\leq$ on $R$ defined above is a partial order, hence $R$ is a WGOMP.

**Example 4.8.** A convenient generalization of Example 4.6 which includes various known examples is as follows. Let $(G,+,0)$ be an abelian group and let $P$ be a nonempty subset of $G$. Assume that there is a partial order $\leq$ on $P$ such that the following two conditions are satisfied for all $a,b,c \in P$:

(o) $a \leq b$ implies $b-a \in P$,
(i) $a \leq b \leq c$ implies $c-b \leq c-a$.

By (o), $0 \in P$. Define a partial binary operation $\ominus$ on $P$ by $(a,b \in P)$: $b \ominus a$ is defined if and only if $a \leq b$ and let $b \ominus a = b-a$. Then the following three conditions are equivalent:

(1) $(P,\leq,0,\ominus)$ is a GDP,
(2) $0$ is a smallest element in $(P,\leq)$,
(3) if $a,b \in P$ and $a \leq b$ then $b-a \leq b$.

Assume that $(P,\leq,0,\ominus)$ is a GDP. Recall that for $a,b \in P$, $a \perp b$ if and only if $a = c \ominus b$ for some $c \in P$. Then the following condition is satisfied:

(4) If $a,b \in P$ then $a \perp b$ if and only if $a+b \in P$ and $a \leq a+b$.

Under the conditions (o) and (i), for every $a \in P$ define $x^{\sharp_a} = a \ominus x$ $(x \in P, x \leq a)$. Then $P$ is a WGOMP if and only if the following two conditions are satisfied for all $a,b,c \in P$:

(ii) $a+b \in P$ and $a \leq a+b$ implies $a \vee b$ exists and $a \vee b = a+b$,
(iii) $a+b, a+c, b+c \in P$, $a \leq a+b, a+c$ and $b \leq b+c$ implies $a+b+c \in P$ and $a \leq a+b+c$.

And $P$ is a GOMP if and only if (ii) and the following condition are satisfied for all $a,b,c \in P$:

(iv) If $a+b, a+c \in P$, $a \leq a+b, a+c$ and $b \vee c$ exists then $a+(b \vee c) \in P$ and $a \leq a+(b \vee c)$.

Proofs of all mentioned statements can be carried on analogously as in Example 4.6. We present some more concrete examples.

(a) Let $R$ be a ring and let $P$ be the set of all idempotents in $R$ (i.e., elements $a \in R$ with $a = a^2$) such that the following binary relation $\leq$ on $P$ is a partial order:

$$a \leq b \text{ if and only if } ab = ba = a.$$

$P$ is nonempty since $0 \in P$. Clearly, $0 \leq a$ for all $a \in P$. If $a,b \in P$ and $a \leq b$ then $(b-a)^2 = b^2 - ab - ba + a^2 = b - a - a + a = b - a$, hence $b - a \in P$. If $a,b,c \in P$ and $a \leq b \leq c$ then $(c-b)(c-a) = c^2 - bc - ca + ba = c - b - a + a = c - b$ and $(c-a)(c-b) = c^2 - ac - cb + ab = c - a - b + a = c - b$, which means that $c - b \leq c - a$. Thus conditions (o) and (i) are satisfied and therefore $P$ is a GDP. According to (4), if $a,b \in P$ then $a \perp b$ if and only if $ab = ba = 0$. It is easy to show that conditions (ii) and (iii) are satisfied (cf. Example 4.7(a)), hence $P$ is a WGOMP, where for every $a \in P$ we define $x^{\sharp_a} = a - x$ $(x \in P, x \leq a)$.

As an example of such a WGOMP we can take the set $P$ of all idempotents of a ring $R$ in Example 4.7(a) (which is an orthomodular group, hence a GOMP) with the partial order $\leq$ on $R$ restricted to the set $P$. Similarly, the set $P$ of all idempotents of a ring $R$ in Example 4.7(c) (which is a WGOMP) with the partial order $\leq$ on $R$ restricted to the set $P$ is also such an example of a WGOMP.

(b) Let $R$ be a $*$-ring and let $P$ be the set of all projections in $R$ (i.e., elements $a \in R$ such that $a^2 = a^* = a$) such that the following binary relation $\leq$ on $P$ is a partial order:

$$a \leq b \text{ if and only if } ab = a.$$

Let us observe that $a, b \in P$ and $a \leq b$ implies $ba = a$. Then $P$ is a GDP since $0 \in P$ and $0 \leq a$ for all $a \in P$, and conditions (o) and (i) are satisfied. By (4), for all $a, b \in P$, $a \perp b$ if and only if $ab = 0$ if and only if $ba = 0$. Conditions (ii) and (iii) are satisfie d, too (cf. Example 4.7(b)), and therefore $P$ is a WGOMP.

The set $P$ of all projections of a $*$-ring $R$ in Example 4.7(b) (which is an orthomodular group, hence a GOMP) with the partial order $\leq$ on $R$ restricted to the set $P$ is such an example of a WGOMP.

The set of all idempotents (projections) of an associative ring ($*$-ring) with 1 is an OMP [**Ka**] ([**Bi**]) and the set of all idempotents (projections) of an associative ring ($*$-ring) which need not have 1 is a WGOMP, see [**M-I**]. Let $R$ be an associative ring ($*$-ring) without 1 and let $\tilde{R}$ denote a unitification of $R$, see [**Be**]. Let $P$ denote the set of all idempotents (projections) of $R$. Then the OMP $\hat{P}$ (cf. Theorem 3.13) is isomorphic with the OMP $\tilde{P}$ of all idempotents (projections) in $\tilde{R}$. Namely, $\tilde{R} = R \times A$, where $A$ is an auxiliary ring ($*$-ring) with a unit, and for all $(a, \alpha)$, $(b, \beta) \in \tilde{R}$, $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$, $(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta)$ (and $(a, \alpha)^* = (a^*, \alpha^*)$). If $(a, \alpha)^2 = (a, \alpha)$ then $(a^2 + 2\alpha a, \alpha^2) = (a, \alpha)$, hence $\alpha^2 = \alpha$ and $a^2 + 2\alpha a = a$ and thus $\alpha = 0$ and $a^2 = a$ or $\alpha = 1$ and $a^2 = -a$. Therefore $\tilde{P} = \{(a, 0) : a \in P\} \cup \{(-a, 1) : a \in P\}$.

**Example 4.9** [**R1**]**.** Another concrete example (which is motivated by a theory of triple systems — alternative and Jordan triples ([**Ba**], [**E, R**], [**L1**], [**L2**], [**L3**], [**M1**], [**M2**]) and which is still very general) of the preceding general example is as follows. Let $(A, +, 0)$ be an abelian group endowed with a ternary operation $(x, y, z) \mapsto (xyz)$ on $A$ which is additive in all three variables such that the following conditions are satisfied for all $a, b, c \in A$:

(1) $((aba)c(aba)) = (a(b(aca)b)a)$,
(2) $((aba)bc) = (a(bab)c)$, $(cb(aba)) = (c(bab)a)$.

A is called a triple group. Elementary consequences of the additivity are the following two properties ($a, b, c \in A$):

$$(0ab) = (a0b) = (ab0) = 0,$$
$$-(abc) = (-a\ bc) = (a\ -b\ c) = (ab\ -c).$$

An element $a \in A$ is called a tripotent if $(aaa) = a$. Let $P$ denote the collection of all tripotents of $A$. Clearly, $0 \in P$. Consider the following binary relation $\leq$ on the set $P$:

$$a \leq b \text{ if and only if } a = (aba) = (bab).$$

Then $\leq$ is reflexive, antisymmetric and $0 \leq a$ for all $a \in P$. If $a, b, c \in P$, $a \leq b$ and $b \leq c$ then $a \leq c$ since by (1),

$$\begin{aligned}(aca) &= ((bab)c(bab)) = (b(a(bcb)a)b) = (b(aba)b) = (bab) = a,\\ (cac) &= (c(bab)c) = (c((bcb)a(bcb))c) = (c(b(c(bab)c)b)c)\\ &= (c(b(cac)b)c) = ((cbc)a(cbc)) = (bab) = a.\end{aligned}$$

Hence $\leq$ is transitive and thus a partial order with a smallest element 0. Let us observe that the following condition is satisfied for all $a, b \in P$ and $c \in A$:

(3) $a \leq b$ implies $(abc) = (bac) = (aac)$ and (cab) = (cba) = (caa).

Namely, if $a, b \in P, a \leq b$ and $c \in A$ then, according to (2), $(abc) = ((aba)bc) = (a(bab)c) = (aac)$ and $(bac) = (b(aba)c) = ((bab)ac) = (aac)$. Similarly, $(cab) = (caa) = (cba)$. In particular, the following condition is satisfied for all $a, b \in P$:

(3*) $a \leq b$ implies $a = (aab) = (baa) = (abb) = (bba)$.

If $a, b \in P$ and $a \leq b$ then by (3*), $(b - a\ b - a\ b - a) = (bbb) - (bba) - (bab) + (baa) - (abb) + (aba) + (aab) - (aaa) = b - a$, which means that $b - a \in P$.

We show that if $a, b, c \in P$ and $a \leq b \leq c$ then $c - b \leq c - a$. Indeed, using conditions (3) and (3*) we obtain $(c - b\ c - a\ c - b) = (ccc) - (ccb) - (cac) + (cab) - (bcc) + (bcb) + (bac) - (bab) = c - b$ and $(c - a\ c - b\ c - a) = (ccc) - (cca) - (cbc) + (cba) - (acc) + (aca) + (abc) - (aba) = c - b$.

Thus conditions (o) and (i) in Example 4.8 are satisfied, hence $P$ is a GDP. According to condition (4) in Example 4.8, for $a, b \in P$, $a \perp b$ if and only if $a + b \in P$ and $a \leq a + b$. Hence, using (3), we obtain for all $a, b \in P$ and $c \in A$:

(4) $a \perp b$ implies $(abc) = (bac) = (cab) = (cba) = 0$.

By the additivity we get the following. For all $a, b \in P$,

$$a + b \in P \text{ if and only if } (aab) + (aba) + (abb) + (baa) + (bab) + (bba) = 0.$$

If $a, b, a + b \in P$ then

$$a \leq a + b \text{ if and only if } (aba) = 0 = (aab) + (baa) + (bab).$$

According to condition (4) it is now clear that for all $a, b \in P$,

(5) $a \perp b$ if and only if $(aba) = (bab) = (aab) = (baa) = (abb) = (bba) = 0$.

Our aim is to prove that the collection of all tripotents of a triple group, partially ordered as above, forms a WGOMP. So, it remains to show that conditions (ii) and (iii) in Example 4.8 are satisfied.

(ii) If $a, b, a+b \in P$ and $a \leq a+b$, i.e. $a \perp b$, then $b \perp a$, hence $b \leq a+b$ and thus $a+b$ is an upper bound of $a$ and $b$. If $c \in P$ and $a, b \leq c$ then by (2),

$$(acb) = ((aca)cb) = (a(cac)b) = (aab) = 0$$

and similarly $(bca) = 0$. Hence we get $(a+b\; c\; a+b) = (aca)+(acb)+(bca)+(bcb) = a+b$ and $(c\; a+b\; c) = (cac)+(cbc) = a+b$, which means that $a+b \leq c$. Therefore $a+b$ is the join of $a$ and $b$ in $(P, \leq)$.

(iii) Using the additivity and conditions (4) and (5) it is easy to see that if $a, b, c \in P$, $a \perp b, a \perp c$ and $b \perp c$ then $a \perp b+c$.

An abelian group $(A, +, 0)$ together with a mapping $A^3 \to A$, $(x, y, z) \mapsto (xyz)$ which is symmetric in the first and third variables and additive in all three variables is called a *Jordan triple group* if the following identities are satisfied:

(J) $(ab(cde)) - (cd(abe)) = ((abc)de) - (c(dab)e)$,

(j) $((aba)ca) = (ab(aca))$.

(J) is known as the *Jordan triple identity*. Let us note that if $A$ satisfies the condition $3x = 0$ implies $x = 0$ for all $x \in A$, then the identity (j) follows from (J) and the symmetry (this can be shown similarly as in [**M2**]). Every Jordan triple group is a triple group. Indeed, replacing $(c, d)$ by $(a, b)$ in (J), we get

$$(ab(abe)) - (ab(abe)) = ((aba)be) - (a(bab)e),$$

which, using the symmetry, implies (2). The identity (1) is known as the fundamental-formula and can be derived similarly as in [**M2**, Theorem 2.2].

By a *Jordan triple* we mean a module $A$ over a ring $R$ with a unit endowed with an operation $A^3 \to A$, $(x, y, z) \mapsto (xyz)$ which is linear in all three variables such that $A$ is a Jordan triple group (cf. [**M2**]). The mapping $A^3 \to A$, $(x, y, z) \mapsto (xyz)$ is called the *Jordan triple product*.

In [**Ba**], a *Jordan triple* is a complex vector space $\mathcal{A}$ equipped with a mapping $\mathcal{A}^3 \to \mathcal{A}$, $(x, y, z) \mapsto (xyz)$ which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies the Jordan triple identity (J). In [**L3**], this system is called a *hermitean Jordan triple*. Every Jordan triple is a Jordan triple group.

A *JB*-triple* $\mathcal{A}$ is a hermitean Jordan triple which is a Banach space such that the mapping from $\mathcal{A}^2$ to the Banach space of bounded linear operators on $\mathcal{A}$, defined on all pairs $(a, b)$ of elements in $\mathcal{A}$ and all $c$ in $\mathcal{A}$, by $D(a, b)c := (abc)$, is continuous and such that, for all elements $a$ in $\mathcal{A}$, $D(a, a)$ is hermitean and with non-negative spectrum and $\|D(a, a)\|$ is equal to $\|a\|^2$. For all elements $a, b$ and $c$ in a JB*-triple $\mathcal{A}$, $\|(abc)\| \leq \|a\|\|b\|\|c\|$ and $\|(aaa)\| = \|a\|^3$. A JB*-triple which is a Banach space dual is called a *JBW*-triple*. In a JBW*-triple $\mathcal{A}$ the Jordan triple product is separately w*-continuous.

The collection of all tripotents in a JBW*-triple together with the partial order defined by Loos has been studied in [**Ba**], [**R2**], where it was shown that the collection of all tripotents in a JBW*-triple is a GOMP.

Indeed, G1 follows by Lemma 3.5(iii) in [**Ba**]. In fact, the interval $[0,w]$ is a complete orthomodular lattice with an orthocomplementation defined by $v^{'w} = w - v$.

To prove G2, observe that if $u \le v \le w$, then $u, v$ are idempotents in a JBW-algebra $\mathcal{A}_2(w)^J_{sa}$,and $v-u$ is the relative complement in the subinterval $[0,v]$ of the orthomodular lattice of the idempotents in $\mathcal{A}_2(w)^J_{sa}$. This gives $v-u=(w-u)\wedge v$.

G3 follows by Corollary 3.7(ii) in [**Ba**].

Condition G4' follows by Corollary 3.10(i) in [**Ba**].

We note that an alternative proof of properties G1, G2 and G3 can be obtained by observing that the partial order introduced by Loos coincides with that we introduced in a triple group, therefore the collection of all tripotents in a JBW*-triple is a WGOMP.

An abelian group $(A,+,0)$ equipped with a mapping $A^3 \to A$, $(x,y,z) \mapsto (xyz)$ which is additive in all three variables is called an *alternative triple group* if the following identities are satisfied:

(A) $(ab(cde)) + (cd(abe)) = ((abc)de) + (c(bad)e)$,
(a1) $((abc)dc) = (ab(cdc))$,
(a2) $(ab(abc)) = ((aba)bc)$.

Putting $a+d$ instead of $a$ in the identity (a2) and then using (a2) and (A) we obtain the following identity (cf. the proof of (1.5) in [**L1**]):

(i) $((abc)bd) = (a(bcb)d)$.

Every alternative triple group is a triple group. Indeed, using the identity (a1) twice and then using (i) we get

$$\begin{aligned}((aba)c(aba)) = (((aba)ca)ba) &= ((ab(aca))ba)\\ &= (a(b(aca)b)a),\end{aligned}$$

which proves the fundamental-formula (1). Putting $a$ instead of $c$ in (i) we get the first identity in (2). The second one is obtained as follows:

$$(cb(aba)) \overset{(a1)}{=} ((cba)ba) \overset{(i)}{=} (c(bab)a).$$

By an *alternative triple* we mean a module $A$ over a ring $R$ with a unit endowed with a mapping $A^3 \to A$, $(x,y,z) \mapsto (xyz)$ which is linear in all three variables such that $A$ is an alternative triple group (cf. [**L2**]). Alternative triples are investigated in [**L1**], where an equivalent system of axioms is dealt with, instead of (a2) the following identity is used:

(ii) $((aba)cd) = (a(cab)d)$.

**Example 4.10.** Using the partial order $\leq$ on the set of all tripotents in a triple group $A$ from Example 4.9 we can obtain generalizations of known results about idempotents (projections) in rings ($*$-rings). Known partial orders on idempotents (projections) in rings ($*$-rings) are extended to tripotents. It follows that the tripotents in special classes of rings and $*$-rings form a WGOMP (as to the examples (a) and (c) below cf. [**Ch**], Proposition 10).

(a) Let $R$ be an alternative ring, i.e. a ring $R$ satisfying $x^2y = x(xy)$ and $yx^2 = (yx)x$ for all $x, y \in R$. Artin's theorem in $R$ says that any subring of $R$ generated by two elements is associative. Other well-known properties are Moufang identities ([**S**], [**Z, S, S, S**]): $(aba)c = a(b(ac))$, $c(aba) = ((ca)b)a$ and $a(bc)a = (ab)(ca)$ for all $a, b, c \in R$.

$R$ together with the mapping $f : R^3 \to R$, $(x, y, z) \mapsto (xy)z$ is a triple group. Clearly, $f$ is additive in all three variables. The first identity in (2) from Example 4.9 is clear for $f$ by Artin's theorem. The second one can be shown as follows. If $a, b, c \in R$ then

$$cb \cdot aba = (cb \cdot a)b \cdot a = (c \cdot bab)a.$$

Using Moufang identities we get for all $a, b, c \in R$,

$$aba \cdot c \cdot aba = a(b \cdot ac) \cdot aba = a \cdot (b \cdot ac)(ab) \cdot a = a(b \cdot aca \cdot b)a,$$

which proves the fundamental-formula (1) in Example 4.9.

The Cayley numbers [**Z, S, S, S**] form an alternative ring which is an example of a triple group which is not an alternative triple group in the sense of the definition from the preceding example. Indeed, conditions (a1), (a2) are satisfied, but (A) is not satisfied (even condition (ii) is not satisfied).

An element $a \in R$ is a tripotent if $f(a, a, a) = a$, i.e. if $a^3 = a$. Every idempotent of $R$ is a tripotent. Let $T$ denote the set of all tripotents in $R$. The partial order $\leq$ on $T$ from Example 4.9 has then the following form:

$$a \leq b \text{ if and only if } a = aba = bab.$$

By Example 4.9, $(T, \leq)$ is a WGOMP. Let us note that the binary relation $\leq$ on $T$ has also the following form (cf. Example 4.8(a)):

$$a \leq b \text{ if and only if } a^2 = ab = ba.$$

Indeed, if $a, b \in T$ and $a \leq b$ then $ab = aba \cdot b = a \cdot bab = a^2$ and $ba = b \cdot aba = bab \cdot a = a^2$. Conversely, if $a, b \in T$ and $ab = ba = a^2$ then $aba = a^2a = a$ and $bab = a^2b = aa^2 = a$, hence $a \leq b$.

(b) Let $R$ be an alternative ring with an involution $^*$. Then $R$ together with the operation $f : R^3 \to R$, $(x, y, z) \mapsto (xy^*)z$ is an alternative triple group (cf. [**L1**]). An element $a \in R$ is a tripotent if $f(a, a, a) = a$, i.e. if $aa^*a = a$. Every projection of $R$ is a tripotent. Let $T$ denote the collection of all tripotents in $R$. Then the partial order $\leq$ on $T$ from Example 4.9 has the following form:

$$a \leq b \text{ if and only if } a = ab^*a = ba^*b.$$

By Example 4.9, since $R$ is a triple group, $(T, \leq)$ is a WGOMP. Let us note that the binary relation $\leq$ on $T$ when restricted to the set $P$ of all projections of $R$ has the following form (cf. Example 4.8(b)):

$$a \leq b \text{ if and only if } ab = a.$$

If $R$ is associative then the relation $\leq$ on $T$ has the following form (cf. Example 4.6(b), 4.7(b)):

$$a \leq b \text{ if and only if } aa^* = ba^* \text{ and } a^*a = a^*b.$$

Namely, if $a, b \in T$ and $a \leq b$ then $aa^* = ba^*b.a^* = b.a^*ba^* = b(ab^*a)^* = ba^*$ and similarly $a^*a = a^*b$. Conversely, if $a, b \in T$, $aa^* = ba^*$ and $a^*a = a^*b$, then $aa^* = ab^*$, hence $ab^*a = aa^*.a = a$ and $ba^*b = aa^*.b = a.a^*b = aa^*a = a$, and thus $a \leq b$.

(c) Let $R$ be a Jordan ring equipped with the mapping $f : R^3 \to R$, $(x, y, z) \mapsto x(yz) - y(xz) + z(xy)$ As known, $R$ is a Jordan triple group (cf. [**M1**]). An element $a \in R$ is a tripotent if $f(a, a, a) = a$, i.e. if $a^3 = a$. Every idempotent of $R$ is a tripotent. Let $T$ denote the set of all tripotents of $R$. The partial order $\leq$ on $T$ from Example 4.9 has the following form:

$$a \leq b \text{ if and only if } a = 2a(ab) - a^2b = 2b(ab) - ab^2.$$

If $a, b \in T$ and $a \leq b$ then by (3*) in Example 4.9, $a = f(a, a, a) = a^2b$ and $a = f(a, b, b) = ab^2$, and by (3) in Example 4.9, $f(a, b, a^2) = f(a, a, a^2)$ from which it follows $ab = a^2$. Thus the relation $\leq$ on $T$ has the following form (cf. Example 4.7(c)):

$$a \leq b \text{ if and only if } ab = a^2 \text{ and } a^2b = ab^2 = a.$$

## 5. Concluding Remarks

An orthomodular lattice, an orthomodular poset and a difference poset have a least and a greatest elements. A generalized orthomodular lattice, a weak generalized orthomodular poset and a generalized difference poset have a least element but need not have a greatest element. It has been shown in [**J1**] that every GOML is an orthomodular ideal of an OML and in [**M-I**] it is proved that every WGOMP is an order ideal of an OMP. We have shown that every GDP is an order ideal of a

DP. The result in [**M-I**] is an extension of the result in [**J1**] from lattices to posets, with a smallest element. Our result is a direct extension of the result in [**M-I**]. A related result, a generalization of the result in [**J1**] in another direction, from lattices with a smallest element to lattices which need not have a smallest element, was obtained by one of the authors in [**H**] for relatively orthomodular lattices. Every relatively orthomodular lattice (ROML) is a dual ideal of a GOML. It appears that a further generalization of the results in [**H**] and in [**M-I**] is possible by a suitable extension of the definition of a WGOMP, which need not have a smallest element. Or, even, it seems to be possible to obtain a common generalization of the results in [**H**] and in the present paper by a suitable modification of the definition of a poset with a difference. The situation is indicated in Fig. 7.

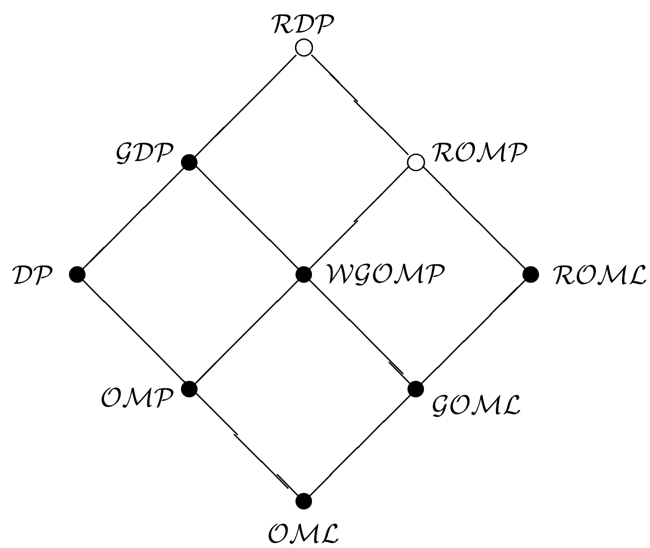


**Fig. 7**

The small filled circles represent the above mentioned known classes of orthomodular structures, the orthomodular lattices ($\mathcal{OML}$), the orthomodular posets ($\mathcal{OMP}$), the difference posets ($\mathcal{DP}$), the generalized orthomodular lattices ($\mathcal{GOML}$), the weak generalized orthomodular posets ($\mathcal{WGOMP}$), the generalized difference posets ($\mathcal{GDP}$) and the relatively orthomodular lattices ($\mathcal{ROML}$). The remaining two small circles represent hypothetic classes of orthomodular structures, relatively orthomodular posets ($\mathcal{ROMP}$) and relatively difference posets ($\mathcal{RDP}$). We will discuss them in a subsequent paper. The whole Fig. 7 depicts an inclusion partial order on the set of these nine classes of orthomodular structures.

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