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DEGREES, NEIGHBOURHOODS, AND CLOSURE OPERATIONS

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Abstract. Closure theorems in graph theory are of the following type: Let G be a graph, $\mathcal{P}$ a graph theoretic property, and let u and v be two non-adjacent vertices of G. If condition $c(u,v)$ holds, then G has property $\mathcal{P}$ if and only if $G+uv$ has $\mathcal{P}$.

We discuss several such results of the above type where the condition $c(u,v)$ refers to neighbourhood properties of u and v.

1. Introduction

Bondy and Chvátal [**2**] extended the classical hamiltonian condition of Ore [**8**] as follows: Given a graph $G=(V_G,E_G)$ of order n and $u,v \in V_G$. If $uv \notin E_G$ and $d(u)+d(v) \geq n$, then G is hamiltonian if and only if $G+uv$ is hamiltonian. In this connection one can define a graph $C_n(G)$, called the n-**closure** of G, as the result of successively joining pairs of non-adjacent vertices with degree sum at least n until no such pair remains. Therefore, G is hamiltonian if $C_n(G)$ is hamiltonian. Moreover, Bondy and Chvátal [**2**] generalized this idea to several graph-theoretic properties.

Inspired by these results many other closure concepts were developed, see for example [**1**], [**3**], [**5**], [**6**], [**7**], [**12**]. Some closure concepts involve information on "local" structure [**3**], [**6**], whereas the others involve information on "global" parameters of G. We refer the reader to a survey [**4**] for other closure concepts.

In this note we derive new closure theorems for some graph-theoretic properties related to cycles, which generalize the corresponding results of Bondy and Chvátal [**2**]. Our results are based on the following graph invariants which turned out to be "congenial" in conjunction with sufficient conditions for the existence of certain cycles in graphs, see e.g. [**9**], [**10**], [**11**].

In a graph G the set of neighbours of a vertex $u \in V_G$ will be denoted by $N(u)$, and the graph induced by the vertices in S, where $S \subseteq V_G$, will be denoted by $G[S]$. Let u and v be two non-adjacent vertices of a graph G. We define $\psi_G(u,v)$ to be the number of components of $G[N(u)]$ containing no neighbour of v. Let

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$\alpha_G(u, v)$ be the number of vertices in G that are adjacent to both u and v, and let $\beta_G(u, v)$ be the number of vertices $\neq v$ that are at distance two from u and are non-adjacent to v. Finally, we define

$$\chi_G(u, v) = \psi_G(u, v) + \text{pos}(\alpha_G(u, v) - \beta_G(u, v) - 1),$$

where $\text{pos}(x) = \max\{x, 0\}$. If no confusion arises, the subscript G will be usually omitted.

To see the significance of ψ and χ let us observe that in any graph of girth at least 5, $\psi(u, v) \geq \delta - 1$ for any pair of non-adjacent vertices u and v, where δ is the minimum degree of the graph. On the other hand if there are two non-adjacent vertices u and v with "many" neighbours in common, then, usually, $\psi(u, v)$ is "small", but $\alpha(u, v) - \beta(u, v)$ can be "large". Thus χ may be well applicable to both sparse and dense graphs, and is well suited for the closure operations.

Let us note that each of the conditions in the next section can be checked in polynomial time. Thus as in **[2]**, our results lead to algorithms which construct the closure based on the conditions in polynomial time. Moreover, if this closure is complete, then for example any hamiltonian cycle in K_n can be transformed into a hamiltonian cycle in the original graph in polynomial time.

2. Results

The following lemma from **[10]** plays an important role in our proofs.

Lemma 1. *Let G be a non-hamiltonian graph with a hamiltonian path $v_1 v_2 \dots v_n$, where $v_1 v_n \notin E_G$. Then*

$$d(v_1) + d(v_n) < n - \max\{\chi(v_1, v_n), \chi(v_n, v_1)\}. \tag{1}$$

Theorem 1. *Let u and v be two non-adjacent vertices of G such that $d(u) + d(v) \geq n - \max\{\chi(u, v), \chi(v, u)\}$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

Proof. If G is hamiltonian, then obviously so is $G + uv$. Now, assume $G + uv$ is hamiltonian, but G is not. Thus G contains a hamiltonian path $v_1 v_2 \dots v_n$, where $v_1 = u$ and $v_n = v$. A contradiction follows from (1). □

Theorem 2. *Let $\phi(u, v) = \max\{\psi(u, v), \psi(v, u)\}$. Let u and v be two non-adjacent vertices of G such that $d(u) + d(v) \geq 2n - s - \phi(u, v)$, where $5 \leq s \leq n$. Then G contains C_s if and only if $G + uv$ contains C_s.*

Proof. If G contains C_s, then obviously $G + uv$ does. Assume now that $G + uv$ contains a cycle C_s but G does not. Thus G contains a path $P = v_1 v_2 \dots v_s$, where

$v_1 = u$ and $v_s = v$. Let H be the graph induced by the vertices of P. Then $H+uv$ is hamiltonian but H is not. We may assume $\max\{\psi(u,v), \psi(v,u)\} = \psi(u,v)$. The other case is analogous. Let $\omega(u,v) = \text{pos}(\psi_G(u,v) - \psi_H(u,v))$. Obviously, $\omega(u,v)$ does not exceed the number of components of $G[N(u)]$ containing at least one vertex outside H and no neighbour of v. Thus, by (1) we have

$$\begin{aligned} d_G(u) + d_G(v) &= d_H(u) + d_H(v) + d_{G-H}(u) + d_{G-H}(v) \\ &< s - \psi_H(u,v) + 2(n-s) - \omega(u,v) \leq 2n - s - \phi(u,v), \end{aligned}$$

a contradiction. □

Theorem 3. *Let* $\phi(u,v) = \min\{\psi(u,v) + pos(\alpha(u,v) - \beta(u,v) - 2),\ \psi(v,u) + pos(\alpha(v,u) - \beta(v,u) - 2)\}$. *Let* u *and* v *be two non-adjacent vertices of a connected graph* G *such that* $d(u) + d(v) \geq n - 1 - \phi(u,v)$. *Then* G *contains a hamiltonian path if and only if* $G + uv$ *contains a hamiltonian path.*

Proof. If G contains a hamiltonian path, then $G + uv$ does so. Assume now, that $G+uv$ contains a hamiltonian path but G does not. Hence $G+uv$ contains a path $P = v_1 v_2 \dots v_i v_{i+1} \dots v_n$, where $u = v_i$ and $v = v_{i+1}$ for some $1 \leq i < n$. Let H be the graph induced by the vertices of P with one extra vertex x and edges xv_1, xu, xv, and xv_n. Obviously, $H + uv$ is hamiltonian but H is not.

Since G is connected, at least one of the edges $v_1 u$ and vv_n is not in G, or $i = 1$ or $n - 1$. Indeed, if both $v_1 u, vv_n \in E_G$, then since there must exist an edge $v_k v_l$, $1 \leq k \leq i$ and $i + 1 \leq l \leq n$, the path $v_{k-1} v_{k-2} \dots v_1 u v_{i-1} v_{i-2} \dots v_k v_l$ $v_{l-1} v_{l-2} \dots v v_n v_{n-1} \dots v_{l+1}$ is a hamiltonian path in G, a contradiction. Thus we assume $v_1 u$ is not an edge of G or $i = 1$.

Since u is not adjacent to v and v_n, we have $\psi_H(u,v) = \psi_G(u,v)$, $\alpha_H(u,v) = \alpha_G(u,v) + 1$, and $\beta_H(u,v) \leq \beta_G(u,v) + 2$. Therefore, by (1) we have

$$\begin{aligned} d_G(u) + 1 + d_G(v) + 1 &= d_H(u) + d_H(v) \\ &< n + 1 - \psi_H(u,v) - \text{pos}(\alpha_H(u,v) - \beta_H(u,v) - 1) \\ &\leq n + 1 - \psi_G(u,v) - \text{pos}(\alpha_G(u,v) - \beta_G(u,v) - 2) \\ &\leq n + 1 - \phi(u,v), \end{aligned}$$

a contradiction. □

Since every graph with a hamiltonian path is connected, the connectivity condition in Theorem 3 cannot be weakened in general.

A graph G is said to be s**-edge hamiltonian** if for each set E of s edges of G that form pairwise disjoint paths in G there exists a hamiltonian cycle in G containing E.

Theorem 4. *Let* $\phi(u,v) = \max\{\psi(u,v), \psi(v,u)\}$. *Let* u *and* v *be two non-adjacent vertices of* G *such that* $d(u) + d(v) \geq n + s - \phi(u,v)$. *Then* G *is* s*-edge hamiltonian if and only if* $G + uv$ *is* s*-edge hamiltonian.*

Proof. If G is s-edge hamiltonian, then so is $G + uv$. Conversely, suppose that $G+uv$ is s-edge hamiltonian but G is not. Then there is a set of s edges E that form pairwise disjoint paths in G, and $G+uv$ has a hamiltonian cycle containing all the edges of E but G does not. Consider the graph H obtained from G by subdividing each edge in E into two. Then $H + uv$ is hamiltonian but H is not. Moreover, $\psi_H(x,y) \geq \psi_G(x,y)$ for any non-adjacent vertices x and y of G. Therefore, by (1) we have

$$d_G(u) + d_G(v) < n + s - \max\{\psi_H(u,v), \psi_H(v,u)\} \leq n + s - \phi(u,v),$$

a contradiction. □

A graph G is defined to be **hamilton-connected** if for each pair of vertices x and y from G there is a hamiltonian path in G joining x and y.

Theorem 5. *Let* $\phi(u,v) = \max\{\psi(u,v) + pos(\alpha(u,v) - \beta(u,v) - 2), \psi(v,u) + pos(\alpha(v,u) - \beta(v,u) - 2)\}$. *Let* u *and* v *be two non-adjacent vertices of a connected graph* G *such that* $d(u) + d(v) \geq n + 1 - \phi(u,v)$. *Then* G *is hamilton-connected if and only if* $G + uv$ *is hamilton-connected.*

Proof. If G is hamilton-connected, then so is $G + uv$. Suppose that $G + uv$ is hamilton-connected but G is not. Thus there are two vertices x and y such that [$xy \in E_G$] ($xy \notin E_G$) and the graph $G + uv$ contains [a hamiltonian cycle containing the edge xy] (a hamiltonian path with end-vertices x and y) but G does not.

Let H be the graph obtained from G by adding extra vertex w and edges xw and yw. Then $H + uv$ is hamiltonian but H is not. We may assume $\phi(u,v) = \psi(u,v) + \mathrm{pos}(\alpha(u,v) - \beta(u,v) - 2)$. Since if $u = x$, then $v \neq y$ and vice versa, we have $\psi_H(u,v) \geq \psi_G(u,v)$, $\alpha_H(u,v) = \alpha_G(u,v)$, and $\beta_H(u,v) \leq \beta_G(u,v) + 1$. Therefore, by (1)

$$\begin{aligned} d_G(u) + d_G(v) &\leq d_H(u) + d_H(v) \\ &< n + 1 - \psi_H(u,v) - \mathrm{pos}(\alpha_H(u,v) - \beta_H(u,v) - 1) \\ &\leq n + 1 - \psi_G(u,v) - \mathrm{pos}(\alpha_G(u,v) - \beta_G(u,v) - 2) \\ &= n + 1 - \phi(u,v), \end{aligned}$$

a contradiction. □

3. Other Closure Operations

Here we compare our results with some existing ones. First let us note that all the conditions in the previous section generalize the corresponding conditions from **[2]**. For example the following is a corollary to Theorem 1.

Corollary 1. ([**2**]) *Let u and v be two non-adjacent vertices of G such that $d(u) + d(v) \geq n$. Then G is hamiltonian if and only if $G + uv$ is hamiltonian.*

In [**5**] closure operations based on neighbourhoods were developed. Our Theorem 3 has the corresponding result from [**5**] as a corollary.

Corollary 2. ([**5**]) *Let u and v be two non-adjacent vertices of a connected graph G such that $|N(u) \cup N(v)| \geq n - 2$. Then G has a hamiltonian path if and only if $G + uv$ has a hamiltonian path.*

Proof. Since $uv \notin E_G$, we must have $|N(u) \cup N(v)| = n - 2$. Now, either $|N(u) \cap N(v)| \geq 1$, or $\psi(u, v), \psi(v, u) \geq 1$. In both cases we have $d(u) + d(v) \geq n - 1 - \min\{\psi(u, v), \psi(v, u)\}$. □

In the rest we show that our results are non-comparable with some other closure operations by giving infinitely many examples. In fact, we provide the examples only for Theorem 1, but for remaining theorems similar examples can be found with some extra effort.

For each $p \geq 3$ we define a graph $\mathcal{F}_p$ of order $2p^2+2p+1$ as the graph consisting of $p + 1$ copies of K_{p+1} with vertex set $\{u_{1,i}, u_{2,i}, \ldots, u_{p+1,i}\}$, $i = 1, 2, \ldots, p + 1$, and of p^2 vertices $v_1, v_2, \ldots, v_{p^2}$. For each vertex $u_{k,l}$, where $k = 2, 3, \ldots, p + 1$, and $l = 1, 2, \ldots, p+1$, we add p^2-p edges of the form $u_{k,l}v_j$, where $j \in \{1, \ldots, p^2\}$. Now distinguish two vertices $u = u_{i,j}$ and $v = u_{k,l}$, where $2 \leq i, k \leq p + 1$, $1 \leq j, l \leq p+1$, and $j \neq l$. Finally, add $2p$ edges (p edges for u and p edges for v) joining u and v with the remaining vertices of the form v_j (vertices previously not joined to u and v, respectively). Thus we have defined $\mathcal{F}_p$ and we have distinguished two vertices u and v in $\mathcal{F}_p$. Let us note that if we add the edges of the for $u_{k,l}v_j$ in some reasonable manner, we can ensure that $\mathcal{F}_p$ is hamiltonian.

One can check that $d(u) = d(v) = p^2 + p$, and $\alpha(u, v) - \beta(u, v) - 1 = p - 1$. Thus we can apply Theorem 1 and ensure that $\mathcal{F}_p$ is hamiltonian if and only if $\mathcal{F}_p + uv$ is hamiltonian.

In [**1**], the following closure operation for hamiltonian cycles is introduced. Let $T_{u,v} = \{x : u, v \notin N(x)\}$, and let $\delta_{u,v} = \min_{x \in T_{u,v}} d(x)$. In [**1**] it is proved that if G is 2-connected and u, v are two non-adjacent vertices of G such that $|T_{u,v}| + 2 \leq \delta_{u,v}$, then G is hamiltonian if and only if $G + uv$ is hamiltonian. Since $|T_{u,v}| = p^2 - 1$ and $\delta_{u,v} = p$ in $\mathcal{F}_p$, we cannot use the condition for vertices u and v in $\mathcal{F}_p$.

In [**3**] the authors prove several theorems similar to the following one: Let x, y, u, and v be four vertices of a 2-connected graph G such that u and v are non-adjacent, and $x, y \in N(u)$. If $N(x) \cup N(y) \subseteq N(v) \cup \{u, v\}$, then G is hamiltonian if and only if $G+uv$ is hamiltonian. One can observe that in $\mathcal{F}_p$ and u, v as defined, for any choice of x and y from $N(u)$ there always exists at least one vertex from $N(x)$ or $N(y)$ that is not in $N(v)$. Hence the condition cannot be used for vertices u and v in $\mathcal{F}_p$.

Finally, we show that the hamiltonicity result from **[12]** is also non-comparable with ours. The condition from **[12]** is: Let u and v be two non-adjacent vertices of G, $R = V_G \setminus (N(u) \cup N(v) \cup \{u, v\})$, $\Theta = n - d(u) - d(v)$, and $R' = \{x \ : \ x \in R, d(x) \geq |R| + \max\{2, \Theta\}\}$. If $d(u) + d(v) \geq n - |R'|$, then G is hamiltonian if , and only if $G + uv$ is hamiltonian. Since $\Theta = 1$, $|R| = p^2 - 1$, and $|R'| = 0$, we cannot use this condition in $\mathcal{F}_p$ for u and v.

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