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# ON A CLASS OF DENSITIES OF SETS OF POSITIVE INTEGERS

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Abstract. A method proposed by R. Alexander in his paper published in Acta Arithmetica XII (1967) enables to obtain various densities of set of positive integers, including asymptotic and logarithmic ones. In our paper some properties of the above mentioned densities are studied and certain earlier results on the asymptotic and logarithmic density are strengthened.

## Introduction and notations

In what follows we assume that $c_n > 0$ $(n = 1, 2, \dots)$ and $\sum\limits_{n=1}^{\infty} c_n = +\infty$. If $A \subseteq \mathbb{N}$, we put

$$h_n(A) = \frac{1}{s_n} \sum_{k=1}^{n} \chi_A(k) c_k \ (n = 1, 2, \dots), \quad \text{where} s_n = c_1 + \dots + c_n \ (n = 1, 2, \dots)$$

and $\chi_A$ is the characteristic function of $A$, i.e. $\chi_A(k) = 1$ if $k \in A$ and $\chi_A(k) = 0$ otherwise.

We set

$$h(A) = \lim_{n \to \infty} h_n(A) \tag{1}$$

whenever the limit on the right-hand side exists.

Observe that the set functions $h_n$ $(n = 1, 2, \dots)$ defined on the set $2^{\mathbb{N}}$ are $\sigma$-additive, while the function $h$ is additive and defined on the class $\mathcal{S}_h$ of all $A \subseteq \mathbb{N}$ for which the limit on the right-hand side of (1) exists.

Taking $c_n = 1$, $c_n = 1/n$ $(n = 1, 2, \dots)$ the function $h$ will mean the asymptotic density $d$, the logarithmic density $\delta$, respectively ($\mathcal{S}_h$ will mean $\mathcal{S}_d$, $\mathcal{S}_\delta$ respectively) (see [**11**, pp. 246–249], [**6**, pp. 21–22, 32–35]). It is well-known that $\mathcal{S}_d \subseteq \mathcal{S}_\delta$ [**6**, p. 34].

We shall use the concept of Baire's metric space. Denote by $\mathcal{P}$ the set of all infinite sequences of natural numbers (we identify the sequence $a_1 < a_2 < \dots < a_n < \dots$ and the set $\{a_1, a_2, \dots, a_n, \dots\}$). If $A = (a_n)$, $B = (b_n)$ belong to

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$\mathcal{P}$, the distance between $A$ and $B$ will be defined by $\rho(A,B) = 0$ if $A = B$, i.e. $a_n = b_n$ for all $n$ and by $\rho(A,B) = 1/\min\{n : a_n \neq b_n\}$ otherwise. The metric space $(\mathcal{P},\rho)$ is complete (see [**10**, p. 95], [**15**]).

Further we recall the concept of porosity of sets in a metric space in consent with [**15**] and [**17**].

Let $(Y,d)$ be a metric space, let $y \in Y$ and $r > 0$. Denote by $B(y,r)$ the ball in $Y$, i.e. $B(y,r) = \{x \in Y; d(x,y) < r\}$. If $M \subseteq Y$, then for $y \in Y$ we set

$$\gamma(y,r,M) = \sup\{t > 0 : (\exists z \in Y)(B(z,t) \subseteq B(y,r)) \wedge (B(z,t) \cap M = \emptyset)\}.$$

If such a $t$ does not exist, we put $\gamma(y,r,M) = 0$. The numbers $\bar{p}(y,M) = \lim_{r\to 0_+}\sup \frac{\gamma(y,r,M)}{r}$, $\underline{p}(y,M) = \lim_{r\to 0_+}\inf \frac{\gamma(y,r,M)}{r}$ are called the upper and lower porosity of $M$ at $y$, respectively.

If $\bar{p}(y,M) = \underline{p}(y,M) = p(y,M)$, then the number $p(y,M)$ is called the porosity of $M$ at $y$.

The numbers $\bar{p}(y,M)$, $\underline{p}(y,M)$ and $p(y,M)$ belong to the interval [0, 1].

A set $M \subseteq Y$ is called porous (very porous) at $y$ if $\bar{p}(y,M) > 0$ ($\underline{p}(y,M) > 0$).

If $c > 0$, then $M$ is called $c$-porous (very $c$-porous) at $y$ provided that $\bar{p}(y,M) \geq c$ ($\underline{p}(y,M) \geq c$).

A set $M \subseteq Y$ is called strongly porous at $y$ if $\underline{p}(y,M) = 1$ (i.e. if $p(y,M) = 1$).

A set $M \subseteq Y$ is called porous, very porous, $c$-porous, very $c$-porous and strongly porous in $Y$ if it is porous, very porous, $c$-porous, very $c$-porous and strongly porous at every $y \in Y$, respectively.

A set $M \subseteq Y$ is called $\sigma$-porous ($\sigma$-very porous) in $Y$ if $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets $M_n$ ($n = 1, 2, \dots$) is porous (very porous) in $Y$.

A set $M \subseteq Y$ is called $\sigma$-$c$-porous, $\sigma$-very $c$-porous and $\sigma$-strongly porous in $Y$ if $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets $M_n$ ($n = 1, 2, \dots$) is $c$-porous, very $c$-porous and strongly porous in $Y$, respectively.

If a set $M$ is porous in $Y$, then it is nowhere-dense in $Y$.

Every $\sigma$-porous set is a set of the first Baire category in $Y$.

Consequently, both the porosity and the $\sigma$-porosity are useful tools to describe the structure of nowhere-dense sets and of sets of the first Baire category more precisely.

## 1. The basic properties of measures $h, h_n$

The measures $h, h_n$ can be viewed as an application of the following summability method to the sequences of numbers 0's and 1's.

The method defined by the matrix

$$C = \begin{pmatrix} c_1/s_1 & & & \\ c_1/s_2, & c_2/s_2 & & \\ \vdots & & & \\ c_1/s_n, & c_2/s_n, & \dots, & c_n/s_n \\ \vdots & & & \end{pmatrix}$$

is said to be $(C)$ method (see [**4**, pp. 72–73] [**13**, p. 4]). It is obvious that the matrix $C$ satisfies the conditions of regularity (see [**13**, p. 69]) and it belongs to the large class of triangular matrices studied in [**9**].

According to the known Steinhaus theorem (see [**4**, p. 93], [**13**, p. 78], [**16**]) there exists a sequence of 0's and 1's which is not summable by the method $(C)$. Such a sequence is the characteristic function of a set from $\mathcal{P}$. Then there is a set $A \in \mathcal{P}$ such that $\chi_A$ is not summable by the method $(C)$ and so $A \notin \mathcal{S}_h$.

Sufficient conditions for the existence of a non-convergent sequence of 0's and 1's which is summable by a matrix method were given in [1] (the considered sequence contains infinitely many 0's and 1's and so, it is a characteristic function of a set $A \in \mathcal{P}$).

Set

$$a_{nk} = \frac{c_k}{s_n} \qquad 1 \le k \le n,$$
$$a_{nk} = 0 \qquad k > n.$$

The sufficient conditions mentioned above are of the form:

(a)
$$\sum_{k=1}^{\infty} |a_{nk}| < +\infty \qquad n = 1, 2, \dots,$$

(b)
$$\lim_{n\to\infty} \max_{1\le k\le n} |a_{nk}| = 0.$$

Since $\sum\limits_{k=1}^{n} (c_k/s_n) = 1$ for all $n$, (a) is fulfilled.

The condition (b) says

(2)
$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{c_k}{s_n} = 0,$$

which can be written as

$$\max_{1\le k\le n} c_k = o(s_n) \ (n \to \infty)$$

((2) holds for instance if the sequence $(c_n)_{n=1}^{\infty}$ is bounded).

We shall show that condition (2) is equivalent to a seemingly stronger condition

(3)
$$\lim_{n\to\infty} \frac{c_n}{s_n} = 0.$$

**Proposition 1.1.** *For every sequence* $(c_n)_{n=1}^{\infty}$, $c_n > 0$, *such that* $\sum\limits_{n=1}^{\infty} c_n = +\infty$, *conditions* (2) *and* (3) *are equivalent.*

*Proof.* We have obviously $\max\{c_k, k \le n\} \ge c_n$. But, then (2) implies (3).

If the sequence $(c_n)_{n=1}^{\infty}$ is bounded, then both limits from conditions (2) and (3) are equal to zero.

Thus, we can suppose that the sequence $(c_n)_{n=1}^{\infty}$ is not bounded. Denote by $i(n)$ the largest index of the maximal element of the finite sequence $c_1, \dots, c_n$ (then $c_{i(n)} = \max\{c_k; k \le n\}$). Since the sequence $(c_n)_{n=1}^{\infty}$ is not bounded, $i(n) \to \infty$ as $n \to \infty$ holds.

Now

$$0 \le \frac{c_{i(n)}}{s_n} \le \frac{c_{i(n)}}{s_{i(n)}}$$

and (3) implies (2). □

We shall summarize our previous considerations.

**Theorem 1.1.** *Let* $c_n > 0$ $(n = 1, 2, \dots)$ *and* $\sum_{n=1}^{\infty} c_n = +\infty$. *Then the following statements are true:*

(i) *there is a set* $A \in \mathcal{P}$ *such that* $A \in \mathcal{P} \backslash \mathcal{S}_h$.

(ii) *If* (3) *is valid, then there exists an* $A \in \mathcal{P}$ *such that* $\mathbb{N} \backslash A$ *is infinite and* $A \in \mathcal{S}_h$.

**Corollary.** *If the assumption of* (ii) *holds, then* $\mathcal{T} \subsetneqq \mathcal{S}_h$ *where* $\mathcal{T}$ *denotes the set of all* $A \in 2^{\mathbb{N}}$ *such that* $A$ *or* $\mathbb{N} \backslash A$ *are finite sets.*

It is well-known that the set of values of the asymptotic density $d$, the logarithmic density $\delta$ as well, fill the interval [0,1] (i.e. $d(\mathcal{S}_d) = [0, 1]$, $\delta(\mathcal{S}_\delta) = [0, 1]$). In this connection we shall show that the density $h$ possesses the same property provided that the sequence $(c_n)_{n=1}^{\infty}$ satisfies (3). In the first place we prove the following auxiliary result.

**Lemma 1.1.** *a) If* $\lim_{n\to\infty} (c_n/s_n) = 0$, *then for every* $A \subseteq \mathbb{N}$ *we have* $\lim_{n\to\infty} h_n(A) - h_{n-1}(A) = 0$.

*b) If there exists* $A \subseteq \mathbb{N}$ *such that* $0 < h(A) < 1$, *then* $\lim_{n\to\infty} (c_n/s_n) = 0$.

*Proof.* From definition of $h$ directly follows that for every $A \subseteq \mathbb{N}$ we have

$$h_n(A) - h_{n-1}(A) = (\chi_A(n) - h_{n-1}(A)) \frac{c_n}{s_n}. \tag{4}$$

This implies (a). For (b) the existence of $h(A)$ implies that $\lim_{n\to\infty} h_n(A) - h_{n-1}(A) = 0$ which is impossible, assuming $\limsup_{n\to\infty} (c_n/s_n) > 0$ on the right-hand side of (4). □

**Theorem 1.2.** *The values of the measure h fill the interval* $[0, 1]$ *if and only if* $\lim_{n\to\infty} (c_n/s_n) = 0$.

*Proof.* 1) Necessarily follows from Lemma 1.1.

2) We shall show that for every $v \in [0, 1]$ there is a set $B \in \mathcal{S}_h$ such that $h(B) = v$.

If $v = 0$, $v = 1$, then it suffices to choose $B = \emptyset$ or $B = \mathbb{N}$ respectively.

Suppose $0 < v < 1$. We shall construct the set $B$ in the form $B = \bigcup_{n=1}^{\infty} (a_n, b_n] \cap \mathbb{N}$ where $a_n, b_n \in \mathbb{N}$, $a_n < b_n < a_{n+1}$. If the intervals $(a_1, b_1], \dots, (a_n, b_n]$ such that $h_{b_n}(B) > v$ are given, then we choose $(a_{n+1}, b_{n+1}]$ such that

$$\begin{aligned} h_{a_{n+1}}(B) &< v \le h_{a_{n+1}-1}(B), \\ h_{b_{n+1}-1}(B) &\le v < h_{b_{n+1}}(B). \end{aligned}$$

By Lemma 1.1(a) we have $h_{a_n}(B) \to v$ , $h_{b_n}(B) \to v$. Since the sequence $h_x(B)$ is monotonous on intervals $[a_n, b_n]$ and $[b_n + 1, a_{n+1} - 1]$ we get $h_n(B) \to v$. □

The previous Theorem 1.2 will be strengthened in the following theorem. Recall that the density $h$ is said to have Darboux property provided that for each $A \subseteq \mathbb{N}$ with $h(A) > 0$ and each $t \in [0, h(A)]$ there exists a set $B \subseteq A$ such that $h(B) = t$ (see [**7**]).

**Theorem 1.3.** *The density* $h$ *has the Darboux property if and only if* $\lim_{n\to\infty}(c_n/s_n) = 0$.

*Proof.* 1) Necessarily follows from Theorem 1.2.

2) Suppose that (3) holds. Let $A = \{a_1 < a_2 < \cdots < a_n < \dots\} \subseteq \mathbb{N}$ be such that $h(A) = a \in [0, 1]$. Let $b \in [0, a]$. We will find a set $B \subseteq A$ such that $h(B) = b$.

If $a = 0$, then any subset $B$ of $A$ has zero density.

Thus we can suppose that $a > 0$. Now let us take the sequence $d_n = c_{a_n}$, $n = 1, 2, \dots$ and consider the density $h'$ based on this sequence. Since $a > 0$ the sequence $(d_n)$ also satisfies (3). Hence, by Theorem 1.2 there exists a set $I \subseteq \mathbb{N}$ such that

$$h'(I) = \frac{b}{a}.$$

We now show, that for the set $B = A_I = \{a_n, n \in I\} \subseteq A$, $h(B) = b$ holds:

$$h_n(B) = \frac{\sum_{k=1}^{n} \chi_B(k)c_k}{s_n} = \frac{\sum_{k=1}^{n} \chi_B(k)c_k}{\sum_{k=1}^{n} \chi_A(k)c_k} \cdot \frac{\sum_{k=1}^{n} \chi_A(k)c_k}{s_n}.$$

Now, for $n \to \infty$ the first factor converges to $h'(I)$, while second one converges to $h(A)$. Thus $h(B) = h'(I) \cdot h(A) = (b/a) \cdot a = b$. □

## 2. Structure of the space $(\mathcal{P}, \rho)$ from the standpoint of the behaviour of the sequence $(h_n(A))_{n=1}^{\infty}$, $A \in \mathcal{P}$

In this part of the paper we shall be concerned with the behaviour of the sequence $(h_n(A))_{n=1}^{\infty}$, where $A \in \mathcal{P}$. We shall deduce certain general and in a certain sense definite result, which enables us to judge the magnitude of the system $\mathcal{S}_h$ ($\mathcal{S}_d$, $\mathcal{S}_\delta$ specially) from topological point of view.

Although the densities $h$ we defined in the first part depend on the choice of series $\sum_{n=1}^{\infty} c_n$, a general result can be proved (Theorem 2.1) for a wide class of these series. We note that from Theorem 1.1 given in [**14**] it follows that $\mathcal{S}_h \cap \mathcal{P}$ is the set of the first Baire category in the space $\mathcal{P}$. Next theorem improves this assertion.

We recall that a number $t \in \mathbb{R}$ is said to be a limit point of a sequence $(a_n)_{n=1}^{\infty}$ ($a_n \in \mathbb{R}$, $n = 1, 2, \dots$) provided that there exists a sequence $n_1 < n_2 < \dots$ such that $a_{n_k} \to t$ $(k \to \infty)$. Denote by $(h_n(A))'_n$ where $A \in \mathcal{P}$, the set of all limit points of the sequence $(h_n(A))_{n=1}^{\infty}$.

First we shall prove an auxiliary result concerning the set $(h_n(A))'_n$.

**Proposition 2.1.** *If* $\lim_{n\to\infty}(c_n/s_n) = 0$*, then for every* $A \subseteq \mathbb{N}$*, the set of all limit points of* $(h_n(A))_{n=1}^{\infty}$ *is connected, i.e. forms an interval.*

*Proof.* Follows from Lemma 1.1(a) and the following theorem of [**3**]:
If $(t_n)_{n=1}^{\infty}$ is a sequence in a metric space $(X,\rho)$ satisfying

i) any subsequence of $(t_n)_{n=1}^{\infty}$ contains a convergent subsequence, and
ii) $\lim_{n\to\infty}\rho(t_n,t_{n-1}) = 0$,

then the set of all limit points of $(t_n)_{n=1}^{\infty}$ is connected in $(X,\rho)$. □

**Theorem 2.1.** *Let* $c_n > 0 \quad (n = 1,2,\dots)$ *and* $\sum_{n=1}^{\infty} c_n = +\infty$*. Let* $(c_n)_{n=1}^{\infty}$ *satisfies* $\lim_{n\to\infty}(c_n/s_n) = 0$*. Then the set of all* $A \in \mathcal{P}$ *with*

$$(h_n(A))'_n = [0,1] \tag{5}$$

*is residual in the space* $\mathcal{P}$*.*

**Corollary.** *The sets* $\mathcal{S}_h \cap \mathcal{P}$*,* $\mathcal{S}_d \cap \mathcal{P}$ *and* $\mathcal{S}_\delta \cap \mathcal{P}$ *are of the first Baire category in the space* $\mathcal{P}$*.*

*Proof of Theorem 2.1.* Put

$$D = \{A \in \mathcal{P} : (h_n(A))'_n = [0,1]\}.$$

Since the set of all limit points of a sequence is closed we have

$$D = \bigcap_{t\in\mathbb{Q}\cap[0,1]} D_t\ , \tag{6}$$

where $\mathbb{Q}$ is the set of all rational numbers and $D_t = \{A \in \mathcal{P} : t \in (h_n(A))'_n\}$.

The set $D_t$ can be expressed in the form

$$D_t = \bigcap_{k=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcup_{n>j}^{\infty} D_{t,k,n}, \tag{7}$$

where

$$D_{t,k,n} = \{A \in \mathcal{P} : |h_n(A) - t| < \frac{1}{k}\}. \tag{7'}$$

For fixed $t,k,n$ the set $D_{t,k,n}$ is evidently open in $\mathcal{P}$. Hence $D_t$ is a $G_\delta$-set (see (7)).

It is easily to see that every set of the form $\{A \in \mathcal{S}_h \cap \mathcal{P};\ h(A) = t\}$ where $t \in [0,1]$ is dense in $\mathcal{P}$. This shows that also $D_t$ is a dense set in $\mathcal{P}$.

Consequently, the set $D_t$ is a dense $G_\delta$-set in $\mathcal{P}$. Therefore the set $D_t$ is residual in $\mathcal{P}$ (see [**8**, p. 49]) and so, the set $D = \bigcap_{t\in\mathbb{Q}\cap[0,1]} D_t$ is residual in $\mathcal{P}$, too. This ends the proof of Theorem 2.1. □

Next result completes Theorem 2.1.

**Theorem 2.1***. *The set $\mathcal{P}\backslash D$ is dense in the space $\mathcal{P}$ and is of the first Baire category in $\mathcal{P}$.*

**Remark.** From the fact that $d$, $\delta$ are special kinds of density $h$ and both satisfy condition (3) it follows that $\mathcal{S}_d \cap \mathcal{P}$ and $\mathcal{S}_\delta \cap \mathcal{P}$ are dense, of the first Baire category in the space $\mathcal{P}$ and so, their complements are residual sets in $\mathcal{P}$.

By the definition of Baire's metric it can be easily seen that each of sets

$$S_1 = \{A \in \mathcal{P} : \lim_{n\to\infty} \sup h_n(A) < 1\},$$
$$S_0 = \{A \in \mathcal{P} : \lim_{n\to\infty} \inf h_n(A) > 0\}$$

is a set of the first Baire category, dense in the space $\mathcal{P}$.

This suggests to investigate the porosity character of them. In this connection we introduce

$$\mathcal{T}_m = \{A \in \mathcal{P} : \lim_{n\to\infty} \sup h_n(A) < 1 - \frac{1}{m}\} \qquad m = 2, 3, \dots,$$
$$\mathcal{T}_{m,p} = \{A \in \mathcal{P} : \forall_{n \ge p} h_n(A) < 1 - \frac{1}{m}\} \qquad p = 1, 2, \dots.$$

It can be easily shown that the following lemma holds.

**Lemma 2.1.** *The following statements are true:*

(i) $$S_1 = \bigcup_{m=2}^{\infty} \mathcal{T}_m$$

(ii) $$\mathcal{T}_m \subseteq \bigcup_{p=1}^{\infty} \mathcal{T}_{m,p} \qquad m = 2, 3, \dots.$$

We shall study the porosity character of the set $\mathcal{T}_{m,p}$ $(m \ge 2)$ at points $A \in \mathcal{P}$ for which

(8) $$\lim_{n\to\infty} \sup h_n(A) = 1$$

holds (i.e. at points of the set $\mathcal{P}\backslash S_1$).

From (8) we obtain that there is a sequence $n_1 < n_2 < \dots < n_k < \dots$ with the property

(9) $$\lim_{k\to\infty} h_{n_k}(A) = 1.$$

Construct the ball $B(A, \delta)$ $(\delta > 0)$ and choose $k \in \mathbb{N}$ such that $1/n_k < \delta$. Then $B(A, 1/n_k) \subseteq B(A, \delta)$. Owing to (9) there is $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $h_{n_k}(A) > 1 - 1/m$ holds. Hence, the intersection of $B(A, 1/n_k)$ and the set $\mathcal{T}_{m,p}$ is empty (we can already assume that $n_k > p$). But, then $\bar{p}(A, \mathcal{T}_{m,p}) = 1$ and by Lemma 2.1 the set $S_1$ is $\sigma$-1-porous at $A$. So we get

**Theorem 2.2.** *The set $S_1$ is $\sigma$-1-porous at every point of the set $\mathcal{P}\backslash S_1$.*

Now we shall deal with the porosity character of the set $S_0$. Set

$$\mathcal{T}'_m = \{A \in \mathcal{P} : \lim_{n\to\infty} \inf h_n(A) > \frac{1}{m}\} \qquad m = 2, 3, \dots,$$

$$\mathcal{T}'_{m,p} = \{A \in \mathcal{P} : \forall_{n \geq p} h_n(A) > \frac{1}{m}\} \qquad p = 1, 2, \dots.$$

It can be easily checked that the following lemma holds.

**Lemma 2.2.** *The following statements are true:*

(i) $$S_0 \subseteq \bigcup_{m=2}^{\infty} \mathcal{T}'_m$$

(ii) $$\mathcal{T}'_m \subseteq \bigcup_{p=1}^{\infty} \mathcal{T}'_{m,p} \qquad m = 2, 3, \dots.$$

**Theorem 2.3.** *The set $S_0$ is $\sigma$-strongly porous in the space $\mathcal{P}$.*

*Proof.* We shall determine the porosity character of the set $\mathcal{T}'_{m,p}$ where $m, p$ are fixed.

Let $A = (a_n)_{n=1}^{\infty}$ the an arbitrary point of $\mathcal{P}$, $0 < \delta < 1$. Choose a $v \in \mathbb{N}$ such that $1/v < \delta \leq 1/(v-1)$ $(v \geq 2)$. We can already suppose that $\delta > 0$ is so small that $v \geq p$.

Choose $D = (d_n)_{n=1}^{\infty}$ where

$$d_n = a_n \qquad n = 1, 2, \dots, v.$$

Then irrespective of the rest terms of $D$ we have

$$D \in B(A, \frac{1}{v}) \subseteq B(A, \delta).$$

Set $t = a_v$. Since $s_t/s_{t+r} \to 0$ $(r \to \infty)$ there is an $r \in \mathbb{N}$ such that

(10) $$\frac{s_t}{s_{t+r}} < \frac{1}{m}.$$

Take

$$d_{v+i} = t + r + i \qquad i = 1, 2, \dots.$$

According to (10) and definition of $D$ we get

(11) $$h_{t+r}(D) = \frac{1}{s_{t+r}} \left( \sum_{k=1}^{t} c_k \chi_D(k) + \sum_{k=t+1}^{t+r} c_k \chi_D(k) \right) < \frac{1}{m}.$$

By the choice of $v$ and from (11) we obtain that $D$ does not belong to $\mathcal{T}'_{m,p}$.

Construct the ball $B(D, 1/(v+1)) \subseteq B(A, \delta)$. If $E \in B(D, 1/(v+1))$, then $E$ and $D$ have the first $v+1$ terms in common and so, $B(D, 1/(v+1)) \cap \mathcal{T}'_{m,p} = \emptyset$. Hence, $\gamma(A, \delta, \mathcal{T}'_{m,p}) \geq 1/(v+1)$ and by the choice of $\delta$ we get

$$\frac{\gamma(A, \delta, \mathcal{T}'_{m,p})}{\delta} \geq \frac{v-1}{v+1} \to 1 \qquad (\delta \to 0_+).$$

In this way $p(A, \mathcal{T}'_{m,p}) = 1$ (i.e. the set $\mathcal{T}'_{m,p}$ is strongly porous in $\mathcal{P}$) and by Lemma 2.2 we get the assertion of Theorem 2.3. □

**Corollary.** *The set $S_0$ is dense, of the first Baire category in the space $\mathcal{P}$.*

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