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ON UNIQUENESS FOR A SYSTEM OF HEAT EQUATIONS COUPLED IN THE BOUNDARY CONDITIONS

M. KORDOŠ

Abstract. We consider the system

$$\begin{array}{llll} u_t = \triangle u, & v_t = \triangle v, & x \in \mathbb{R}^N_+, & t > 0, \\ -\dfrac{\partial u}{\partial x_1} = v^p, & -\dfrac{\partial v}{\partial x_1} = u^q, & x_1 = 0, & t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \mathbb{R}^N_+, & \end{array}$$

where $\mathbb{R}^N_+ = \{(x_1, x') : x' \in \mathbb{R}^{N-1}, x_1 > 0\}$, p, q are positive numbers, and functions u_0, v_0 in the initial conditions are nonnegative and bounded. We show that nonnegative solutions are unique if $pq \geqslant 1$ or if (u_0, v_0) is nontrivial. In the case of zero initial data and $pq < 1$, we find all nonnegative nontrivial solutions.

1. Introduction

In this paper we study the uniqueness of nonnegative classical solutions of the system

$$\begin{array}{llll} u_t = \triangle u, & v_t = \triangle v, & x \in \mathbb{R}^N_+, & t > 0, \\ -\dfrac{\partial u}{\partial x_1} = v^p, & -\dfrac{\partial v}{\partial x_1} = u^q, & x_1 = 0, & t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \mathbb{R}^N_+, & \end{array} \tag{1.1}$$

where $\mathbb{R}^N_+ = \{(x_1, x') : x' \in \mathbb{R}^{N-1},\ x_1 > 0\}$, $N \geqslant 1$, $p > 0$, $q > 0$, and both u_0, v_0 are nonnegative bounded functions satisfying the compatibility conditions

$$-\frac{\partial u_0}{\partial x_1} = v_0^p \qquad \text{and} \qquad -\frac{\partial v_0}{\partial x_1} = u_0^q \quad \text{at} \quad x_1 = 0.$$

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In order to motivate our results, we recall a paper by Fujita and Watanabe [**6**], in which they studied the Cauchy-Dirichlet problem

$$
(1.2)\qquad
\begin{aligned}
u_t - \triangle u &= u^p, && x \in \Omega, && t > 0,\\
u(x,0) &= u_0(x), && x \in \Omega, &&\\
u(x,t) &= 0, && x \in \partial\Omega, && t \geqslant 0,
\end{aligned}
$$

where $p > 0$, u_0 is a continuous, nonnegative and bounded real function, and Ω is a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$. They showed that uniqueness fails when $p < 1$.

Analogous results for systems were obtained by Escobedo and Herrero. In [**4**] they investigated the initial value problem for a weakly coupled system on the whole space

$$
(1.3)\qquad
\begin{aligned}
u_t &= \triangle u + v^p, & v_t &= \triangle v + u^q, && x \in \mathbb{R}^N, && t > 0,\\
u(x,0) &= u_0(x), & v(x,0) &= v_0(x), && x \in \mathbb{R}^N,
\end{aligned}
$$

with $N \geqslant 1$, $p > 0$, $q > 0$, and where u_0 and v_0 are nonnegative, continuous, and bounded real functions. They showed that solutions of (1.3) are unique if $pq \geqslant 1$ or if one of the initial functions u_0, v_0 is different from zero. They also characterized the whole set of solutions emanating from the initial value $(u_0, v_0) = (0,0)$ when $0 < pq < 1$. In this case, the set of nontrivial nonnegative solutions of (1.3) is given by

$$
u(\cdot,t;s) = c_1(t-s)_+^{\alpha_1}, \qquad v(\cdot,t;s) = d_1(t-s)_+^{\beta_1},
$$

where $(r)_+ = \max\{r, 0\}$, $s \geqslant 0$,

$$
\alpha_1 = \frac{p+1}{1-pq}, \qquad \beta_1 = \frac{q+1}{1-pq},
$$

and c_1, d_1 depend on p and q only.

In [**5**] they proved the corresponding result for the bounded domain version of the problem (1.3). Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$. They considered the following Cauchy-Dirichlet problem

$$
(1.4)\qquad
\begin{aligned}
u_t - \triangle u &= v^p, && x \in \Omega, && t > 0,\\
v_t - \triangle v &= u^q, && x \in \Omega, && t > 0,\\
u = v &= 0, && x \in \partial\Omega, && t \geqslant 0,\\
u(x,0) = u_0(x), \quad v(x,0) &= v_0(x), && x \in \Omega,
\end{aligned}
$$

where $p > 0$, $q > 0$, and u_0, v_0 are nonnegative, continuous, and bounded real functions. They showed again that solutions of (1.4) are unique if $pq \geqslant 1$ or if the initial data u_0, v_0 are nontrivial, and they also characterized the set of solutions with zero initial value $(u_0, v_0) = (0,0)$ when $pq < 1$. In the latter case, the set of nonnegative solutions of (1.3) consists of (i) the trivial solution $u(x,t) = v(x,t) = 0$, (ii) a solution $U(x,t)$, $V(x,t)$ such that $U(x,t) > 0$ and $V(x,t) > 0$ for any $x \in \Omega$ and $t > 0$, (iii) a monoparametric family $U_s(x,t)$, $V_s(x,t)$, where $U_s(x,t) = U(x,(t-s)_+)$, $V_s(x,t) = V(x,(t-s)_+)$, $s > 0$, and $(r)_+ = \max\{r, 0\}$.

A nonuniqueness result for the system (1.1) is obtained by Deng, Fila, and Levine in [**3**] where they constructed a nontrivial solution with zero initial data and $pq < 1$ in the dimension $N = 1$. It is a self-similar solution of the form

$$u(x_1, t) = t^\alpha f(y), \qquad v(x_1, t) = t^\beta g(y), \qquad \text{for } y = \frac{x_1}{\sqrt{t}}, \quad t > 0,$$

with

$$\alpha = \frac{1+p}{2(1-pq)} = \frac{\alpha_1}{2}, \qquad \beta = \frac{1+q}{2(1-pq)} = \frac{\beta_1}{2},$$

where f, $g > 0$ solve the corresponding initial value problem

$$\begin{aligned} f''(y) + \frac{y}{2} f'(y) - \alpha f(y) &= 0, \\ g''(y) + \frac{y}{2} g'(y) - \beta g(y) &= 0 \qquad \text{for } y > 0, \\ f'(0) &= -\, g^p(0), \\ g'(0) &= -\, f^q(0), \end{aligned}$$

and where (f, g) decays to $(0, 0)$ as $y \to \infty$. We have (see Theorem 3.5 in [**3**])

$$\begin{aligned} f(y) &= c_2\, \mathrm{e}^{-\frac{y^2}{4}}\, U\left(\frac{1}{2} + \alpha, \frac{1}{2}, \frac{y^2}{4}\right), \\ g(y) &= d_2\, \mathrm{e}^{-\frac{y^2}{4}}\, U\left(\frac{1}{2} + \beta, \frac{1}{2}, \frac{y^2}{4}\right), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} c_2 &= \pi^{-\frac{1}{2}} \left(\frac{\Gamma(\frac{1}{2} + \beta)}{\Gamma(1+\beta)}\right)^{\frac{p}{1-pq}} \left(\frac{\Gamma(\frac{1}{2} + \alpha)}{\Gamma^{pq}(1+\alpha)}\right)^{\frac{1}{1-pq}}, \\ U(a, b, r) &= \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-rt}\, t^{a-1} (1+t)^{b-a-1} dt, \end{aligned}$$

and d_2 is obtained from c_2 by the interchange of α with β and p with q.

Wang, Xie, and Wang showed in [**9**] besides the blow-up estimates also the uniqueness of the trivial solution of (1.1) in the case $pq \geqslant 1$ with trivial initial data $(u_0, v_0) \equiv (0, 0)$, and Lin generalized this result for the corresponding system of n equations in [**8**].

The bounded domain version of the problem (1.1) was discussed by Cortazar, Elgueta, and Rossi. In [**2**] they considered the system

$$\begin{aligned} u_t &= \triangle u, & v_t &= \triangle v & &\text{in} \quad \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} &= v^p, & \frac{\partial v}{\partial \nu} &= u^q & &\text{on} \quad \partial\Omega \times (0, T), \\ u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x) & &\text{in} \quad \Omega, \end{aligned} \tag{1.6}$$

with smooth initial data $u_0 \geqslant 0$ and $v_0 \geqslant 0$, $p > 0$, $q > 0$, and ν being the outer normal to $\partial\Omega$. Their result for (1.6) takes the same form as for (1.4) in [**5**].

Finally, a uniqueness result is showed in [**7**] for the system

$$
(1.7)\qquad
\begin{array}{llll}
u_t = \triangle u + v^p, & v_t = \triangle v, & x \in \mathbb{R}^N_+, & t > 0,\\
-\dfrac{\partial u}{\partial x_1} = 0, & -\dfrac{\partial v}{\partial x_1} = u^q, & x_1 = 0, & t > 0,\\
u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \mathbb{R}^N_+, &
\end{array}
$$

with $N \geqslant 1$, $p > 0$, $q > 0$, and u_0, v_0 nonnegative, smooth, and bounded functions satisfying the compatibility condition. The nonnegative solutions are unique if $pq \geqslant 1$ while a nontrivial nonnegative solution is constructed with vanishing initial values when $pq < 1$.

In [**3**], Deng, Fila, and Levine studied also the large time behaviour of nonnegative solutions of (1.1). They proved that if $pq \leqslant 1$, every nonnegative solution is global. Set, when $pq > 1$,

$$\alpha_2 = -\alpha, \qquad \beta_2 = -\beta.$$

They showed that if $\max(\alpha_2, \beta_2) \geqslant N/2$, then all nontrivial nonnegative solutions are nonglobal; if $\max(\alpha_2, \beta_2) < N/2$ there exist both global and nonglobal nonnegative solutions.

The purpose of this paper is to complete the uniqueness result for (1.1), which has the same form as for (1.3) in [**4**]. We prove the following

Theorem.
(i) *Let $pq \geqslant 1$. The system* (1.1) *has then a unique solution.*
(ii) *Let $pq < 1$ and $(u_0, v_0) \not\equiv (0,0)$. The system* (1.1) *has then a unique solution.*
(iii) *Let $pq < 1$ and $(u_0, v_0) \equiv (0,0)$. The set of nontrivial nonnegative solutions of* (1.1) *is then given by the family*

$$
(1.8)\qquad
\begin{array}{l}
\tilde{u}(x,t;s) = (t-s)_+^{\alpha} f(y),\\
\tilde{v}(x,t;s) = (t-s)_+^{\beta} g(y),
\end{array}
\qquad
y = \begin{cases} \dfrac{x_1}{\sqrt{t-s}} & \text{if} \quad t > s,\\ 0 & \text{otherwise,} \end{cases}
$$

where $(r)_+ = \max\{r, 0\}$, $s \geqslant 0$, $\alpha = \dfrac{1+p}{2(1-pq)}$, $\beta = \dfrac{1+q}{2(1-pq)}$, and f, g are given in (1.5).

We prove the parts (i), (ii), (iii) in Sections 2, 3, and 4 respectively.

2. Proof of Part (i)

Similarly as in [**3**], we denote

$$
\begin{aligned}
G_N(x,y;t) &= (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right),\\
H_N(x,y;t) &= G_N(x,y;t) + G_N(x,-y;t),\\
H_1(x_1,y_1;t) &= \frac{1}{2}(\pi t)^{-\frac{1}{2}}\left(\exp\left(-\frac{|x_1-y_1|^2}{4t}\right) + \exp\left(-\frac{|x_1+y_1|^2}{4t}\right)\right),\\
R(x_1,t) &= H_1(x_1,0;t) = (\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x_1^2}{4t}\right)
\end{aligned}
$$

for $t>0$, x, $y \in \mathbb{R}^N$, x_1, $y_1 \in \mathbb{R}$, x', $y' \in \mathbb{R}^{N-1}$, and $x=(x_1,x')$, $y=(y_1,y')$. We use these functions to define several operators for $w \in L^1_{loc}(\mathbb{R}^N_+)$, namely

$$
\begin{aligned}
\mathcal{S}_N(t)w(x) &= \int_{\mathbb{R}^N} G_N(x,y;t)w(y)dy,\\
\mathcal{S}_{N-1}(t)w(x_1,x') &= \int_{\mathbb{R}^{N-1}} G_{N-1}(x',y';t)w(x_1,y')dy',\\
\mathcal{T}(t)w(x) &= \int_{\mathbb{R}_+} H_1(x_1,y_1;t)w(y_1,x')dy_1,\\
\mathcal{R}(t)w(x) &= R(x_1,t)\mathcal{S}_{N-1}(t)w(0,x').
\end{aligned}
$$

These integral operators allow us to write the variation of constants formulae for solutions of (1.1)

$$
u(x,t) = \mathcal{T}(t)\mathcal{S}_{N-1}(t)u_0(x) + \int_0^t \mathcal{R}(t-\eta)v^p(x,\eta)d\eta, \tag{2.1a}
$$

$$
v(x,t) = \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x) + \int_0^t \mathcal{R}(t-\eta)u^q(x,\eta)d\eta. \tag{2.1b}
$$

It is possible to prove the local (in time) existence of the solution for given L^∞ initial values using (2.1) and the contraction mapping principle. Since the solution does not have to exist globally in this case (see [**3**]), we define a strip $S_T = \mathbb{R}^N_+ \times (0,T)$ for any $0 < T \leqslant \infty$.

We point out several useful relationships. One can easily check that for $w \in L^1_{loc}(\mathbb{R}^N_+)$, $s,t>0$, the equalities

$$
\begin{aligned}
\mathcal{T}(t)\mathcal{S}_{N-1}(t)w &= \mathcal{S}_{N-1}(t)\mathcal{T}(t)w,\\
\mathcal{S}_{N-1}(t)\mathcal{S}_{N-1}(s)w &= \mathcal{S}_{N-1}(t+s)w
\end{aligned}
$$

hold. We use them later without referring as well as Jensen's inequality in the following two forms

$$\text{if} \quad r \geqslant 1 \qquad \text{then} \qquad \left(\int_0^t f(s)ds\right)^r \leqslant t^{r-1}\int_0^t f^r(s)ds,$$

$$\text{if} \quad r \leqslant 1 \qquad \text{then} \qquad \int_0^t f^r(s)ds \leqslant t^{1-r}\left(\int_0^t f(s)ds\right)^r.$$

We prove the following formulation of Part (i).

Proposition 2.1. *If (u,v) and $(\bar{u},\bar{v})$ are two solutions of the problem* (1.1) *with $pq \geqslant 1$ in some strip S_T, then $(u,v)=(\bar{u},\bar{v})$ in S_T.*

Proof. We omit the standard argument when both nonlinearities are Lipschitz continuous, i.e., $p,q \geqslant 1$ (cf. Preliminaries in [**5**]). Since the system (1.1) is symmetric in the sense of interchanging p and q, we may assume $p<1$ (i.e., $q>1$) for definiteness without loss of generality. We adapt the argument from the proof of Lemma 2 in [**4**].

Let $\tau \in (0,T)$ be an arbitrary time and let $0 \leqslant s \leqslant \eta \leqslant t \leqslant \tau$ be always ordered this way in further discussion. We fix $(x,\eta) \in S_\tau$ and define a functional $g(\cdot)(x,\eta) : L^\infty(S_\tau) \to \mathbb{R}$

$$\begin{aligned} g(w)(x,\eta) &= \mathcal{T}(\eta)\mathcal{S}_{N-1}(\eta)v_0(x) + \int_0^\eta \mathcal{R}(\eta-s)w^q(x,s)ds,\\ f(\xi) &= \xi^p, \qquad \xi>0, \end{aligned}$$

so that we obtain by the mean value theorem for $f \circ g$

$$\begin{aligned} V(x,\eta) &= (v^p-\bar{v}^p)(x,\eta) = (g(u)(x,\eta))^p - (g(\bar{u})(x,\eta))^p \\ &= pq\,(g(w)(x,\eta))^{p-1}\int_0^\eta \mathcal{R}(\eta-s)\left(w^{q-1}(u-\bar{u})\right)(x,s)ds \end{aligned} \tag{2.2}$$

for some w between u and $\bar{u}$. More precisely, we write

$$w(\cdot,s) = \rho(x,\eta)u(\cdot,s) + (1-\rho(x,\eta))\bar{u}(\cdot,s)$$

where $0<\rho(x,\eta)<1$. We also define $F(t) = \sup\{\|(u-\bar{u})(\cdot,\eta)\|_\infty : \eta \in [0,t]\}$, and by Hölder's inequality we derive (since $\frac{1}{q} \leqslant p < 1$)

$$\begin{aligned} |V(x,\eta)| &\leqslant pqF(\eta)\left(\int_0^\eta \mathcal{R}(\eta-s)w^q(x,s)ds\right)^{p-1} \\ &\qquad \times \int_0^\eta \mathcal{R}(\eta-s)w^{q-1}(x,s)ds \\ &\leqslant pqF(\eta)\left(2^{\frac{1}{q}}\pi^{-\frac{1}{2q}}\eta^{\frac{1}{2q}}\right)\left(\int_0^\eta \mathcal{R}(\eta-s)w^q(x,s)ds\right)^{p-1+1-\frac{1}{q}} \\ &\leqslant pq2^p\pi^{-\frac{p}{2}}U^{pq-1}F(\eta)\eta^{\frac{p}{2}}, \end{aligned} \tag{2.3}$$

where U is the upper bound of w in $\mathbb{R}^N_+ \times [0,\tau]$. Hence, applying the solution formulae (2.1), we obtain for any $x \in \mathbb{R}^N_+$, $\eta \in [0,t]$

$$
\begin{aligned}
|u-\bar{u}|(x,\eta) &\leqslant \int_0^\eta \mathcal{R}(\eta-s)|V(x,s)|ds \\
&\leqslant pq2^p\pi^{-\frac{p}{2}}U^{pq-1}F(\eta)\int_0^\eta s^{\frac{p}{2}}ds \leqslant Kt^{\frac{1+p}{2}}F(t),
\end{aligned} \tag{2.4}
$$

where the constant K depends on p, q, and on the bounds of u and $\bar{u}$ in $\mathbb{R}^N_+ \times [0,\tau]$. The supremum property implies $F(t) \leqslant Kt^{\frac{1+p}{2}}F(t)$ on $[0,\tau]$, and thus $F(t)=0$ for $t \in (0, K^{-\frac{2}{1+p}})$. Since the system is autonomous, finite iterating of the argument yields $u=\bar{u}$ in $\mathbb{R}^N_+ \times [0,\tau]$. The equality $v=\bar{v}$ follows consequently from (2.1). □

3. Proof of Part (ii)

In this section we establish an estimate for the nontrivial nonnegative solutions of (1.1) when $pq<1$ that we will use in Section 4 as well. We also prove Part (ii) of our main result.

Let us introduce further notation for convenience. We set $b(\gamma)=B(\frac{1}{2}+\gamma,\frac{1}{2})$ for $\gamma>-\frac{1}{2}$ where $B(x,y)$ is the Beta function. Thus we have for $t>0$

$$
\int_0^t (t-\eta)^{-\frac{1}{2}}\eta^\gamma d\eta = t^{\frac{1}{2}+\gamma}B\left(1+\gamma,\frac{1}{2}\right) = t^{\frac{1}{2}+\gamma}b\left(\frac{1}{2}+\gamma\right). \tag{3.1}
$$

Remark 3.1. Notice that $\dfrac{1}{2}+p\beta=\alpha$ and $\dfrac{1}{2}+q\alpha=\beta$, which will be richly used in the iteration arguments.

Remark 3.2. We recall also a standard auxiliary result that can be proved by standard arguments. For $t>0$ and a solution (u,v) of (1.1) with $u_0 \not\equiv 0$, there exist γ, $\sigma>0$ such that

$$
u(x,t) \geqslant \gamma\, e^{-\sigma|x|^2}, \qquad x \in \mathbb{R}^N_+. \tag{3.2}
$$

Lemma 3.3. *If (u,v) is a solution of the system (1.1) with nontrivial initial condition $(u_0,v_0)\not\equiv(0,0)$ and $pq<1$, then*

$$
\begin{aligned}
u(0,x';t) &\geqslant Ct^\alpha, \\
v(0,x';t) &\geqslant Dt^\beta,
\end{aligned} \qquad x' \in \mathbb{R}^{N-1},\ t>0, \tag{3.3}
$$

where $C=\pi^{-\alpha}b^{\frac{1}{1-pq}}(\alpha)b^{\frac{p}{1-pq}}(\beta)$ and $D=\pi^{-\beta}b^{\frac{q}{1-pq}}(\alpha)b^{\frac{1}{1-pq}}(\beta)$.

Proof. We adapt the arguments from the proof of Lemma 2 in [**4**]. First we obtain the estimate assuming $u_0(0,x') \geqslant \gamma\, e^{-\sigma|x'|^2}$, $x' \in \mathbb{R}^{N-1}$ for some γ, $\sigma>0$. Since (cf. (2.13) in [**3**])

$$
\mathcal{S}_{N-1}(t)\, e^{-\sigma|x'|^2} = (1+4\sigma t)^{-\frac{N-1}{2}}\, e^{-\frac{\sigma}{1+4\sigma t}|x'|^2},
$$

we have

$$
\mathcal{S}_{N-1}(t-\eta)\, e^{-\sigma|x'|^2} \geqslant (1+4\sigma t)^{-\frac{N-1}{2}}\, e^{-\sigma|x'|^2} \tag{3.4}
$$

for $0 \leqslant \eta \leqslant t$. We use (3.4) and the solution formulae (2.1) to get partial estimates for u and v on the boundary $x_1 = 0$. In the first step, we obtain

$$\begin{aligned} u(0,x';t) &\geqslant (\mathcal{T}(t)\mathcal{S}_{N-1}(t)u_0)(0,x') \geqslant \gamma(1+4\sigma t)^{-\frac{N-1}{2}} \,\mathrm{e}^{-\sigma|x'|^2}, \\ v(0,x';t) &\geqslant \int_0^t \left(\mathcal{R}(t-\eta)u^q\right)(0,x';\eta)d\eta \\ &\geqslant 2\pi^{-\frac{1}{2}}\gamma^q(1+4\sigma t)^{-\frac{N-1}{2}q}(1+4\sigma qt)^{-\frac{N-1}{2}} \,\mathrm{e}^{-\sigma q|x'|^2}\, t^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (2.1) again yields

$$\begin{aligned} u(0,x';t) \geqslant &\int_0^t \left(\mathcal{R}(t-\eta)v^p\right)(0,x';\eta)d\eta \\ \geqslant &\, 2^p\pi^{-\frac{1+p}{2}}\gamma^{pq}(1+4\sigma t)^{-\frac{N-1}{2}pq}(1+4\sigma qt)^{-\frac{N-1}{2}p}(1+4\sigma pqt)^{-\frac{N-1}{2}} \\ &\times \mathrm{e}^{-\sigma pq|x'|^2}\, b\left(\frac{1+p}{2}\right) t^{\frac{1+p}{2}}, \\ v(0,x';t) \geqslant &\int_0^t \left(\mathcal{R}(t-\eta)u^q\right)(0,x';\eta)d\eta \\ \geqslant &\, 2^{pq}\pi^{-\frac{1+q+pq}{2}}\gamma^{pq^2}(1+4\sigma t)^{-\frac{N-1}{2}pq^2}(1+4\sigma qt)^{-\frac{N-1}{2}pq}(1+4\sigma pqt)^{-\frac{N-1}{2}q} \\ &\times (1+4\sigma pq^2t)^{-\frac{N-1}{2}} \,\mathrm{e}^{-\sigma pq^2|x'|^2}\, b^q\left(\frac{1+p}{2}\right) b\left(\frac{1+q+pq}{2}\right) t^{\frac{1+q+pq}{2}}. \end{aligned}$$

By induction, we obtain

$$\begin{aligned} u(0,x';t) &\geqslant 2^{p(pq)^{k-1}}\gamma^{(pq)^k}\,\mathrm{e}^{-\sigma(pq)^k|x'|^2}\,K_k(t)C_k t^{\alpha_k}, \\ v(0,x';t) &\geqslant 2^{(pq)^k}\gamma^{(pq)^k q}\,\mathrm{e}^{-\sigma(pq)^k q|x'|^2}\,L_k(t)D_k t^{\beta_k}, \end{aligned} \qquad k \in \mathbb{N}, \tag{3.6}$$

where (using also that $b(\gamma)$ is decreasing)

$$\begin{aligned} K_k(t) &= \prod_{j=0}^{k}\left(1+4\sigma t(pq)^j\right)^{-\frac{N-1}{2}(pq)^{k-j}} \prod_{j=1}^{k}\left(1+4\sigma t(pq)^{j-1}q\right)^{-\frac{N-1}{2}p(pq)^{k-j}}, \\ L_k(t) &= \prod_{j=0}^{k}\left(1+4\sigma t(pq)^j\right)^{-\frac{N-1}{2}(pq)^{k-j}q} \prod_{j=0}^{k}\left(1+4\sigma t(pq)^j q\right)^{-\frac{N-1}{2}(pq)^{k-j}}, \\ C_k &= \pi^{-\alpha_k}\prod_{j=1}^{k} b^{(pq)^{k-j}}\left(\alpha_j\right) \prod_{j=1}^{k-1} b^{p(pq)^{k-j-1}}\left(\beta_j\right) \\ &\geqslant \pi^{-\alpha_k} b^{\frac{1-(pq)^k}{1-pq}}\left(\alpha_k\right) b^{p\frac{1-(pq)^{k-1}}{1-pq}}\left(\beta_{k-1}\right), \end{aligned}$$

$$\begin{aligned} D_k &= \pi^{-\beta_k}\prod_{j=1}^{k} b^{(pq)^{k-j}q}\left(\alpha_j\right) \prod_{j=1}^{k} b^{(pq)^{k-j}}\left(\beta_j\right) \\ &\geqslant \pi^{-\beta_k} b^{q\frac{1-(pq)^k}{1-pq}}\left(\alpha_k\right) b^{\frac{1-(pq)^k}{1-pq}}\left(\beta_k\right), \end{aligned}$$

and $\alpha_k = \alpha(1 - (pq)^k) \geqslant \alpha_j$, $\beta_k = \beta(1 - (pq)^k) + \frac{(pq)^k}{2} \geqslant \beta_j$ for $1 \leqslant j \leqslant k$. For $\theta \in (0,1)$, any positive number ξ, and any real number ζ, we have

$$\lim_{k\to\infty} \prod_{j=0}^{k} (1+\xi\theta^j)^{\zeta\theta^{k-j}} = 1, \tag{3.7}$$

and therefore

$$\lim_{k\to\infty} K_k(t) = 1, \qquad \lim_{k\to\infty} L_k(t) = 1.$$

The argument proving (3.7) for ζ negative runs as follows

$$0 \geqslant \ln \prod_{j=0}^{k} \left(1+\xi\theta^j\right)^{\zeta\theta^{k-j}} = \zeta \sum_{j=0}^{k} \theta^{k-j} \ln\left(1+\xi\theta^j\right) \geqslant \xi\zeta \sum_{j=0}^{k} \theta^k \underset{k\to\infty}{\longrightarrow} 0.$$

It is also obvious that

$$\liminf_{k\to\infty} C_k \geqslant C, \qquad \liminf_{k\to\infty} D_k \geqslant D.$$

Letting $k \to \infty$ in (3.6), we obtain (3.3) for considered initial condition.

Now we generalize the estimate for any nontrivial initial data $u_0 \not\equiv 0$ using Remark 3.2. We take arbitrary $\varepsilon > 0$ and set $u_\varepsilon(\cdot,t) = u(\cdot,t+\varepsilon)$, $v_\varepsilon(\cdot,t) = v(\cdot,t+\varepsilon)$. The autonomous nature of the system (1.1) implies

$$u_\varepsilon(x,t) = \mathcal{T}(t)\mathcal{S}_{N-1}(t)u_\varepsilon(x,0) + \int_0^t \mathcal{R}(t-\eta)v_\varepsilon^p(x,\eta)d\eta,$$

where $u_\varepsilon(0,x';0) > \gamma\,\mathrm{e}^{-\sigma|x'|^2}$ for some positive numbers γ and σ. Therefore $u_\varepsilon(t) \geqslant Ct^\alpha$, and accordingly

$$u(0,x';t) \geqslant C(t-\varepsilon)^\alpha.$$

Thus (3.3) holds for any $u_0 \not\equiv 0$, since ε is arbitrary. Obviously, the assumption $u_0 \not\equiv 0$ is made without loss of generality. □

Proposition 3.4. *If (u,v) and $(\bar{u},\bar{v})$ are two solutions of the problem* (1.1) *with nontrivial initial condition $(u_0,v_0) \not\equiv (0,0)$ and $pq < 1$, then $(u,v) = (\bar{u},\bar{v})$.*

Proof. We use the contradiction argument from the proof of Lemma 3 in [**4**]. We make the assumption $0 < p < 1$ without loss of generality and introduce notation $f_+ = \max\{f,0\}$ and $\|f(t)\| = \sup\{|f_+(0,x';t)| : x' \in \mathbb{R}^{N-1}\}$. Suppose that $(u,v) \neq (\bar{u},\bar{v})$. Then we can find $t > 0$ such that without loss of generality, we may assume $\|(u-\bar{u})(\eta)\| \leqslant \|(u-\bar{u})(t)\| > 0$ for $0 \leqslant \eta \leqslant t$.

(a) We start with the symmetric case $0 < q < 1$. We use $|\xi^r - \zeta^r| \leqslant |\xi - \zeta|^r$ for nonnegative ξ, ζ and $r \in (0,1)$, and obtain

$$\begin{aligned}\|(u-\bar{u})(t)\| &\leqslant \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} \left(\int_0^\eta (\pi(\eta-s))^{-\frac{1}{2}} \|(u-\bar{u})(s)\|^q ds\right)^p d\eta \\ &\leqslant \|(u-\bar{u})(t)\|^{pq} 2^p \pi^{-\frac{1+p}{2}} b\left(\frac{1+p}{2}\right) t^{\frac{1+p}{2}},\end{aligned}$$

so that

$$\|(u-\bar{u})(t)\| \leqslant Pt^{\alpha}, \qquad P = 2^{\frac{p}{1-pq}}\pi^{-\alpha}b^{\frac{1}{1-pq}}\left(\frac{1+p}{2}\right) \tag{3.8}$$

holds. The mean value theorem for $g(\xi)=\xi^r$, $\xi>0$, $r\in\{p,q\}$ gives

$$\begin{aligned}(u^q-\bar{u}^q)(0,x';t) &= qw^{q-1}(0,x';t)(u-\bar{u})(0,x';t),\\ (v^p-\bar{v}^p)(0,x';t) &= pz^{p-1}(0,x';t)(v-\bar{v})(0,x';t),\end{aligned} \tag{3.9}$$

where w, z are between u and $\bar{u}$, v and $\bar{v}$, respectively, and fulfil therefore

$$\begin{aligned}w^{q-1}(0,x';s) &\leqslant C^{q-1}s^{\alpha(q-1)},\\ z^{p-1}(0,x';\eta) &\leqslant D^{p-1}\eta^{\beta(p-1)}\end{aligned}$$

by Lemma 3.3 and by the fact that both p, $q\in(0,1)$. Notice also that

$$C^{q-1}D^{p-1} = \pi b^{-1}(\alpha)b^{-1}(\beta).$$

By solution formulae (2.1), inequalities

$$\begin{aligned}(u-\bar{u})_+(0,x';t) &\leqslant \int_0^t \pi^{-\frac12}(t-\eta)^{-\frac12}\mathcal{S}_{N-1}(t-\eta)\left(v^p-\bar{v}^p\right)_+(0,x';\eta)d\eta,\\ (v-\bar{v})_+(0,x';t) &\leqslant \int_0^t \pi^{-\frac12}(t-\eta)^{-\frac12}\mathcal{S}_{N-1}(t-\eta)\left(u^q-\bar{u}^q\right)_+(0,x';\eta)d\eta\end{aligned}$$

hold. We use (3.9) and obtain

$$\begin{aligned}\|(u-\bar{u})(t)\| \leqslant pq\pi^{-1}D^{p-1}C^{q-1}&\int_0^t (t-\eta)^{-\frac12}\eta^{\beta(p-1)}\\ &\left(\int_0^\eta(\eta-s)^{-\frac12}s^{\alpha(q-1)}\|(u-\bar{u})(s)\|ds\right)d\eta.\end{aligned} \tag{3.10}$$

By (3.8), we see that the right-hand side of (3.10) is integrable. Moreover, combining (3.8) with (3.10) yields

$$\begin{aligned}\|(u-\bar{u})(t)\| &\leqslant pq\pi^{-1}D^{p-1}C^{q-1}P\int_0^t(t-\eta)^{-\frac12}\eta^{\beta(p-1)}\left(\int_0^\eta(\eta-s)^{-\frac12}s^{\alpha q}ds\right)d\eta\\ &= pq\pi^{-1}D^{p-1}C^{q-1}b(\beta)P\int_0^t(t-\eta)^{-\frac12}\eta^{\beta p}d\eta = pqPt^{\alpha}.\end{aligned} \tag{3.11}$$

Iterating this procedure k times, we obtain

$$\|(u-\bar{u})(t)\| \leqslant (pq)^k Pt^{\alpha}, \qquad k\in\mathbb{N}. \tag{3.12}$$

(b) Before completing the proof, we apply the arguments from the proof of Lemma 3 in [**4**] to get the estimate (3.12) for $q\geqslant 1$ as well. For an arbitrary

$\theta \in (0,1)$, using the inequalities $u \leqslant \bar{u} + (u-\bar{u})_+$ and $u^\theta \leqslant \bar{u}^\theta + (u-\bar{u})_+^\theta$, we obtain

$$\begin{aligned}
v(x,t) &= \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&\quad + \int_0^t \int_{\mathbb{R}^{N-1}} \left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{q-\theta}{q}} u^{q-\theta}(x_1,y';\eta)\\
&\qquad \left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{\theta}{q}} u^{\theta}(x_1,y';\eta)dy'd\eta\\
&\leqslant \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&\quad + \int_0^t \int_{\mathbb{R}^{N-1}} \left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{q-\theta}{q}} u^{q-\theta}(x_1,y';\eta)\\
&\qquad \left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{\theta}{q}} \bar{u}^{\theta}(x_1,y';\eta)dy'd\eta\\
&\quad + \int_0^t \int_{\mathbb{R}^{N-1}} \left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{q-\theta}{q}} u^{q-\theta}(x_1,y';\eta)\\
&\qquad \left(R(x_1,t-\eta)G_{N-1}(x',y';t-\eta)\right)^{\frac{\theta}{q}} (u-\bar{u})_+^{\theta}(x_1,y';\eta)dy'd\eta.
\end{aligned}$$

We apply Hölder's inequality twice to get

$$\begin{aligned}
v(x,t) &\leqslant \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&\quad + \int_0^t \left(\mathcal{R}(t-\eta)u^q(x,\eta)\right)^{\frac{q-\theta}{q}} \left(\mathcal{R}(t-\eta)\bar{u}^q(x,\eta)\right)^{\frac{\theta}{q}} d\eta\\
&\quad + \int_0^t \left(\mathcal{R}(t-\eta)u^q(x,\eta)\right)^{\frac{q-\theta}{q}} \big(\mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)\big)^{\frac{\theta}{q}} d\eta\\
&\leqslant \mathcal{T}(t)\mathcal{S}_{N-1}(t)v_0(x)\\
&\quad + \left(\int_0^t \mathcal{R}(t-\eta)u^q(x,\eta)d\eta\right)^{\frac{q-\theta}{q}} \left(\int_0^t \mathcal{R}(t-\eta)\bar{u}^q(x,\eta)d\eta\right)^{\frac{\theta}{q}}\\
&\quad + \left(\int_0^t \mathcal{R}(t-\eta)u^q(x,\eta)d\eta\right)^{\frac{q-\theta}{q}} \left(\int_0^t \mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)d\eta\right)^{\frac{\theta}{q}},
\end{aligned}$$

and using $\chi+\xi^{1-\gamma}\zeta^\gamma \leqslant (\chi+\xi)^{1-\gamma}(\chi+\zeta)^\gamma$ for any nonnegative χ, ξ, ζ, and $\gamma \in (0,1)$ yields

$$v(x,t) \leqslant v^{\frac{q-\theta}{q}}(x,t)\bar{v}^{\frac{\theta}{q}} + v^{\frac{q-\theta}{q}}(x,t)\left(\int_0^t \mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)d\eta\right)^{\frac{\theta}{q}}.$$

We set $\theta = pq$ and obtain

$$(v^p - \bar{v}^p)(x,t) \leqslant \left(\int_0^t \mathcal{R}(t-\eta)(u-\bar{u})_+^q(x,\eta)d\eta\right)^p, \tag{3.13}$$

that we use to get (3.8) for $q \geqslant 1$.

Now we need an inequality like (3.10), such that its combining with (3.8) yields (3.11). As in Section 2, we set

$$g(w)(x,t) = \mathcal{T}(t)\mathcal{S}_{N-1}v_0(x) + \int_0^t \mathcal{R}(t-\eta)w^q(x,\eta)d\eta, \qquad f(\xi) = \xi^p,$$

and by the mean value theorem for $f \circ g$, we write (using assumption $0 < p < 1$ as well)

$$(3.14) \qquad \begin{aligned} (u-\bar{u})(x,t) \leqslant pq \int_0^t \mathcal{R}(t-\eta) \left(\int_0^\eta \mathcal{R}(\eta - s) w^q(x,s) ds \right)^{p-1} \\ \left(\int_0^\eta \mathcal{R}(\eta - s)(w^{q-1}(u-\bar{u}))(x,s) ds \right) d\eta \end{aligned}$$

for some $w(\cdot,t) = \rho(x,s)u(\cdot,t) + (1-\rho(x,s))\bar{u}(\cdot,t)$, where $0 < \rho(x,s) < 1$. We also have by Hölder's inequality

$$(3.15) \qquad \begin{aligned} &\int_0^\eta \mathcal{R}(\eta - s)(w^{q-1}(u-\bar{u}))(x,s)ds \\ &\leqslant \left(\int_0^\eta \mathcal{R}(\eta - s) w^q(x,s) ds \right)^{\frac{q-1}{q}} \left(\int_0^\eta \mathcal{R}(\eta - s)|u-\bar{u}|^q(x,s) ds \right)^{\frac{1}{q}}, \end{aligned}$$

and since $w^q(0,x';s) \geqslant C^q s^{\alpha q}$, $pq - 1 < 0$, we derive from inequalities (3.14), (3.15) that

$$(3.16) \qquad \begin{aligned} \|(u-\bar{u})(t)\| &\leqslant pq \int_0^t (\pi(t-\eta))^{-\frac{1}{2}} \left(\int_0^\eta \mathcal{R}(\eta - s) C^q s^{\alpha_2 q} ds \right)^{\frac{pq-1}{q}} \\ &\quad \left(\int_0^\eta (\pi(\eta - s))^{-\frac{1}{2}} \|(u-\bar{u})(s)\|^q ds \right)^{1/q} d\eta \\ &= pqb^{-1}(\alpha)\, b^{-\frac{1}{q}}(\beta) \int_0^t (t-\eta)^{-\frac{1}{2}} \eta^{-\frac{1+q}{2q}} \\ &\quad \left(\int_0^\eta (\eta - s)^{-\frac{1}{2}} \|(u-\bar{u})(s)\|^q ds \right)^{\frac{1}{q}} d\eta. \end{aligned}$$

It takes the role of (3.10) in the iterating procedure, because combining (3.16) with (3.8) yields

$$\begin{aligned} \|(u-\bar{u})(t)\| &\leqslant pqb^{-1}(\alpha)\, b^{-\frac{1}{q}}(\beta)\, Pb^{\frac{1}{q}}(\beta) \int_0^t (t-\eta)^{-\frac{1}{2}} \eta^{-\frac{1+q}{2q}} \eta^{\frac{1+q}{2(1-pq)q}} d\eta \\ &= pqPt^\alpha, \end{aligned}$$

which is exactly (3.11), hence (3.12) does hold for $q \geqslant 1$ as well.

The final steps are obvious. Letting $k \to \infty$ in (3.12) implies $u = \bar{u}$ on the boundary $x_1 = 0$, and the contradiction argument is finished. □

4. Proof of Part (iii)

In this section we generalize the nontrivial one dimensional solution constructed in [**3**] for $pq < 1$ and trivial initial data $(u_0, v_0) \equiv (0, 0)$ to dimensions $N > 1$. Then we show that the members of the family (1.8) are the only solutions of (1.1) in this case, which completes the proof of Theorem.

Proposition 4.1. *Every member of the family* (1.8) *solves the problem* (1.1) *with trivial initial condition* $(u_0, v_0) \equiv (0, 0)$ *and* $pq < 1$.

Proof. The generalization of one dimensional solution constructed in [**3**] to higher dimensions is very simple. Obviously, the members of the family (1.8) fulfil (1.1) with $pq < 1$ and trivial initial condition when $t \neq s$. We only need to show that

$$\lim_{t \to s^+} \tilde{u}_t(x, t; s) = 0, \qquad x \in \mathbb{R}^N_+,\ s \in [0, \infty). \tag{4.1}$$

We use the facts (cf. [**1**])

$$-U_r(a, b, r) = aU(1 + a, 1 + b, r), \quad U(a, b, r) = r^{-a}(1 + O(r^{-1})) \quad \text{for } r \to \infty$$

to write

$$\begin{aligned}
\tilde{u}_t(x, t; s) &= c_2 \alpha\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}} (t - s)^{\alpha - 1} U\left(\frac{1}{2} + \alpha, \frac{1}{2}, \frac{x_1^2}{4(t - s)}\right) \\
&\quad + c_2 \frac{x_1^2}{4}\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}} (t - s)^{\alpha - 2} U\left(\frac{1}{2} + \alpha, \frac{1}{2}, \frac{x_1^2}{4(t - s)}\right) \\
&\quad + c_2 \frac{(1 + 2\alpha)x_1^2}{8}\, \mathrm{e}^{-\frac{x_1^2}{4(t-s)}} (t - s)^{\alpha - 2} U\left(\frac{3}{2} + \alpha, \frac{3}{2}, \frac{x_1^2}{4(t - s)}\right) \\
&= \mathrm{e}^{-\frac{x_1^2}{4(t-s)}} (t - s)^{2\alpha - \frac{3}{2}} \left(\varphi_2(x_1) + O(t - s)\right) \qquad \text{for } t \to s^+.
\end{aligned}$$

We see that (4.1) holds, i.e., $(\tilde{u}(x, t; s), \tilde{v}(x, t; s))$ solves (1.1) with $(u_0, v_0) \equiv (0, 0)$ and $pq < 1$. □

Proposition 4.2. *If* (u, v) *is a nontrivial nonnegative solution of the problem* (1.1) *with trivial initial condition* $(u_0, v_0) \equiv (0, 0)$ *and* $pq < 1$*, then there exists* $s \geqslant 0$ *such that* $(u, v) = (\tilde{u}(\cdot; s), \tilde{v}(\cdot; s))$ *where* $(\tilde{u}, \tilde{v})$ *is given in* (1.8).

Proof. First we observe that in terms of function b from (3.1) and constants C, D from (3.3), we have

$$\begin{aligned}
c_2 &= \pi^{-\alpha} b^{\frac{p}{1-pq}}(\beta) b^{\frac{pq}{1-pq}}(\alpha) \Gamma\left(\frac{1}{2} + \alpha\right) = C\Gamma\left(\frac{1}{2} + \alpha\right) b^{-1}(\alpha), \\
d_2 &= D\Gamma\left(\frac{1}{2} + \beta\right) b^{-1}(\beta), \\
b(\gamma) &= \int_0^1 s^{\gamma - \frac{1}{2}}(1 - s)^{-\frac{1}{2}} ds = \int_0^\infty t^{\gamma - \frac{1}{2}}(1 + t)^{-1-\gamma} dt \\
&= \Gamma\left(\frac{1}{2} + \gamma\right) U\left(\frac{1}{2} + \gamma, \frac{1}{2}, 0\right)
\end{aligned} \tag{4.2}$$

for $\gamma > 0$, and thus $f(0) = C$, $g(0) = D$. We apply the idea from Lemma 4 in [**4**]. Without loss of generality, we assume that there are $t > 0$ and $x \in \mathbb{R}^N_+$ such that $v(x,t) = \int_0^t \mathcal{R}(t-\eta)u^q(x,\eta)d\eta > 0$. We define τ as follows

$$\tau = \inf\{t > 0 : u(0,x';t) > 0,\ x' \in \mathbb{R}^{N-1}\}.$$

By standard results, $u(x,t)$, $v(x,t) > 0$ for any $x \in \mathbb{R}^N_+$ and $t > \tau$. Now we take $\bar{t} > \tau$ and set

$$\bar{u}(x,t) = u(x,\bar{t}+t), \qquad \bar{v}(x,t) = v(x,\bar{t}+t).$$

Obviously, $(\bar{u},\bar{v})$ solves (1.1) and $\bar{u}(x,0)$, $\bar{v}(x,0) > 0$, and according to Lemma 3.3,

$$u(0,x';\bar{t}+t) \geqslant Ct^\alpha, \qquad v(0,x';\bar{t}+t) \geqslant Dt^\beta$$

for any $x' \in \mathbb{R}^{N-1}$ and $t \geqslant 0$. This implies

$$u(0,x';t) \geqslant C(t-\tau)_+^\alpha, \quad v(0,x';t) \geqslant D(t-\tau)_+^\beta \quad x' \in \mathbb{R}^{N-1},\ t \geqslant 0. \tag{4.3}$$

Now let $T > 0$ be arbitrary and $M(T) > 0$ be such that

$$\|u(0,\cdot;s)\|_\infty \leqslant M(T)\|u(0,\cdot;t)\|_\infty, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

By (2.1),

$$\|u(0,\cdot;t)\|_\infty \leqslant \pi^{-\frac{1+p}{2}} \int_0^t (t-\eta)^{-\frac{1}{2}} \left(\int_0^\eta (\eta-s)^{-\frac{1}{2}} \|u(0,\cdot;s)\|_\infty^q ds\right)^p d\eta, \tag{4.4}$$

and therefore,

$$\|u(0,\cdot;t)\|_\infty \leqslant 2^{\frac{p}{1-pq}} \pi^{-\alpha} b^{\frac{1}{1-pq}} \left(\frac{1+p}{2}\right) M^{\frac{pq}{1-pq}}(T) t^\alpha = P(T)t^\alpha. \tag{4.5}$$

The usual iteration argument (combining with (4.4)) yields

$$\|u(0,\cdot;t)\|_\infty \leqslant P^{(pq)^k}(T) \pi^{-\alpha(1-(pq)^k)} b^{p\frac{1-(pq)^k}{1-pq}}(\beta) b^{\frac{1-(pq)^k}{1-pq}}(\alpha) t^\alpha. \tag{4.6}$$

We get an analogous result for v the same way, and letting $k \to \infty$, $T \to \infty$, we arrive at

$$\begin{aligned} u(0,x';t) &\leqslant \|u(0,\cdot;t)\|_\infty \leqslant Ct^\alpha, \\ v(0,x';t) &\leqslant \|v(0,\cdot;t)\|_\infty \leqslant Dt^\beta, \end{aligned} \qquad x' \in \mathbb{R}^{N-1},\ t \geqslant 0. \tag{4.7}$$

When $\tau > 0$, we take $0 < \underline{t} < \tau$ and define

$$\underline{u}(x,t) = u(x,\underline{t}+t), \qquad \underline{v}(x,t) = v(x,\underline{t}+t).$$

A simple contradiction argument implies that $u(\underline{t}) = v(\underline{t}) \equiv 0$, and therefore $(\underline{u},\underline{v})$ solves (1.1) with trivial initial data. From (4.7) we obtain

$$u(0,x';\underline{t}+t) \leqslant Ct^\alpha, \qquad v(0,x';\underline{t}+t) \leqslant Dt^\beta$$

for any $x' \in \mathbb{R}^{N-1}$ and $t \geqslant 0$. This implies

$$u(0,x';t) \leqslant C(t-\tau)_+^\alpha, \quad v(0,x';t) \leqslant D(t-\tau)_+^\beta, \quad x' \in \mathbb{R}^{N-1},\ t \geqslant 0, \tag{4.8}$$

and, by (4.7), it holds for $\tau = 0$ as well.

We conclude from (4.3) and (4.8) that

$$u(0,x';t) = C(t-\tau)_+^{\alpha} = \tilde{u}(0,x',t;\tau),$$
$$v(0,x';t) = D(t-\tau)_+^{\beta} = \tilde{v}(0,x',t;\tau)$$

for $x' \in \mathbb{R}^{N-1}$ and $t \geqslant 0$. In other words, for any nontrivial nonnegative solution of (1.1) there exists a member of the family (1.8) such that they equal on the boundary $x_1 = 0$ in any time $t \geqslant 0$. Hence they are identical everywhere by the maximum principle. □

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M. Kordoš, Institute of Applied Mathematics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia, *e-mail*: `kordos@fmph.uniba.sk`