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# GENERALIZED WARPED PRODUCT MANIFOLDS AND BIHARMONIC MAPS

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Abstract. In this paper, we present some new properties for biharmonic and conformal biharmonic maps between generalized warped product manifolds.

## 1. Introduction

Biharmonic maps are critical points of bi-energy functional defined on the space of smooth maps between Riemannian manifolds, introduced by Eells and Sampson in 1964, which is a generalization of harmonic maps [**7**].
If $\varphi\colon (M,g) \to (N,h)$ is a smooth map between Riemannian manifolds then the tension field of $\varphi$ is defined as

$$\tau(\varphi) = \operatorname{trace}_g \nabla \mathrm{d}\varphi.$$

Then $\varphi$ is called harmonic if the tension field vanishes. The equivalent definition is that $\varphi$ is a critical point of the energy functional

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where $e(\varphi) = \frac{1}{2}\operatorname{trace}_g(\varphi^* h)$ is called energy density of $\varphi$. If $M$ is not compact then the energy $E(\varphi)$ may be defined on its compact subsets.

**Definition 1.** A map $\varphi\colon (M,g) \to (N,h)$ between Riemannian manifolds is called *biharmonic* if it is a critical point of the *bi-energy* functional:

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g$$

(or over any compact subset $K \subset M$).

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bi-tension field

$$(1) \qquad \tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -\big(\Delta^\varphi \tau(\varphi) + \operatorname{trace}_g R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi\big),$$

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where $R^N$ is the curvature tensor field on $N$ and $J_\varphi$ is the Jacobi operator defined by

$$J_\varphi \colon \Gamma(\varphi^{-1}(TN) \to \Gamma(\varphi^{-1}(TN)$$
$$V \mapsto \Delta^\varphi V + \mathrm{trace}_g R^N(V, \mathrm{d}\varphi)\mathrm{d}\varphi. \tag{2}$$

(One can refer to [**1**] [**5**] [**6**] [**9**] [**11**] for more details)

## 2. Some results on generalized warped product manifolds

In this section, we give the definition and some geometric properties of generalized warped product manifolds. For more detail see [**3**] [**4**] [**8**] [**13**].

**Definition 2** ([**4**])**.** Let $(M^m, g)$ and $(N^n, h)$ be two Riemannian manifolds, and $f \colon M \times N \to \mathbb{R}$ be a smooth positive function. The generalized warped metric on $M \times_f N$ is defined by

$$G_f = \pi^* g + (f)^2 \eta^* h \tag{3}$$

where $\pi \colon (x, y) \in M \times N \to x \in M$ and $\eta \colon (x, y) \in M \times N \to y \in N$ are the canonical projections. For all $X, Y \in T(M \times N)$, we have

$$G_f(X, Y) = g(\mathrm{d}\pi(X), \mathrm{d}\pi(Y)) + (f)^2 h(\mathrm{d}\eta(X), \mathrm{d}\eta(Y)).$$

By $X \wedge_{G_{f^2}} Y$, we denote the linear map

$$Z \in \mathcal{H}(M) \times \mathcal{H}(N) \to \Big(X \wedge_{G_{f^2}} Y\Big)Z = G_{f^2}\Big(Z, Y\Big)X - G_{f^2}\Big(Z, X\Big)Y. \tag{4}$$

**Proposition 1** ([**13**])**.** *Let $(M^m, g)$ and $(N^n, h)$ be two Riemannian manifolds. If $\overline{\nabla}$ denotes the Levi-Civita connection and $\overline{R}$ the curvature tensor on $(M \times_f N, G_f)$, then for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$, we have*

$$\begin{aligned} \overline{\nabla}_X Y - \nabla_X Y = {} & X(\ln f)(0, Y_2) + Y(\ln f)(0, X_2) \\ & - \frac{1}{2} h(X_2, Y_2)(\mathrm{grad}_M f^2, \frac{1}{f^2}\mathrm{grad}_N f^2) \end{aligned} \tag{5}$$

*and*

$$\begin{aligned} \overline{R}(X,Y)Z - R(X,Y)Z = {} & (\nabla^M_{Y_1} \mathrm{grad}_M \ln f + Y_1(\ln f)\,\mathrm{grad}_M \ln f, 0) \wedge_{G_f} (0, X_2)Z \\ & - (\nabla^M_{X_1} \mathrm{grad}_M \ln f + X_1(\ln f)\,\mathrm{grad}_M \ln f, 0) \wedge_{G_f} (0, Y_2)Z \\ & + \frac{1}{f^2}\Big[(0, \nabla^N_{Y_2} \mathrm{grad}_N \ln f - Y_2(\ln f)\mathrm{grad}_N \ln f) \wedge_{G_f} (0, X_2) \\ & - (0, \nabla^N_{X_2} \mathrm{grad}_N \ln f - X_2(\ln f)\mathrm{grad}_N \ln f) \wedge_{G_f} (0, Y_2) \\ & - (f^2 |\,\mathrm{grad}_M \ln f\,|^2 + |\,\mathrm{grad}_N \ln f\,|^2)(0, X_2) \wedge_{G_f} (0, Y_2)\Big]Z \\ & + \Big[X_1(Z_2(\ln f)) + X_2(Z_1(\ln f))\Big](0, Y_2) \\ & - \Big[Y_1(Z_2(\ln f)) + Y_2(Z_1(\ln f))\Big](0, X_2) \end{aligned} \tag{6}$$

where $X=(X_1,X_2)$, $\quad Y=(Y_1,Y_2)$, $\quad \nabla_X Y=(\nabla^M_{X_1}Y^1,\nabla^N_{X_2}Y^2)$ and $R(X,Y)Z=(R^M(X_1,Y_1)Z_1,R^N(X_2,Y_2)Z_2)$.

**Proposition 2** ([**8**]). *Let $(M^m,g)$ and $(N^n,h)$ be two Riemannian manifolds and $f\colon M\times N\to\mathbb{R}$ be smooth positive function. The Ricci curvature of the generalized warped product manifolds $(M\times_f N,G_f)$ is given by the following formulas:*

$$\begin{aligned}
\mathrm{Ric}((X_1,0),(Y_1,0)) &= \mathrm{Ric}^M(X_1,Y_1)-\mathrm{ng}(\nabla^M_{X_1}\mathrm{grad}_M\ln f+X_1(\ln f)\,\mathrm{grad}_M\ln f,Y_1)\\
\mathrm{Ric}((X_1,0),(0,Y_2)) &= -\,nX_1(Y_2(\ln f))\\
\mathrm{Ric}((0,X_2),(Y_1,0)) &= h(X_2,\mathrm{grad}_N(Y_1(\ln f)))-nX_2(Y_1(\ln f))\\
\mathrm{Ric}((0,X_2),(0,Y_2)) &= \mathrm{Ric}^N(X_2,Y_2)+(2-n)h(\nabla^N_{X_2}\mathrm{grad}_N\ln f,Y_2)\\
&\quad +(2-n)\big[h(X_2,Y_2)\mid\mathrm{grad}_N\ln f\mid^2-X_2(\ln f)h(\mathrm{grad}_N\ln f,Y_2)\big]\\
&\quad +h(X_2,Y_2)\big[nf^2\mid\mathrm{grad}_M\ln f\mid^2-\Delta_N(\ln f)-f^2\Delta_M(\ln f)\big]
\end{aligned}$$

*for all $X_1,Y_1\in\mathcal{H}(M)$ and $X_2,Y_2\in\mathcal{H}(N)$.*

**Proposition 3** ([**3**]). *If $\varphi\colon P\to M$ and $\psi\colon P\to N$ are regular maps. Then the tension field of $\phi\colon x\in(P^p,\ell)\longrightarrow(\phi(x)=(\varphi(x),\psi(x))\in(M\times_f N,G_f)$ is given by the following relation*

$$\begin{aligned}
\tau(\phi) &= \Big(\tau(\varphi),\tau(\psi)\Big)+2\Big(0,\mathrm{d}\psi(\mathrm{grad}_P(\ln f\circ\phi))\Big)\\
&\quad -e(\psi)\Big(\mathrm{grad}_M f^2,\frac{1}{f^2}\mathrm{grad}_N f^2\Big).
\end{aligned}\tag{7}$$

**Corollary 1** ([**3**]). *Let $(M^m,g)$ be a Riemannian manifold and $f\colon(x,y)\in M\times M\to f(x,y)\in\mathbb{R}$ be smooth positive function. Then the tension field of the map*

$$\begin{aligned}
\phi\colon (M,g) &\longrightarrow (M\times_f M,G_f)\\
x &\longmapsto (x,x)
\end{aligned}$$

*is given by*

$$\begin{aligned}
\tau(\phi) &= \big\{-\frac{m}{2}(\mathrm{grad}_x f^2,0)+2(0,\mathrm{grad}_x\ln f)\\
&\qquad +(2-m)(0,\mathrm{grad}_y\ln f)\big\}\circ\phi\\
&= \{-m.f^2(\mathrm{grad}_x\ln f,0)+2(0,\mathrm{grad}_x\ln f)\\
&\qquad +(2-m)(0,\mathrm{grad}_y\ln f)\}\circ\phi.
\end{aligned}\tag{8}$$

**Proposition 4** ([**3**]). *The tension field of $\phi\colon(M\times_f N,G_f)\longrightarrow(P,k)$ is given by*

$$\begin{aligned}
\tau(\phi) &= \tau(\phi_M)+nd\phi_M(\mathrm{grad}_M\ln f)\\
&\qquad +\frac{1}{f^2}\{\tau(\phi_N)+(n-2)d\phi_N(\mathrm{grad}_N\ln f)\}
\end{aligned}\tag{9}$$

*where $\phi_M\colon x\in M\to\phi_M(x)=\phi(x,y)\in P$ and $\phi_N\colon y\in N\to\phi_N(y)=\phi(x,y)\in P$.*

**Proposition 5** ([**3**]). *If $\varphi\colon M \to M$ and $\psi\colon N \to N$ are harmonic maps, then the tension fields of*

$$\begin{aligned}\phi\colon (M \times_f N, G_f) &\longrightarrow (M \times N, G)\\ (x,y) &\longmapsto (\varphi(x), \psi(y))\end{aligned}$$

*is given by the following formula*

$$\tau(\phi) = n(\mathrm{d}\varphi(\operatorname{grad}_M \ln f), 0) + \frac{(n-2)}{f^2}(0, \mathrm{d}\psi(\operatorname{grad}_N \ln f)). \tag{10}$$

## 3. Biharmonic maps on generalized warped product manifolds

### 3.1. Biharmonicity conditions of the inclusion $\bar{\phi}\colon (N,h) \longrightarrow (M^m \times_f N^n, G_f)$

**Theorem 1.** *Let $(M^m, g)$ and $(N^n, h)$ be two Riemannian manifolds and $x_0$ be an arbitrary point of $M$. Then the tension and the bitension fields of the inclusion*

$$\begin{aligned}\bar{\phi}\colon (N,h) &\longrightarrow (M \times_f N, G_f)\\ y &\longmapsto (x_0, y)\end{aligned} \tag{11}$$

*are given by:*

*i)* $\tau(\bar{\phi}) = \{-n\,\mathrm{e}^{2\gamma}(\operatorname{grad}_M \gamma, 0) + (2-n)(0, \operatorname{grad}_N \gamma)\} \circ \bar{\phi}$.

*ii)*

$$\begin{aligned}\tau_2(\bar{\phi}) = \Big\{ &-\frac{n^2\,\mathrm{e}^{4\gamma}}{2}(\operatorname{grad}_M(|\operatorname{grad}_M \gamma|^2), 0)\\ &+ (n-2)(0, \operatorname{grad}_N(\Delta_N(\gamma)) + 2\operatorname{Ricci}_N(\operatorname{grad}_N \gamma))\\ &- \mathrm{e}^{2\gamma}\Big[2n^2\,\mathrm{e}^{2\gamma} \mid \operatorname{grad}_M \gamma \mid^2 - 4\Delta_N\gamma\Big](\operatorname{grad}_M \gamma, 0)\\ &- \mathrm{e}^{2\gamma}\Big[(n^2 - 4n - 4) \mid \operatorname{grad}_N \gamma \mid^2\Big](\operatorname{grad}_M \gamma, 0)\\ &+ \Big[(2-n)^2 \mid \operatorname{grad}_N \gamma \mid^2 - 2(2-n)\Delta_N\gamma\Big](0, \operatorname{grad}_N \gamma)\\ &+ \Big[2n(n-4)\,\mathrm{e}^{2\gamma} \mid \operatorname{grad}_M \gamma \mid^2\Big](0, \operatorname{grad}_N \gamma)\\ &+ n\,\mathrm{e}^{2\gamma}\Big[(0, \operatorname{grad}_N(\mid \operatorname{grad}_M \gamma \mid^2) + \operatorname{trace}_N\big((\operatorname{grad}_M \gamma)(\star(\gamma))(0,\star)\big)\Big]\\ &+ \frac{(n-2)(6-n)}{2}(0, \operatorname{grad}_N(\mid \operatorname{grad}_N \gamma \mid^2)\Big\} \circ \bar{\phi}\end{aligned} \tag{12}$$

*where $f(x,y) = \mathrm{e}^{\gamma(x,y)}$.*

*Proof.* From Proposition 3, we obtain

i) $\tau(\bar{\phi}) = \{-n\,\mathrm{e}^{2\gamma}(\operatorname{grad}_M \gamma, 0) + (2-n)(0, \operatorname{grad}_N \gamma)\} \circ \bar{\phi}$.

ii) Let $y \in N$ and $(F_i)_i$ be a local orthonormal frame on $(N^n, h)$ such that

$$(\nabla_{F_i} F_j)_y = 0 \quad (1 \le i, j \le n).$$

Using the general formula of bitension field

$$\tau_2(\bar{\phi}) = -\operatorname{tr}_h(\nabla^{\bar{\phi}})^2 \tau(\bar{\phi}) - \operatorname{tr}_h \overline{R}(\tau(\bar{\phi}), d\bar{\phi})d\bar{\phi}. \tag{13}$$

and Proposition 1, we have:

- $\nabla^{\bar{\phi}}_{F_i}\tau(\bar{\phi})$

$$\begin{aligned}(14)\quad &= -n\,\mathrm{e}^{2\gamma}\Big[2F_i(\gamma)(\mathrm{grad}_M\gamma,0)+\mid\mathrm{grad}_M\gamma\mid^2(0,F_i)\Big]\\ &\quad+(2-n)\Big[(0,\nabla^N_{F_i}\mathrm{grad}_N\gamma)+\mid\mathrm{grad}_N\gamma\mid^2(0,F_i)\Big]\\ &\quad-(2-n)\,\mathrm{e}^{2\gamma}F_i(\gamma)(\mathrm{grad}_M\gamma,0).\end{aligned}$$

- $tr_h(\nabla^{\bar{\phi}})^2\tau(\bar{\phi})$

$$\begin{aligned}(15)\quad &= -n\,\mathrm{e}^{2\gamma}\Big[(0,\mathrm{grad}_N(\mid\mathrm{grad}_M\gamma\mid^2)+(6-n)\mid\mathrm{grad}_M\gamma\mid^2\mathrm{grad}_N\gamma)\\ &\quad+(2-n)\big(0,tr_h(\nabla^N)^2\mathrm{grad}_N\gamma+2\,\mathrm{grad}_N(\mid\mathrm{grad}_N\gamma\mid^2)\big)\\ &\quad+(2-n)\Big[(2-n)\mid\mathrm{grad}_N\gamma\mid^2-\mathrm{e}^{2\gamma}\mid\mathrm{grad}_M\gamma\mid^2-\Delta(\gamma)\Big](0,\mathrm{grad}_N\gamma)\\ &\quad-\mathrm{e}^{2\gamma}\Big[4\Delta_N(\gamma)-n^2\,\mathrm{e}^{2\gamma}\mid\mathrm{grad}_M\gamma\mid^2-(n^2-4n-4)\mid\mathrm{grad}_N\gamma\mid^2\Big].\end{aligned}$$

- $\sum_i \overline{R}((\mathrm{e}^{2\gamma}\mathrm{grad}_M\gamma,0),(0,F_i))(0,F_i)$

$$\begin{aligned}(16)\quad &= -n\,\mathrm{e}^{4\gamma}\left(\frac{1}{2}\mathrm{grad}_M(\mid\mathrm{grad}_M\gamma\mid^2)+\mid\mathrm{grad}_M\gamma\mid^2\mathrm{grad}_M\gamma,0\right)\\ &\quad+\mathrm{e}^{2\gamma}\sum_i(\mathrm{grad}_M\gamma)(F_i(\gamma))(0,F_i).\end{aligned}$$

- $\sum_i \overline{R}((0,\mathrm{grad}_N\gamma),(0,F_i))(0,F_i)$

$$\begin{aligned}(17)\quad &= (0,\mathrm{Ricci}_N(\mathrm{grad}_N\gamma)+\frac{2-n}{2}\mathrm{grad}_N(\mid\mathrm{grad}_N\gamma\mid^2))\\ &\quad+\Big[(1-n)\,\mathrm{e}^{2\gamma}\mid\mathrm{grad}_M\gamma\mid^2-\Delta_N(\gamma)\Big](0,\mathrm{grad}_N\gamma).\end{aligned}$$

Substituting (15), (16) and (17) in (13), we obtain the formula (12). □

**Remarks.**

1) If $\dim N = 2$, then
$\tau(\bar{\phi}) = -2\,\mathrm{e}^{2\gamma}(\mathrm{grad}_M\gamma,0)$
and

$$\begin{aligned}\tau_2(\bar{\phi}) = &-2\,\mathrm{e}^{4\gamma}(\mathrm{grad}_M(\mid\mathrm{grad}_M\gamma\mid^2),0)+8\,\mathrm{e}^{2\gamma}\mid\mathrm{grad}_M\gamma\mid^2(0,\mathrm{grad}_N\gamma)\\ &-\mathrm{e}^{2\gamma}\Big[8\,\mathrm{e}^{2\gamma}\mid\mathrm{grad}_M\gamma\mid^2-4\Delta_N(\gamma)-8\mid\mathrm{grad}_N\gamma\mid^2\Big](\mathrm{grad}_M\gamma,0)\\ &+2\,\mathrm{e}^{2\gamma}(0,\mathrm{grad}_N(\mid\mathrm{grad}_M\gamma\mid^2)).\end{aligned}$$

2) If $\gamma\in C^\infty(M)$, $\big(\gamma(x,y)=\gamma(x),\quad \forall(x,y)\in M\times N\big)$, then
$\tau(\bar{\phi}) = -n\,\mathrm{e}^{2\gamma}(\mathrm{grad}_M\gamma,0)$

and

$$\tau_2(\bar{\phi}) = -2\,\mathrm{e}^{4\gamma}\big(\operatorname{grad}_M(|\operatorname{grad}_M\gamma|^2) - 4\,|\operatorname{grad}_M\gamma|^2\,(\operatorname{grad}_M\gamma, 0).$$

The results coincide with the formulas obtained in [**1**].

3) If $\gamma \in C^\infty(N)$, $\big(\gamma(x,y) = \gamma(y), \quad \forall (x,y) \in M\times N\big)$, then

$$\tau(\bar{\phi}) = (2-n)(0, \operatorname{grad}_N \ln f)$$

and

$$\begin{aligned}\tau_2(\bar{\phi}) &= (n-2)\big(0, \operatorname{grad}_N(\Delta(\gamma)) + 2\,\mathrm{Ricci}(\operatorname{grad}_N\gamma\big)\\ &\quad + (n-2)\Big[(2-n)\,|\operatorname{grad}_N\gamma|^2 - 2\Delta_N(\gamma)\Big](0, \operatorname{grad}_N\gamma)\\ &\quad + \frac{(n-2)(6-n)}{2}(0, \operatorname{grad}_N(|\operatorname{grad}_N\gamma|^2))\end{aligned}$$

### 3.2. Biharmonicity conditions of $\phi$: $(M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$

**Lemma 1.** *Let $\lambda \in C^\infty(M\times N)$ be a smooth function and $\sigma \in \Gamma(\phi^{-1}TP)$. Then*

$$\begin{aligned}J_\phi(\lambda\sigma) &= \lambda J_{\phi_M}(\sigma) + \Delta_M(\lambda)\sigma + 2\nabla^{\phi_M}_{\operatorname{grad}_M\lambda}\sigma\\ &\quad + n\Big[(\operatorname{grad}_M\gamma)(\lambda)\sigma + \lambda\nabla^{\phi_M}_{\operatorname{grad}_M\gamma}\sigma\Big]\\ &\quad + \mathrm{e}^{-2\gamma}\Big[\lambda J_{\phi_N}(\sigma) + \Delta_N(\lambda)\sigma + 2\nabla^{\phi_N}_{\operatorname{grad}_N\lambda}\sigma\Big]\\ &\quad + (n-2)\,\mathrm{e}^{-2\gamma}\Big[(\operatorname{grad}_N\gamma)(\lambda)\sigma + \lambda\nabla^{\phi_N}_{\operatorname{grad}_N\gamma}\sigma\Big]\end{aligned} \tag{18}$$

*where* $f(x,y) = \mathrm{e}^{\gamma(x,y)}$.

*Proof.* Let $(E_i)_{i=1}^m$ and $(F_j)_{j=1}^n$ be a local orthonormal frame on $M$ and $N$, respectively. From the expression of Jacobi operator (formula (2)), we have

$$J_\phi(\lambda\sigma) = \operatorname{trace}_{G_f}(\nabla^\phi)^2(\lambda\sigma) + \operatorname{trace}_{G_f} R^p(\lambda\sigma, \mathrm{d}\phi)\mathrm{d}\phi. \tag{19}$$

By calculating each term, we obtain:

$$\begin{aligned}\operatorname{trace}_{G_f}(\nabla^\phi)^2(\lambda\sigma) &= \sum_{i=1}^m\Big[\nabla^\phi_{(E_i,0)}\nabla^\phi_{(E_i,0)}\lambda\sigma - \nabla^\phi_{\overline{\nabla}_{(E_i,0)}(E_i,0)}\lambda\sigma\Big]\\ &\quad + \sum_{j=1}^n\Big[\frac{1}{f}\nabla^\phi_{(0,F_j)}\frac{1}{f}\nabla^\phi_{(0,F_j)}\lambda\sigma - \nabla^\phi_{\overline{\nabla}_{\frac{1}{f}(0,F_j)}\frac{1}{f}(0,F_j)}\lambda\sigma\Big],\end{aligned} \tag{20}$$

$$\sum_{i=1}^m \nabla^\phi_{(E_i,0)}\nabla^\phi_{(E_i,0)}\lambda\sigma = \Delta_M(\lambda)\sigma + 2\nabla^{\phi_M}_{\operatorname{grad}_M\lambda}\sigma + \lambda\nabla^{\phi_M}_{E_i}\nabla^{\phi_M}_{E_i}\sigma, \tag{21}$$

(22)

$$\begin{aligned}\sum_{j=1}^n \frac{1}{f}\nabla^\phi_{(0,F_j)}\frac{1}{f}\nabla^\phi_{(0,F_j)}\lambda\sigma &= \frac{1}{f^2}\Big[\Delta_N(\ln f)\sigma - (\operatorname{grad}_N\ln f)(\lambda)\sigma\\ &\quad - \lambda\nabla^{\phi_N}_{\operatorname{grad}_N\ln f}\sigma + 2\nabla^{\phi_N}_{\operatorname{grad}_N\lambda}\sigma + \lambda\nabla^{\phi_N}_{F_j}\nabla^{\phi_N}_{F_j}\sigma\Big],\end{aligned}$$

$$\sum_{j=1}^{n}\overline{\nabla}_{\frac{1}{f}(0,F_j)}\frac{1}{f}(0,F_j)=\frac{1-n}{f^2}(0,\operatorname{grad}_N\ln f)-n(\operatorname{grad}_M\ln f,0),\tag{23}$$

$$\begin{aligned}-\sum_{j=1}^{n}\nabla^{\phi}_{\overline{\nabla}_{\frac{1}{f}(0,F_j)}\frac{1}{f}(0,F_j)}\lambda\sigma=\frac{n-1}{f^2}&\Big[(\operatorname{grad}_N\ln f)(\lambda)\sigma+\lambda\nabla^{\phi_N}_{\operatorname{grad}_N\ln f}\sigma\Big]\\&+n\Big[(\operatorname{grad}_M\ln f)(\lambda)\sigma+\lambda\nabla^{\phi_M}_{\operatorname{grad}_M\ln f}\sigma\Big],\end{aligned}\tag{24}$$

$$\begin{aligned}&\operatorname{trace}_{G_f}R^p(\lambda\sigma,\mathrm{d}\phi)\mathrm{d}\phi\\&\qquad=\lambda\operatorname{trace}_g R^p(\sigma,\mathrm{d}\phi_M)\mathrm{d}\phi_M+\frac{\lambda}{f^2}\operatorname{trace}_h R^p(\sigma,\mathrm{d}\phi_N)\mathrm{d}\phi_N.\end{aligned}\tag{25}$$

Substuting (21), (22) and (24) in (20), and summing with (25), we obtain the formula (18). □

**Theorem 2.** *Let $(M^m,g)$, $(N^n,h)$ and $(P^p,k)$ be Riemannian manifolds and $f\colon M\times N\to\mathbb{R}$ be a smooth positive function. Then the bitension fields of $\phi\colon(M^m\times_f N^n,G_f)\longrightarrow(P^p,k)$ is given by the following*

$$\begin{aligned}\tau_2(\phi)=\;&\tau_2(\phi_M)-nJ_{\phi_M}(\mathrm{d}\phi_M(\operatorname{grad}_M\gamma))-n\nabla^{\phi_M}_{\operatorname{grad}_M\gamma}V\\&+\mathrm{e}^{-4\gamma}\Big[\tau_2(\phi_N)-(n-2)J_{\phi_N}(\mathrm{d}\phi_N(\operatorname{grad}_N\gamma))-(n-6)\nabla^{\phi_N}_{\operatorname{grad}_N\gamma}W\\&-\big(2(4-n)\mid\operatorname{grad}_N\gamma\mid^2-2\Delta_N(\gamma)\big)W\Big]\\&-\mathrm{e}^{-2\gamma}\Big[J_{\phi_N}(V)+(n-2)\nabla^{\phi_N}_{\operatorname{grad}_N\gamma}V+J_{\phi_M}(W)+(n-4)\nabla^{\phi_M}_{\operatorname{grad}_M\gamma}W\\&+\big(2(2-n)\mid\operatorname{grad}_M\gamma\mid^2-2\Delta_M(\gamma)\big)W\Big]\end{aligned}\tag{26}$$

*where $V=\tau(\phi_M)+n\mathrm{d}\phi_M(\operatorname{grad}_M\gamma)$, $W=\tau(\phi_N)+(n-2)\mathrm{d}\phi_N(\operatorname{grad}_N\gamma)$ and $f=\mathrm{e}^{\gamma}$.*

*Proof.* From formulas (1) and (2), we have

$$\begin{aligned}J_\phi(V)=\tau_2(\phi_M)&+nJ_{\phi_M}(\mathrm{d}\phi_M(\operatorname{grad}_M\gamma))-n\nabla^{\phi_M}_{\operatorname{grad}_M\gamma}V\\&+\mathrm{e}^{-2\gamma}J_{\phi_N}(V)-(n-2)\,\mathrm{e}^{-2\gamma}\,\nabla^{\phi_N}_{\operatorname{grad}_N\gamma}V.\end{aligned}\tag{27}$$

From Lemma 18, we obtain

$$\begin{aligned}J_\phi(\mathrm{e}^{-2\gamma}W)=\;&\mathrm{e}^{-4\gamma}\Big[\tau_2(\phi_N)+(n-2)J_{\phi_N}(d\phi_N(\operatorname{grad}_N\gamma))\\&-(n-6)\nabla^{\phi_N}_{\operatorname{grad}_N\gamma}W-2\big((4-n)\mid\operatorname{grad}_N\gamma\mid^2-\Delta_N(\gamma)\big)W\Big]\\&+\mathrm{e}^{-2\gamma}\Big[J_{\phi_M}(W)-(n-4)\nabla^{\phi_M}_{\operatorname{grad}_M\gamma}W\Big]\\&-2\,\mathrm{e}^{-2\gamma}\Big[(2-n)\mid\operatorname{grad}_M\gamma\mid^2-\Delta_M(\gamma)\Big]W\end{aligned}\tag{28}$$

using Proposition 4 and summing the formulas (27) and (28), Theorem 2 follows. □

**Particular cases**

- If $f \in C^\infty(M)$, then

$$\begin{aligned}\tau_2(\phi) = \tau_2(\phi_M) &+ \mathrm{e}^{-4\gamma}\,\tau_2(\phi_N) - nJ_{\phi_M}(\mathrm{d}\phi_M(\mathrm{grad}_M\gamma)) - n\nabla^{\phi_M}_{\mathrm{grad}_M\gamma}V\\ &- (n-4)\,\mathrm{e}^{-2\gamma}\,\nabla^{\phi_M}_{\mathrm{grad}_M\gamma}\tau(\phi_N) + 4\,\mathrm{e}^{-2\gamma}\,\Delta_M(\gamma)\tau(\phi_N)\\ &- \mathrm{e}^{-2\gamma}\Big[J_{\phi_N}(V) + J_{\phi_M}(\tau(\phi_N)) + 2(2-n)\mid\mathrm{grad}_M\gamma\mid^2\tau(\phi_N)\Big]\end{aligned}$$

- If $f \in C^\infty(M)$ and $\phi\colon (x,y)\in M\times N \to x \in M$ is the first projection, then $V = n\cdot\mathrm{grad}(\gamma)$ and $\tau_2(\phi) = -n\Big(J_{\phi_M}(\mathrm{grad}(\gamma)) + \dfrac{n}{2}\,\mathrm{grad}(|\,\mathrm{grad}\,\gamma|^2\Big)\circ\phi$, we find the result obtained in [**1**]
- If $f \in C^\infty(N)$, then

$$\begin{aligned}\tau_2(\phi) = \tau_2(\phi_M) &- \mathrm{e}^{-2\gamma}\Big[J_{\phi_M}(W) + J_{\phi_N}(\tau(\phi_M)) + (n-2)\nabla^{\phi_N}_{\mathrm{grad}_N\gamma}\tau(\phi_M)\Big]\\ &+ \mathrm{e}^{-4\gamma}\Big[\tau_2(\phi_N) - (n-2)J_{\phi_N}(\mathrm{d}\phi_N(\mathrm{grad}_N\gamma)) - (n-6)\nabla^{\phi_N}_{\mathrm{grad}_N\gamma}W\\ &- (2(4-n)\mid\mathrm{grad}_N\gamma\mid^2 - 2\Delta_N(\gamma))W\Big].\end{aligned}$$

- If $\varphi\colon (M,g) \longrightarrow (P,k)$ be regular map and $\phi(x,y) = \varphi(x)$, then

$$\tau_2(\phi) = \tau_2(\varphi) - nJ_\varphi(\mathrm{d}\varphi(\mathrm{grad}_M\gamma)) - n\nabla^{\varphi}_{\mathrm{grad}_M\gamma}V. \tag{29}$$

From Proposition 4 and Lemma 1, we deduce the following.

**Theorem 3.** *Let $\varphi\colon (M,g) \longrightarrow (P,\ell)$ be a conformal map with dilation $\lambda$. Then the bitension field of $\phi\colon (x,y)\in (M\times_f N, G_f) \longrightarrow \phi(x,y) = \varphi(x) \in (P,\ell)$, is given by*

$$\tau_2(\phi) = -J_\varphi(\mathrm{d}\varphi(\mathrm{grad}_M\ln(\mu))) - n\nabla^{\varphi}_{\mathrm{grad}_M\ln f}\mathrm{d}\varphi(\mathrm{grad}_M\ln(\mu)) \tag{30}$$

*where $\mu = \lambda^{2-m}f^n$.*

**Theorem 4.** *Let $f \in C^\infty(M)$, thus the domain of $\phi$ is a warped product, and $\varphi\colon (M^m,g) \longrightarrow (P^m,\ell)$ $(m \geq 3)$ be a conformal map with dilation $\lambda$. Then $\phi\colon (x,y) \in (M\times_f N, G_f) \longrightarrow \phi(x,y) = \varphi(x) \in (P,\ell)$ is biharmonic map if and only if the following equation is verified*

$$\begin{aligned}0 = {}& \mathrm{grad}(\Delta\ln\lambda^{2-m}f^n) + 2\,\mathrm{Ricci}^M(\mathrm{grad}\ln\lambda^{2-m}f^n)\\ &+ 2n(2-m)\nabla_{\mathrm{grad}\ln f}\,\mathrm{grad}\ln\lambda + 4n\nabla_{\mathrm{grad}\ln\lambda}\,\mathrm{grad}\ln f\\ &+ \frac{n^2}{2}\,\mathrm{grad}(\mid\mathrm{grad}\ln f\mid^2) + \frac{(6-m)(2-m)}{2}\,\mathrm{grad}(\mid\mathrm{grad}\ln\lambda\mid^2)\\ &+ \big[(2-m)^2\mid\mathrm{grad}\ln\lambda\mid^2 - n^2\mid\mathrm{grad}\ln f\mid^2 + 2\Delta(\ln\lambda^{2-m}f^n)\big]\,\mathrm{grad}\ln\lambda\\ &+ 2n\big[(2-m)\mid\mathrm{grad}\ln\lambda\mid^2 + nd\ln f(\mathrm{grad}\ln\lambda)\big]\,\mathrm{grad}\ln f\end{aligned} \tag{31}$$

*where* grad, $\Delta$ *and* $\nabla$ *are evaluated on* $M$.

For the proof of Theorem 4, we need the following two lemmas.

**Lemma 2.** *Let $\varphi\colon (M,g) \longrightarrow (P,\ell)$ be a conformal map with dilation $\lambda$ and $f \in C^\infty(M)$. Then for any vector field $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned}\ell(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) = {}& \lambda^2 \mathrm{d}f(\operatorname{grad}\ln\lambda) g(X,Y) + \lambda^2 g(\nabla_X \operatorname{grad} f, Y) \\ & + \lambda^2[X(\ln\lambda)Y(f) - X(f)Y(\ln\lambda)].\end{aligned} \tag{32}$$

*Proof.*

$$\begin{aligned}&\ell(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) - \ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X)) \\ &= X(\lambda^2 g(\operatorname{grad} f, Y)) - \ell(\mathrm{d}\varphi(\operatorname{grad} f), \nabla_X \mathrm{d}\varphi(Y)) - Y(\lambda^2 g(\operatorname{grad} f, X)) \\ &\quad + \ell(\mathrm{d}\varphi(\operatorname{grad} f), \nabla_Y \mathrm{d}\varphi(X)) \\ &= X(\lambda^2) g(\operatorname{grad} f, Y) + \lambda^2 g(\nabla_X \operatorname{grad} f, Y) + \lambda^2 g(\operatorname{grad} f, \nabla_X Y) - Y(\lambda^2) g(\operatorname{grad} f, X) \\ &\quad - \lambda^2 g(\nabla_Y \operatorname{grad} f, X) - \lambda^2 g(\operatorname{grad} f, \nabla_Y X) - \lambda^2 g(\operatorname{grad} f, [X,Y])\end{aligned}$$

from which we have

$$\begin{aligned}&\ell(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) = \\ &\ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X)) + 2\lambda^2[X(\ln\lambda)Y(f) \quad - Y(\ln\lambda)X(f)]\end{aligned} \tag{33}$$

On the other hand,

$$\begin{aligned}\ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X)) &= \ell(\nabla \mathrm{d}\varphi(\operatorname{grad} f, Y), \mathrm{d}\varphi(X)) + \lambda^2 g(\nabla_Y \operatorname{grad} f, X) \\ &= \ell(\nabla_{\operatorname{grad} f} \mathrm{d}\varphi(Y), \mathrm{d}\varphi(X)) - \lambda^2 g(\nabla_{\operatorname{grad} f} Y, X) \\ &\quad + \lambda^2 g(\nabla_Y \operatorname{grad} f, X) \\ &= \operatorname{grad} f(\lambda^2 g(X,Y)) - \ell(\mathrm{d}\varphi(Y), \nabla_{\operatorname{grad} f} \mathrm{d}\varphi(X)) \\ &\quad - \lambda^2 g(\nabla_{\operatorname{grad} f} Y, X) + \lambda^2 g(\nabla_Y \operatorname{grad} f, X) \\ &= 2\lambda^2 df(\operatorname{grad}\lambda) g(X,Y) + \lambda^2 g(\nabla_{\operatorname{grad} f} X, Y) \\ &\quad + \lambda^2 g(\nabla_Y \operatorname{grad} f, X) - \ell(\mathrm{d}\varphi(Y), \nabla \mathrm{d}\varphi(X, \operatorname{grad} f)) \\ &\quad - \lambda^2 g(Y, \nabla_{\operatorname{grad} f} X)\end{aligned}$$

$$\begin{aligned}\ell(\nabla_Y \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(X)) = {}& 2\lambda^2 df(\operatorname{grad}\lambda) g(X,Y) \\ & + 2\lambda^2 g(\nabla_Y \operatorname{grad} f, X) - \ell(\mathrm{d}\varphi(Y), \nabla_X \mathrm{d}\varphi(\operatorname{grad} f))\end{aligned} \tag{34}$$

Substituting (34) in (33) we obtain

$$\begin{aligned}h(\nabla_X \mathrm{d}\varphi(\operatorname{grad} f), \mathrm{d}\varphi(Y)) = {}& \lambda^2 df(\operatorname{grad}\ln\lambda) g(X,Y) + \lambda^2 g(\nabla_X \operatorname{grad} f, Y) \\ & + \lambda^2[X(\ln\lambda)Y(f) - X(f)Y(\ln\lambda)]\end{aligned}$$

□

**Lemma 3.** *Let $\varphi\colon (M,g) \longrightarrow (P,\ell)$ be conformal map with dilation $\lambda$ and $f \in C^\infty(M)$. Then for any vector field $X \in \Gamma(TM)$, we have*

$$\begin{aligned}h(\langle \nabla \mathrm{d}\varphi, \nabla df\rangle, \mathrm{d}\varphi(X)) = {}& 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) \\ & - \lambda^2 \Delta(f) g(\operatorname{grad}\ln\lambda, X)\end{aligned} \tag{35}$$

*where*

$$\langle \nabla \mathrm{d}\varphi, \nabla \mathrm{d}f\rangle = \mathrm{trace}_g \nabla \mathrm{d}\varphi(*, \nabla_* \operatorname{grad} f)) = \sum_i \nabla \mathrm{d}\varphi(e_i, \nabla_{e_i} \operatorname{grad} f))$$

$(e_i)_{i=1}^m$ *is a local orthonormal frame on* $M$.

*Proof.* For any vector field $X \in \Gamma(TM)$, summing over the index $i$, we obtain

$$\begin{aligned}
&h(\langle \nabla \mathrm{d}\varphi, \nabla df\rangle, \mathrm{d}\varphi(X)) \\
&\quad= h(\nabla_{e_i} \mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(X)) - h(\mathrm{d}\varphi(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(X)) \\
&\quad= e_i(\lambda^2 g(\nabla_{e_i} \operatorname{grad} f, X)) - \lambda^2 g(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f, X) \\
&\qquad - h(\mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \nabla_{e_i} \mathrm{d}\varphi(X)) \\
&\quad= 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) + \lambda^2 g(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f, X) \\
&\qquad + \lambda^2 g(\nabla_{e_i} \operatorname{grad} f, \nabla_{e_i} X) - h(\mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \nabla \mathrm{d}\varphi(e_i, X)) \\
&\qquad - \lambda^2 g(\nabla_{e_i} \operatorname{grad} f, \nabla_{e_i} X) - \lambda^2 g(\nabla_{e_i}\nabla_{e_i} \operatorname{grad} f, X) \\
&\quad= 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) + h(\nabla_X \mathrm{d}\varphi(\nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(e_i)) \\
&\qquad - X(\lambda^2\Delta(f)) \\
&\quad= 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) + h(\nabla \mathrm{d}\varphi(X, \nabla_{e_i} \operatorname{grad} f), \mathrm{d}\varphi(e_i)) \\
&\qquad - X(\lambda^2\Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\quad= 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - X(\lambda^2\Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\qquad + h(\nabla_{\nabla_{e_i} \operatorname{grad} f} \mathrm{d}\varphi(X), \mathrm{d}\varphi(e_i)) - \lambda^2 g(\nabla_{\nabla_{e_i} \operatorname{grad} f} X, e_i) \\
&\quad= 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - X(\lambda^2\Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\qquad + \nabla_{e_i} \operatorname{grad} f(\lambda^2 g(X, e_i)) - h(\mathrm{d}\varphi(X), \nabla_{\nabla_{e_i} \operatorname{grad} f} \mathrm{d}\varphi(e_i)) \\
&\qquad - \lambda^2 g(\nabla_{\nabla_{e_i} \operatorname{grad} f} X, e_i) \\
&\quad= 2\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - X(\lambda^2\Delta(f)) + \lambda^2 g(\nabla_X \nabla_{e_i} \operatorname{grad} f, e_i) \\
&\qquad - h(\mathrm{d}\varphi(X), \nabla \mathrm{d}\varphi(e_i, \nabla_{e_i} \operatorname{grad} f)) + \nabla_{e_i} \operatorname{grad} f(\lambda^2) g(X, e_i)
\end{aligned}$$

from which

$$2h(\langle \nabla \mathrm{d}\varphi, \nabla df\rangle, \mathrm{d}\varphi(X)) = 4\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda} \operatorname{grad} f, X) - 2\lambda^2\Delta(f) g(\operatorname{grad}\ln\lambda, X)$$

□

*Proof of Theorem 4.* From formula (30), the bitension field of $\phi$ is given by

$$\tau_2(\phi) = -J_\varphi(\mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu))) - n\nabla^\varphi_{\operatorname{grad}_M \ln f} \mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu))$$

where $\mu = \lambda^{2-m} f^n$. Then $\phi$ is a biharmonic map if and only if

$$\ell(\tau_2(\phi), d\phi(X)) = 0$$

for each $X \in \Gamma(T(M \times N))$. We have

$$\begin{aligned}
J_\varphi(\mathrm{d}\varphi(\operatorname{grad}_M \ln(\mu))) &= \mathrm{d}\varphi(\operatorname{grad}_M \Delta_M(\ln\mu)) + 2\mathrm{d}\varphi(\mathrm{Ricci}^M(\operatorname{grad}_M \ln\mu)) \\
&\quad + \nabla^M_{\operatorname{grad}_M \ln\mu} \tau(\varphi) + 2\langle \nabla^M \mathrm{d}\varphi, \nabla(d^M \ln\mu)\rangle
\end{aligned}$$

(see [**12**, formula (2.47)]), hence

$$\begin{aligned}
&\ell(\tau_2(\phi), d\phi(X)) \\
&\quad= \underbrace{\ell(\mathrm{d}\varphi(\operatorname{grad}\Delta(\ln\mu)), \mathrm{d}\phi(X))}_{T_1} + 2\underbrace{\ell(\mathrm{d}\varphi(\operatorname{Ricci}^M(\operatorname{grad}\ln\mu)), \mathrm{d}\phi(X))}_{T_2} \\
&\qquad+ \underbrace{\ell(\nabla_{\operatorname{grad}\ln\mu}\tau(\varphi), \mathrm{d}\phi(X))}_{T_3} + \underbrace{2\ell(\langle\nabla\mathrm{d}\varphi, \nabla d\ln\mu\rangle, \mathrm{d}\phi(X))}_{T_4} \\
&\qquad+ \underbrace{n.\ell(\nabla^{\varphi}_{\operatorname{grad}_M\ln f}\mathrm{d}\varphi(\operatorname{grad}_M\ln(\mu)), \mathrm{d}\phi(X))}_{T_5}.
\end{aligned} \tag{36}$$

Calculating each term of the above equation, we get

$$T_1 = \lambda^2 g(\operatorname{grad}\Delta(\ln\mu), X_1) = \lambda^2 g(\operatorname{grad}\Delta(\ln\lambda^{2-m}f^n), X_1)$$
$$T_2 = 2\lambda^2 g(\operatorname{Ricci}^M(\operatorname{grad}\ln\lambda^{2-m}f^n), X_1).$$

From formula (32) of Lemma 2, we obtain

$$\begin{aligned}
T_3 &= \ell(\nabla_{\operatorname{grad}\ln\mu}\tau(\varphi), \mathrm{d}\phi(X)) \\
&= (2-m)\ell(\nabla_{\operatorname{grad}\ln\mu}\mathrm{d}\varphi(\operatorname{grad}\ln\lambda), \mathrm{d}\phi(X)) \\
&= \lambda^2(2-m) \mid \operatorname{grad}\ln\lambda \mid^2 g(\operatorname{grad}\ln\mu, X_1) \\
&\quad+ n(2-m)g(\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda) \\
&\quad+ \frac{(2-m)^2}{2}g(\operatorname{grad}(\mid \operatorname{grad}\ln\lambda \mid^2), X_1)
\end{aligned}$$

and

$$\begin{aligned}
T_5 = \lambda^2\Big[&n[(2-m) \mid \operatorname{grad}\ln\lambda \mid^2 + 2d^M\ln f(\operatorname{grad}\ln\lambda)]g(\operatorname{grad}\ln f, X_1) \\
&+ \frac{n^2}{2}g(\operatorname{grad}(\mid \operatorname{grad}\ln f \mid^2), X_1) + n(2-m)g(\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda, X_1) \\
&- n^2 \mid \operatorname{grad}\ln f \mid^2 g(\operatorname{grad}\ln\lambda, X_1),
\end{aligned}$$

using formula (35) of Lemma 3, we deduce

$$\begin{aligned}
T_4 &= 2\ell(\langle\nabla\mathrm{d}\varphi, \nabla d\ln\mu\rangle, \mathrm{d}\phi(X)) \\
&= 4n\lambda^2 g(\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f, X_1) + 2(2-m)\lambda^2 g(\operatorname{grad}(\mid \operatorname{grad}\ln\lambda \mid^2), X_1) \\
&\quad- 2\lambda^2\Delta(\ln\mu)g(\operatorname{grad}\ln\lambda, X_1)
\end{aligned}$$

Substituting $T_1$, $T_2$, $T_3$, $T_4$ and $T_5$ in (36), we obtain

$$\begin{aligned}
0 = \;& \operatorname{grad}(\Delta\ln\lambda^{2-m}f^n) + 2\operatorname{Ricci}^M(\operatorname{grad}\ln\lambda^{2-m}f^n) \\
&+ 2n(2-m)\nabla_{\operatorname{grad}\ln f}\operatorname{grad}\ln\lambda + 4n\nabla_{\operatorname{grad}\ln\lambda}\operatorname{grad}\ln f \\
&+ \frac{n^2}{2}\operatorname{grad}(\mid \operatorname{grad}\ln f \mid^2) + \frac{(6-m)(2-m)}{2}\operatorname{grad}(\mid \operatorname{grad}\ln\lambda \mid^2) \\
&+ \big[(2-m)^2 \mid \operatorname{grad}\ln\lambda \mid^2 - n^2 \mid \operatorname{grad}\ln f \mid^2 + 2\Delta(\ln\lambda^{2-m}f^n)\big]\operatorname{grad}\ln\lambda \\
&+ 2n\big[(2-m) \mid \operatorname{grad}\ln\lambda \mid^2 + nd^M\ln f(\operatorname{grad}\ln\lambda)\big]\operatorname{grad}\ln f
\end{aligned}$$

□

From Theorem 4, we deduce the following corollary.

**Corollary 2.** *Let* $\varphi\colon (M^m, g) \longrightarrow (P^m, \ell)$ $(m \geq 3)$ *be a conformal map with dilation* $\lambda$*. If* $\varphi$ *is a biharmonic, not harmonic map, then*

$$\phi\colon (x, y) \in (M \times_f N, G_f) \longrightarrow \phi(x, y) = \varphi(x) \in (P, \ell)$$

*is biharmonic if and only if the following equation*

$$\begin{aligned} 0 = \ & \operatorname{grad}(\Delta \ln f) + 2 \operatorname{Ricci}^M(\operatorname{grad} \ln f) + 2(2-m)\nabla_{\operatorname{grad} \ln f} \operatorname{grad} \ln \lambda \\ & + 2(2-m) \mid \operatorname{grad} \ln \lambda \mid^2 \operatorname{grad} \ln f - 2\Delta(\ln f) \operatorname{grad} \ln \lambda \\ & + 2nd \ln f(\operatorname{grad} \ln \lambda) \operatorname{grad} \ln f - n \mid \operatorname{grad} \ln f \mid^2 \operatorname{grad} \ln \lambda \\ & + 4\nabla_{\operatorname{grad} \ln \lambda} \operatorname{grad} \ln f + \frac{n}{2} \operatorname{grad}(\mid \operatorname{grad} \ln f \mid^2) \end{aligned}$$

*is verified.*

**Example 1.** We consider the inversion map $\varphi\colon \mathbb{R}^m - \{0\} \longrightarrow \mathbb{R}^m - \{0\}$ defined by

$$\varphi(x) = \frac{x}{\mid x \mid^2}$$

$\varphi$ is a conformal map with the dilation

$$\lambda(x) = \frac{1}{\mid x \mid^2} = \frac{1}{r^2}.$$

$\varphi$ is biharmonic not harmonic map if and only if $m = 4$ (see [**12**]). Let

$$\begin{aligned} \phi\colon (\mathbb{R}^4 - \{0\}) \times_f N^n & \longrightarrow (\mathbb{R}^4 - \{0\}) \\ (x, y) & \longmapsto \frac{x}{\mid x \mid^2} \end{aligned}$$

and $f = e^{\alpha(r)}$, where $r = |x|$ and $\alpha \in C^\infty([0, +\infty[, \mathbb{R})$. We have:

$$\begin{aligned} \operatorname{grad} \ln f &= \alpha' \frac{\partial}{\partial r} \\ \mid \operatorname{grad} \ln f \mid^2 &= (\alpha')^2 \\ \operatorname{grad}(\mid \operatorname{grad} \ln f \mid^2) &= 2\alpha'\alpha'' \frac{\partial}{\partial r} \\ \Delta \ln f &= \alpha'' + \frac{3}{r}\alpha' \\ \operatorname{grad}(\Delta \ln f) &= \left(\alpha''' + \frac{3}{r}\alpha'' - \frac{3}{r^2}\alpha'\right) \frac{\partial}{\partial r}. \end{aligned}$$

Let $\ln \lambda = \beta(r)$. So $\phi$ is biharmonic if and only if $\alpha$ satisfies the following ordinary differential equation

$$\alpha''' + n\alpha'\alpha'' - \frac{1}{r}\alpha'' - \frac{15}{r^2}\alpha' - \frac{2n}{r}(\alpha')^2 = 0. \tag{37}$$

From which we obtain

$$f(x) = |x|^{-\frac{4}{n}}.$$

From Theorem 2, we deduce the following theorem.

**Theorem 5.** *Let $\psi\colon (N,h) \longrightarrow (P,\ell)$ be a regular map and $f\colon M\times N\to\mathbb{R}$ be a smooth positive function, then the bitension field of*

$$\phi\colon (x,y)\in (M\times_f N, G_f) \longrightarrow \phi(x,y)=\psi(y)\in (P,\ell)$$

*is given by*

$$\begin{aligned}\tau_2(\phi) = {} & + \frac{1}{f^4}\Big[\tau_2(\psi) - (n-2)J_\psi(\mathrm{d}\psi(\mathrm{grad}_N \ln f)) - (n-6)\nabla^{\psi}_{\mathrm{grad}_N \ln f} W \\ & - \big(2(4-n) \mid \mathrm{grad}_N \ln f \mid^2 - 2\Delta_N(\ln f)\big)W\Big] \\ & - \frac{2}{f^2}\Big[(2-n) \mid \mathrm{grad}_M \ln f \mid^2 - \Delta_M(\ln f)\Big]W.\end{aligned}$$

**Corollary 3.** *If $\psi$ is a conformal map with dilation $\mu$, then*

$$\begin{aligned}\tau_2(\phi) = {} & - \frac{n-2}{f^4}\left[J_\psi\left(\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right)\right)\right. \\ & + (n-6)\nabla^{\phi_N}_{\mathrm{grad}_N \ln f}\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right) \\ & \left. + \big(2(4-n) \mid \mathrm{grad}_N \ln f \mid^2 - 2\Delta_N(\ln f)\big)\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right)\right] \\ & - \frac{2(n-2)}{f^2}\Big[(2-n) \mid \mathrm{grad}_M \ln f \mid^2 - \Delta_M(\ln f)\Big]\mathrm{d}\psi\left(\mathrm{grad}_N\left(\ln\frac{f}{\mu}\right)\right).\end{aligned}$$

**Corollary 4.** *The tension and bitension fields of the second projection $\eta$ are given by the following formulae*

$$\tau(\eta) = \frac{n-2}{f^2}\,\mathrm{grad}_N \ln f$$

*and*

$$\begin{aligned}\tau_2(\eta) = {} & - \frac{n-2}{f^4}\Big[\mathrm{grad}_N(\Delta_N(\ln f)) + 2\,\mathrm{Ricci}(\mathrm{grad}_N)\ln f \\ & + \frac{n-6}{2}\,\mathrm{grad}_N(\mid \mathrm{grad}_N \ln f \mid^2) + \big((4-n) \mid \mathrm{grad}_N \ln f \mid^2\Big] \\ & + \frac{n-2}{f^4}\Delta_N(\ln f)\big)\,\mathrm{grad}_N \ln f + \frac{2(n-2)^2}{f^2} \mid \mathrm{grad}_M \ln f \mid^2 \\ & + \frac{2(n-2)}{f^2}\Delta_M(\ln f)\,\mathrm{grad}_N \ln f.\end{aligned}$$

**Theorem 6.** *Let $\psi\colon N\to N$ be a harmonic map, then the bitension field of $\phi\colon (x,y)\in (M\times_f N, G_f) \longrightarrow (x,\psi(y))\in (M\times N, G)$ is given by the following* *formula*

(38)

$$\begin{aligned}\tau_2(\phi) = & -\Big(n.\operatorname{grad}_M(\Delta(\ln f)) + 2n\operatorname{Ricci}^M(\operatorname{grad}_M \ln f), 0\Big)\\ & - \frac{n^2}{2}\Big(\operatorname{grad}_M(\mid \operatorname{grad}_M \ln f \mid^2), 0\Big)\\ & + \frac{n-2}{f^4}\Big(0, J_\psi(\mathrm{d}\psi(\operatorname{grad}_N \ln f)) - (n-6)\nabla^\psi_{\operatorname{grad}_N \ln f}\mathrm{d}\psi(\operatorname{grad}_N \ln f)\Big)\\ & + \frac{2(n-2)}{f^4}\Big[\Delta_N(\ln f) + f^2\Delta_M(\ln f) + (n-4)\mid \operatorname{grad}_N \ln f \mid^2\\ & + (n-2)\mid \operatorname{grad}_M \ln f \mid^2\Big](0, \mathrm{d}\psi(\operatorname{grad}_N \ln f))\end{aligned}$$

*Proof.* From Proposition 5, we obtain

$$\tau(\phi) = n(\operatorname{grad}_M \ln f, 0) + \frac{n-2}{f^2}(0, \mathrm{d}\psi(\operatorname{grad}_N \ln f)).$$

(39)

$$\begin{aligned}&\operatorname{trace}_{G_f}(\nabla^\phi)^2(\tau(\phi))\\ &\quad = n.\operatorname{trace}_g(\nabla^M)^2\Big(\operatorname{grad}_M \ln f, 0\Big)\\ &\qquad + \frac{n^2}{2}\Big(\operatorname{grad}_M(\mid \operatorname{grad}_M \ln f \mid^2), 0\Big)\\ &\qquad + \frac{n-2}{f^4}\Big(0, \operatorname{trace}_h(\nabla^\psi)^2\mathrm{d}\psi(\operatorname{grad}_N \ln f)\Big)\\ &\qquad + \frac{(n-2)(n-6)}{f^4}\Big(0, \nabla^\psi_{\operatorname{grad}_N \ln f}\mathrm{d}\psi(\operatorname{grad}_N \ln f)\Big)\\ &\qquad - \frac{2(n-2)}{f^4}\Big[\Delta_N(\ln f) + (n-4)\mid \operatorname{grad}_N \ln f \mid^2\\ &\qquad + f^2\Delta_M(\ln f) + (n-2)\mid \operatorname{grad}_M \ln f \mid^2\Big](0, \mathrm{d}\psi(\operatorname{grad}_N \ln f))\end{aligned}$$

and

(40)

$$\begin{aligned}trG_f\widetilde{R}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi = & \; n(\operatorname{Ricci}^M(\operatorname{grad}_M \ln f), 0)\\ & + \frac{n-2}{f^4}(0, tr_h R^N(\mathrm{d}\psi(\operatorname{grad}_N \ln f))).\end{aligned}$$

Substituting (39) and (40) in Jacobi formula

$$J_\phi(\tau(\phi)) = -\operatorname{trace}_{G_f}(\nabla^\phi)^2(\tau(\phi)) - \operatorname{trace} G_f\widetilde{R}(\tau(\phi), \mathrm{d}\phi)\mathrm{d}\phi,$$

we deduce formula (5). □

**Corollary 5.** *If $N$ is a surface of dimension 2 ($\dim N = 2$ ), then the bitension field of $\phi$ is given by*

$$\tau_2(\phi) = -2\Big(\operatorname{grad}_M(\Delta_M(\ln f)) + 2\operatorname{Ricci}^M(\operatorname{grad}_M \ln f) + \operatorname{grad}_M(\mid \operatorname{grad}_M \ln f \mid^2), 0\Big).$$

**Example 2.** Let $M=\mathbb{R}^n-\{0\}$, $\dim N=2$ and $\psi\colon N\to N$ be a harmonic map. Then the tension and the bitension fields of

$$\begin{aligned}\phi\colon \mathbb{R}^n-\{0\}\times_f N &\longrightarrow \mathbb{R}^n-\{0\}\times N\\ (x,y)&\longmapsto (x,\psi(y))\end{aligned}$$

are given by the following equations:

$$\begin{aligned}\tau(\phi)&=2(\operatorname{grad}_M \ln f,0)\\ \tau_2(\phi)&=2(\operatorname{grad}_M(\Delta_M(\ln f))+\operatorname{grad}_M(|\operatorname{grad}_M \ln f\,|^2)),\end{aligned}$$

hence $\phi$ is biharmonic not harmonic if and only if

$$\begin{cases}\Delta_M(\ln f)+|\operatorname{grad}_M \ln f\,|^2=\beta(y) & \text{(independent of } x),\\ \operatorname{grad}_M \ln f\neq 0\end{cases}.$$

If $f\in C^\infty(\mathbb{R}^n-\{0\})$, thus the domain of $\phi$ is a warped product, such as $\ln f$ is a radial function ($\ln f=\alpha(|x|)$), then $\phi$ is biharmonic not harmonic if and only if

$$f(x)=k|x|^{(2-n)},\quad (k\neq 0)$$

where $|x|=\sqrt{x_1^2+\ldots+x_n^2}$

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