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COMPOSITION OPERATOR ON THE SPACE OF FUNCTIONS TRIEBEL-LIZORKIN AND BOUNDED VARIATION TYPE

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Abstract. For a Borel-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ and

$$\sup_{t>0} t^{-1} \int_{\mathbb{R}} \sup_{|h|\le t} |f'(x+h) - f'(x)|^p \,\mathrm{d}x < +\infty, \qquad (0 < p < +\infty),$$

we study the composition operator $T_f(g) := f \circ g$ on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ in the case $0 < s < 1 + (1/p)$.

1. Introduction and the main result

The study of the composition operator $T_f\colon g \to f \circ g$ associated to a Borel-measurable function $f\colon \mathbb{R} \to \mathbb{R}$ on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$, consists in finding a characterization of the functions f such that

$$T_f(F^s_{p,q}(\mathbb{R}^n)) \subseteq F^s_{p,q}(\mathbb{R}^n). \tag{1.1}$$

The investigation to establish (1.1) was improved by several works, for example the papers of Adams and Frazier [**1, 2**], Brezis and Mironescu [**6**], Maz'ya and Shaposnikova [**9**], Runst and Sickel [**12**] and [**10**]. There were obtained some necessary conditions on f; from which we recall the following results. *For $s > 0$, $1 < p < +\infty$ and $1 \le q \le +\infty$*

- *if T_f takes $L_\infty(\mathbb{R}^n) \cap F^s_{p,q}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$, then f is locally Lipschitz continuous.*
- *if T_f takes the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$, then f belongs locally to $F^s_{p,q}(\mathbb{R})$.*

The first assertion is proved in [**3**, Theorem 3.1]. The proof of the second assertion can be found in [**12**, Theorem 2, 5.3.1].

Bourdaud and Kateb [**4**] introduced the functions class $U^1_p(\mathbb{R})$, the set of Lipschitz continuous functions f such that their derivatives, in the sense of distributions, satisfy

$$A_p(f') := \Big(\sup_{t>0} t^{-1} \int_{\mathbb{R}} \sup_{|h|\le t} |f'(x+h) - f'(x)|^p \,\mathrm{d}x\Big)^{1/p} < +\infty, \tag{1.2}$$

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and are endowed with the seminorm

$$\|f\|_{U_p^1(\mathbb{R})} := \inf(\|g\|_\infty + A_p(g)),$$

where the infimum is taken over all functions g such that f is a primitive of g. In [**4**] the authors, proved the acting of the operator T_f on Besov space $B^s_{p,q}(\mathbb{R}^n)$ for $1 \le p < +\infty$, $1 < s < 1 + (1/p)$ and $f \in U_p^1(\mathbb{R})$ with $f(0) = 0$. In [**5**] the same result holds for $0 < s < 1 + (1/p)$.

In this work we will study the composition operator T_f on $F^s_{p,q}(\mathbb{R}^n)$ for a function f which belongs to $U_p^1(\mathbb{R})$, then we will obtain a result of type (1.1). To do this, we introduce the set $\mathcal{V}_p(\mathbb{R}^n)$ of the functions $g\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\|g\|_{\mathcal{V}_p(\mathbb{R}^n)} := \sum_{j=1}^n \Big(\int_{\mathbb{R}^{n-1}} \|g_{x'_j}\|^p_{BV_p^1(\mathbb{R})} \mathrm{d}x'_j \Big)^{1/p} < +\infty$$

where $BV_p^1(\mathbb{R})$ is the Wiener space of the primitives of functions of bounded p-variation (see Subsection 2.2 below for the definition) and

$$g_{x'_j}(y) := g(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n), \qquad y \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{1.3}$$

We will prove the following statement.

Theorem 1.1. *Let $0 < p, q < +\infty$ and $0 < s < 1 + (1/p)$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \le c\,\|f\|_{U_p^1(\mathbb{R})} \Big(\|g\|_p + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)} \Big) \tag{1.4}$$

holds for all functions $g \in L_p(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ and all $f \in U_p^1(\mathbb{R})$ satisfying $f(0) = 0$. Moreover, for all such f, the operator T_f takes $L_p(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$.

Remark. (i) Since $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$, then T_f maps from $F^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$ under the assumptions of Theorem 1.1.
(ii) Since the Bessel potential spaces $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$, $1 < p < \infty$, Theorem 1.1 covers the results of composition operators in case $H^s_p(\mathbb{R}^n)$ instead of $F^s_{p,q}(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2 we collect some properties of the needed function spaces $F^s_{p,q}(\mathbb{R}^n)$ and $BV_p^1(\mathbb{R})$. Section 3 is devoted to the proof of the main result where in a first step we study the case of 1-dimensional which is the main tool when we prove Theorem 1.1. Also, our proof uses various Sobolev and Peetre embeddings, Fubini and Fatou properties, etc. In Section 4 we give some corollaries and prove the sharp estimate of (1.4).

Notation. We work with functions defined on the Euclidean space $\mathbb{R}^n$. All spaces and functions are assumed to be real-valued. We denote by $C_b(\mathbb{R}^n)$ the Banach space of bounded continuous functions on $\mathbb{R}^n$ endowed with the supremum. Let $\mathcal{D}(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$) denotes the C^∞-functions with compact support (resp. the Schwartz space of all C^∞ rapidly decreasing functions and its topological dual). With $\|\cdot\|_p$ we denote the L_p-norm. We define the differences by $\Delta_h f := f(\cdot + h) - f$ for all $h \in \mathbb{R}^n$. If E is a Banach function space on $\mathbb{R}^n$, we denote by $E^{\ell oc}$ the collection of all functions f such that $\varphi f \in E$ for all

$\varphi \in \mathcal{D}(\mathbb{R}^n)$. As usual, constants $c, c_1, \dots$ are strictly positive and depend only on the fixed parameters n, s, p, q; their values may vary from line to line.

2. Function spaces

2.1. Triebel-Lizorkin spaces

Let $0 < a \le \infty$. For all measurable functions f on $\mathbb{R}^n$, we set

$$M^{s,u,a}_{p,q}(f) := \Big(\int_{\mathbb{R}^n} \Big(\int_0^a t^{-sq} \Big(\frac{1}{t^n} \int_{|h| \le t} |\Delta_h f(x)|^u \, \mathrm{d}h \Big)^{q/u} \frac{\mathrm{d}t}{t} \Big)^{p/q} \mathrm{d}x \Big)^{1/p}.$$

Definition 2.1. Let $0 < p < +\infty$ and $0 < q \le +\infty$. Let s be a real satisfying

$$1 < s < 2 \qquad \text{and} \qquad s > n \max\Big(\frac{1}{p} - 1, \frac{1}{q} - 1 \Big).$$

Then, a function $f \in L_p(\mathbb{R}^n)$ belongs to $F^s_{p,q}(\mathbb{R}^n)$ if

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} := \|f\|_p + \sum_{j=1}^n M^{s-1,1,\infty}_{p,q}(\partial_j f) < +\infty.$$

The set $F^s_{p,q}(\mathbb{R}^n)$ is a quasi Banach space for the quasi-norm defined above. For the equivalence of the above definition with other characterizations we refer to [**15**, Theorem 3.5.3] from which we recall the following statement.

Proposition 2.2. *Let $0 < p < +\infty$ and $0 < q, u \le +\infty$. Let s be a real satisfying*

$$1 < s < 2 \qquad \text{and} \qquad s > n \max\Big(\frac{1}{p} - \frac{1}{u}, \frac{1}{q} - \frac{1}{u} \Big).$$

Then, a function $f \in L_p(\mathbb{R}^n)$ belongs to $F^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\|f\|_p + M^{s,u,\infty}_{p,q}(f) < +\infty \tag{2.1}$$

and the expression (2.1) *is an equivalent quasi-norm in $F^s_{p,q}(\mathbb{R}^n)$. Moreover, this assertion remains true if one replaces $M^{s,u,\infty}_{p,q}$ by $M^{s,u,a}_{p,q}$ for any fixed $a > 0$.*

The argument of the equivalence of above quasi-norms that we can replace the integration for $t \in]0, +\infty[$ by $t \le a$ for a fixed positive number a is the part of the integral for which $t > a$ can be easily estimated by the L_p-norm.

Embeddings. Triebel-Lizorkin spaces are spaces of equivalence classes w.r.t. almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function. Later on we need the following continuous embeddings:

(i) The spaces $F^s_{p,q}(\mathbb{R}^n)$ are monotone with respect to s and q, more exactly $F^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow F^t_{p,q}(\mathbb{R}^n) \hookrightarrow F^t_{p,\infty}(\mathbb{R}^n)$ if $t < s$ and $0 < q \le \infty$.
(ii) With Besov spaces, we have $B^s_{p,1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n)$.
(iii) If either $s > n/p$ or $s = n/p$ and $0 < p \le 1$, then $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$.

For various further embeddings we refer to [**14**, 2.3.2, 2.7.1] or [**12**, 2.2.2, 2.2.3].

The Fatou property. Well-known the Triebel-Lizorkin space has the Fatou property, cf. [**8**]. We will briefly recall it. Any $f \in F^s_{p,q}(\mathbb{R}^n)$ can be approximated (in the weak sense in $\mathcal{S}'(\mathbb{R}^n)$) by a sequence $(f_j)_{j\geq 0}$ such that any f_j is an entire function of exponential type

$$f_j \in F^s_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad \limsup_{j\to+\infty} \| f_j\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\, \| f\|_{F^s_{p,q}(\mathbb{R}^n)}$$

with a positive constant c independent of f. Vice versa, if for a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, there exists a sequence $(f_j)_{j\geq 0}$ such that

$$f_j \in F^s_{p,q}(\mathbb{R}^n) \qquad \text{and} \qquad A := \limsup_{j\to+\infty} \| f_j\|_{F^s_{p,q}(\mathbb{R}^n)} < +\infty\,,$$

and $\lim_{j\to+\infty} f_j = f$ in the sense of distributions, then f belongs to $F^s_{p,q}(\mathbb{R}^n)$ and there exists a constant $c > 0$ independent of f such that $\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq cA$.

2.2. Functions of bounded variation

For a function $g\colon \mathbb{R} \to \mathbb{R}$, we set

$$\nu_p(g) := \sup \Big(\sum_{k=1}^{N} |g(b_k) - g(a_k)|^p \Big)^{1/p}, \tag{2.2}$$

taken over all finite sets $\{]a_k, b_k[\,;\, k = 1, \ldots, N\}$ of pairwise disjoint open intervals. A function g is said to be of *bounded p-variation* if $\nu_p(g) < +\infty$. Clearly, by considering a finite sequence with only two terms, we obtain $|g(x) - g(y)| \leq \nu_p(g)$, for all $x, y \in \mathbb{R}$, hence g is a bounded function. The set of (generalized) primitives of functions of bounded p-variation is denoted by $BV^1_p(\mathbb{R})$ and endowed with the seminorm

$$\|f\|_{BV^1_p(\mathbb{R})} := \inf\, \nu_p(g),$$

where the infimum is taken over all functions g whose f is the primitive. For more details about this space we refer to [**11**] or [**5**]. However, we need to recall some embeddings

$$BV^1_p(\mathbb{R}) \hookrightarrow U^1_p(\mathbb{R}) \tag{2.3}$$

(equality in case $p = 1$), see [**5**, Theorem 5] for the proof which is given for $1 < p < +\infty$ and can be easily extended to $0 < p \leq 1$, see also [**7**, Theorem 9.3]. The Peetre embedding theorem

$$\dot{B}^{1+(1/p)}_{p,1}(\mathbb{R}) \hookrightarrow BV^1_p(\mathbb{R}) \hookrightarrow \dot{B}^{1+(1/p)}_{p,\infty}(\mathbb{R})\,, \quad (1 \leq p < +\infty), \tag{2.4}$$

where the dotted space is the *homogeneous* Besov space.

Example. Let $\alpha \in \mathbb{R}$. We put $u_\alpha(x) := |x + \alpha| - |\alpha|$ for all $x \in \mathbb{R}$, and

$$f_\alpha(x, y) := u_\alpha(x)\chi_{[0,1]}(y) + u_\alpha(y)\chi_{[0,1]}(x), \quad \forall x, y \in \mathbb{R},$$

where $\chi_{[0,1]}$ denotes the indicatrix function of $[0, 1]$. Clearly that $\nu_p(u'_\alpha) = 2$ and $\|\chi_{[0,1]}\|_{BV^1_p(\mathbb{R})} = 0$. Then it holds $f_\alpha \in \mathcal{V}_p(\mathbb{R}^2)$ with $\|f_\alpha\|_{\mathcal{V}_p(\mathbb{R}^2)} = 4$. The $\mathcal{V}_p(\mathbb{R}^n)$ space is defined in Section 1.

3. Proof of the result

Theorem 1.1 can be obtained from the following statement.

Proposition 3.1. *Let $0 < p, q < +\infty$, $0 < u < \min(p, q)$ and $0 < s < 1/p$. Then there exists a constant $c > 0$ such that the inequality*

$$M^{s,u,\infty}_{p,q}((f \circ g)') \leq c \|f\|_{U^1_p(\mathbb{R})} \|g\|_{BV^1_p(\mathbb{R})} \tag{3.1}$$

holds for all $f \in U^1_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and all real analytic functions g in $BV^1_p(\mathbb{R})$.

Proof. For a better readability we split our proof in two steps.
Step 1. Let us prove

$$M^{s,u,a}_{p,q}((f \circ g)') \leq c\, a^{(1/p)-s} \|f\|_{U^1_p(\mathbb{R})} \|g\|_{BV^1_p(\mathbb{R})} \tag{3.2}$$

for all $a > 0$ and all $f \in U^1_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and all real analytic functions g in $BV^1_p(\mathbb{R})$.

Assume first $a = 1$. By the assumptions on f and g it holds $(f \circ g)' = (f' \circ g)\, g'$. We have $\|(f \circ g)'\|_\infty \leq \|f'\|_\infty \|g'\|_\infty$ and

$$|\Delta_h((f' \circ g)g')(x)| \leq \|f'\|_\infty |\Delta_h g'(x)| + |g'(x)|\, |\Delta_h(f' \circ g)(x)|.$$

Hence

$$M^{s,u,1}_{p,q}((f \circ g)') \leq \|f'\|_\infty M^{s,u,1}_{p,q}(g') + V(f; g),$$

where

$$V(f;g) := \Big(\int_{\mathbb{R}} \Big(\int_0^1 t^{-sq} \Big(\frac{1}{t} \int_{-t}^{t} |\Delta_h(f' \circ g)(x)|^u |g'(x)|^u \mathrm{d}h \Big)^{q/u} \frac{\mathrm{d}t}{t} \Big)^{p/q} \mathrm{d}x \Big)^{1/p}. \tag{3.3}$$

Estimate of $M^{s,u,1}_{p,q}(g')$. By writing $\int_0^1 \cdots = \sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} \cdots$ and by an elementary computation, we have

$$\begin{aligned} \int_0^1 t^{-sq} \Big(\frac{1}{t} \int_{-t}^{t} |\Delta_h g'(x)|^u \,\mathrm{d}h \Big)^{q/u} \frac{\mathrm{d}t}{t} &\leq c_1 \sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} t^{-sq} \sup_{|h| \leq t} |\Delta_h g'(x)|^q \frac{\mathrm{d}t}{t} \\ &\leq c_2 \sum_{j=0}^\infty 2^{jsq} \sup_{|h| \leq 2^{-j}} |\Delta_h g'(x)|^q. \end{aligned}$$

Let $\alpha := \min(1, p/q)$. By using the monotonicity of the ℓ_r-norms (i.e. $\ell_1 \hookrightarrow \ell_{1/\alpha}$) and by the Minkowski inequality w.r.t $L_{p/(\alpha q)}$, since $q < +\infty$, we obtain

$$\begin{aligned} M^{s,u,1}_{p,q}(g') &\leq c_1 \Big(\int_{\mathbb{R}} \Big(\sum_{j=0}^\infty 2^{js\alpha q} \sup_{|h| \leq 2^{-j}} |\Delta_h g'(x)|^{\alpha q} \Big)^{p/(\alpha q)} \mathrm{d}x \Big)^{1/p} \\ &\leq c_2 \Big(\sum_{j=0}^\infty 2^{js\alpha q} \Big(\int_{\mathbb{R}} \sup_{|h| \leq 2^{-j}} |\Delta_h g'(x)|^p \,\mathrm{d}x \Big)^{(\alpha q)/p} \Big)^{1/(\alpha q)} \\ &\leq c_3 \Big(\sum_{j=0}^\infty 2^{j(s-(1/p))\alpha q} \Big)^{1/(\alpha q)} \|g\|_{U^1_p(\mathbb{R})}. \end{aligned}$$

From the embedding (2.3) and the assumption on s, the desired estimate holds.

Estimate of $V(f;g)$. In (3.3) the integral with respect to h can be limited to the interval $[0,t]$ denoting the corresponding expression by $V_+(f;g)$. Let us notice that the estimate with respect to $[-t,0]$ will be completely similar.

Again, by applying the Minkowski inequality twice, it holds

$$\begin{aligned}
&V_+(f;g)\\
&\le \Big(\int_{\mathbb{R}}\Big(\int_0^1\Big(\int_h^1 t^{-(s+(1/u))q}\,|\Delta_h(f'\circ g)(x)|^q\,|g'(x)|^q\,\frac{\mathrm{d}t}{t}\Big)^{u/q}\mathrm{d}h\Big)^{p/u}\mathrm{d}x\Big)^{1/p}\\
&\le \Big(\int_0^1\Big(\int_{\mathbb{R}}|\Delta_h(f'\circ g)(x)|^p\,|g'(x)|^p\,\mathrm{d}x\Big)^{u/p}\Big(\int_h^\infty t^{-(s+(1/u))q}\,\frac{\mathrm{d}t}{t}\Big)^{u/q}\mathrm{d}h\Big)^{1/u}\\
&\le c\Big(\int_0^1 h^{-(su+1)}\Big(\int_{\mathbb{R}}|\Delta_h(f'\circ g)(x)|^p\,|g'(x)|^p\,\mathrm{d}x\Big)^{u/p}\mathrm{d}h\Big)^{1/u}.
\end{aligned}$$

Case 1: Assume that g' *does not vanish on* $\mathbb{R}$. By the Mean Value Theorem and by the change of variable $y=g(x)$, we find

$$\begin{aligned}
&V_+(f;g)\\
&\le c_1\|g'\|_\infty^{1-(1/p)}\Big(\int_0^1 h^{-(su+1)}\Big(\int_{\mathbb{R}}\sup_{|v|\le h\|g'\|_\infty}|f'(v+y)-f'(y)|^p\mathrm{d}y\Big)^{u/p}\mathrm{d}h\Big)^{1/u}\\
&\le c_2\|f\|_{U_p^1(\mathbb{R})}\,\|g'\|_\infty\Big(\int_0^1 h^{u((1/p)-s)-1}\mathrm{d}h\Big)^{1/u}\\
&\le c_3\,\|f\|_{U_p^1(\mathbb{R})}\,\|g\|_{BV_p^1(\mathbb{R})}\,.
\end{aligned}$$

Case 2: Assume that the set of zeros of g' *is nonempty.* Then it is a discrete set whose complement in $\mathbb{R}$ is the union of a family $(I_l)_l$ of open disjoint intervals. For any $h>0$, we denote by $I'_{l,h}$ the set of $x\in I_l$ whose distance to the boundary of I_l is greater than h. We set

$$I''_{l,h} := I_l\setminus I'_{l,h}\quad\text{and}\quad g_l := g_{|_{I_l}}.$$

Clearly the function g_l is a diffeomorphism of I_l onto $g(I_l)$. Let us notice that $I'_{l,h}$ is an open interval, possibly empty. In case it is not empty, we have

$$|\,g(g_l^{-1}(y)+h)-y\,|\le h\sup_{I_l}|g'|,\qquad \forall y\in g_l(I'_{l,h}). \tag{3.4}$$

The set $I''_{l,h}$ is an interval of length at most $2h$ or the union of two such intervals, and g' vanishes at one of the endpoints of these or those intervals.

We write $V_+(f;g)\le V_1(f;g)+V_2(f;g)$, where

$$V_1(f;g) := \Big(\int_0^1 h^{-(su+1)}\Big(\sum_l\int_{I'_{l,h}}|\Delta_h(f'\circ g)(x)|^p\,|g'(x)|^p\,\mathrm{d}x\Big)^{u/p}\mathrm{d}h\Big)^{1/u}$$

and $V_2(f;g)$ is defined in the same way by replacing $I'_{l,h}$ by $I''_{l,h}$.

Estimate of $V_1(f;g)$. By the change of variable $y = g_l(x)$ and by (3.4), we deduce

$$\begin{aligned} V_1(f;g) &\le \Bigg(\int_0^1 h^{-(su+1)} \Bigg(\sum_l \sup_{I_l} |g'|^{p-1} \\ &\quad\times \int_{g(I'_{l,h})} \sup_{|v|\le h\sup_{I_l}|g'|} |f'(v+y)-f'(y)|^p\,\mathrm{d}y\Bigg)^{u/p} \mathrm{d}h\Bigg)^{1/u} \\ &\le c_1\,\|f\|_{U_p^1(\mathbb{R})}\Bigg(\sum_l \sup_{I_l}|g'|^p\Bigg)^{1/p}\Bigg(\int_0^1 h^{u((1/p)-s)-1}\,\mathrm{d}h\Bigg)^{1/u} \\ &\le c_2\,\|f\|_{U_p^1(\mathbb{R})}\Bigg(\sum_l \sup_{I_l}|g'|^p\Bigg)^{1/p}. \end{aligned}$$

Hence it suffices to show

$$\Big(\sum_l \sup_{t\in I_l} |g'(t)|^p\Big)^{1/p} \le c\,\|g\|_{BV_p^1}. \tag{3.5}$$

Indeed, by the assumption on g, for any I_l there exists $\xi_l \in I_l$ such that

$$|g'(\xi_l)| = \sup_{t\in I_l} |g'(t)|.$$

Furthermore, set β_l the right endpoint of I_l. The open intervals $\{\,]\xi_l,\beta_l[\,\}_l$ are pairwise disjoint. Then the assertion (3.5) follows from

$$\sum_l \sup_{t\in I_l} |g'(t)|^p = \sum_l |g'(\xi_l)-g'(\beta_l)|^p \le \nu_p(g')^p.$$

(See (2.2) for the definition of ν_p).

Estimate of $V_2(f;g)$. Using both the elementary inequality $|\Delta_h(f'\circ g)(x)| \le 2\|f'\|_\infty$ and the properties of $I''_{l,h}$, it holds

$$\begin{aligned} V_2(f;g) &\le c_1\,\|f'\|_\infty\Bigg(\sum_l \sup_{I_l}|g'|^p\Bigg)^{1/p}\Bigg(\int_0^1 h^{u((1/p)-s)-1}\,\mathrm{d}h\Bigg)^{1/u} \\ &\le c_2\,\|f\|_{U_p^1(\mathbb{R})}\|g\|_{BV_p^1(\mathbb{R})}. \end{aligned}$$

Hence we obtain (3.2) with $a = 1$. We put $g_\lambda(x) := g(\lambda x)$ for all $x \in \mathbb{R}$ and all $\lambda > 0$. Then (3.2) can be obtained for all $a > 0$ since $\|g_a\|_{BV_p^1(\mathbb{R})} = a\|g\|_{BV_p^1(\mathbb{R})}$ and

$$M_{p,q}^{s,u,a}((f\circ g)') = a^{(1/p)-s-1}\, M_{p,q}^{s,u,1}((f\circ g_a)').$$

Step 2: Proof of (3.1). Let $a > 0$. Let f and g be as in Proposition 3.1. By Proposition 2.2 it holds

$$M_{p,q}^{s,u,\infty}((f\circ g)') \le \|(f\circ g)'\|_{F^s_{p,q}(\mathbb{R})} = \|(f\circ g)'\|_p + M_{p,q}^{s,u,a}((f\circ g)').$$

Applying (3.2), we obtain

$$M_{p,q}^{s,u,\infty}((f\circ g)') \le \|f'\|_\infty\|g'\|_p + c_1\,a^{(1/p)-s}\,\|f\|_{U_p^1(\mathbb{R})}\,\|g\|_{BV_p^1(\mathbb{R})} \tag{3.6}$$

with a positive constant c_1 depending only on s, p and q (see the end of Step 1). Now, by replacing g by g_λ in (3.6), (g_λ is defined in Step 1), and by using the equality

$$M^{s,u,\infty}_{p,q}\big((f\circ g_\lambda)'\big) = \lambda^{s+1-(1/p)}\, M^{s,u,\infty}_{p,q}((f\circ g)'),$$

we deduce

$$\begin{aligned} &M^{s,u,\infty}_{p,q}((f\circ g)') \\ &\le \lambda^{-s}\|f'\|_\infty\|g'\|_p + c_1 a^{(1/p)-s}\lambda^{(1/p)-s}\|f\|_{U^1_p(\mathbb{R})}\|g\|_{BV^1_p(\mathbb{R})} \end{aligned} \tag{3.7}$$

for all $a, \lambda > 0$. Taking $a = 1/\lambda$. Now letting $\lambda \to +\infty$ in (3.7), we obtain the desired result. □

Remark. Proposition 3.1 is also valid in the n-dimensional case. The inequality (3.1) becomes

$$M^{s-1,u,\infty}_{p,q}(\partial_j(f\circ g)) \le c\|f\|_{U^1_p(\mathbb{R})}\,\|g\|_{\mathcal{V}_p(\mathbb{R}^n)}, \qquad (j = 1,\dots,n)$$

for all $f \in U^1_p(\mathbb{R}) \cap C^1(\mathbb{R})$ and all real analytic functions g in $\mathcal{V}_p(\mathbb{R}^n)$.

Proof of Theorem 1.1. *Step 1.* Observe that the conditions $f(0) = 0$ and $f' \in L_\infty(\mathbb{R})$ imply

$$\|f\circ g\|_p \le \|f'\|_\infty\, \|g\|_p$$

which is sufficient for the estimate $T_f(g)$ with respect to $L_p(\mathbb{R}^n)$-norm.

Step 2: The case $1 < s < 1 + (1/p)$ *and* $n = 1$. We first consider a function $f \in U^1_p(\mathbb{R})$, of class C^1 and a function g real analytic in $L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R})$. By Proposition 3.1, it holds

$$\|f\circ g\|_{F^s_{p,q}(\mathbb{R})} \le c\,\|f\|_{U^1_p(\mathbb{R})}\Big(\|g\|_p + \|g\|_{BV^1_p(\mathbb{R})}\Big). \tag{3.8}$$

Now we prove (3.8) in the general case. Let $g \in L_p(\mathbb{R}) \cap BV^1_p(\mathbb{R})$ and $f \in U^1_p(\mathbb{R})$. We introduce a function $\rho \in \mathcal{D}(\mathbb{R})$ satisfying $\rho(0) = 1$, and we set $\varphi_j(x) := 2^{jn}\mathcal{F}^{-1}\rho(2^j x)$ for all $x \in \mathbb{R}$ and all $j \in \mathbb{N}$; here $\mathcal{F}^{-1}\rho$ denotes the inverse Fourier transform of ρ. We set also

$$f_j := \varphi_j * f - \varphi_j * f(0) \quad \text{and} \quad g_j := \varphi_j * g.$$

Then the function g_j is real analytic and $g_j \to g$ in $L_p(\mathbb{R})$. We have also

$$\|g_j\|_{BV^1_p(\mathbb{R})} \le c\,\|g\|_{BV^1_p(\mathbb{R})}\ , \qquad \forall j \in \mathbb{N}. \tag{3.9}$$

To prove (3.9), let $\{]a_k, b_k[,\ k = 1,\dots,N\}$ be a set of pairwise disjoint intervals. By the Minkowski inequality, it holds

$$\begin{aligned} &\Big(\sum_{k=1}^N\Big|\int_{\mathbb{R}}\varphi_j(y)\,\Big(g'(b_k-y) - g'(a_k-y)\Big)\,\mathrm{d}y\Big|^p\Big)^{1/p} \\ &\qquad \le \int_{\mathbb{R}}|\varphi_j(y)|\Big(\sum_{k=1}^N\Big|g'(b_k-y) - g'(a_k-y)\Big|^p\Big)^{1/p}\,\mathrm{d}y. \end{aligned}$$

Now, for all $y \in \mathbb{R}$, the intervals $]a_k - y, b_k - y[$ $(k = 1, \ldots, N)$ are pairwise disjoint. Then

$$\Big(\sum_{k=1}^{N} |g_j'(b_k) - g_j'(a_k)|^p\Big)^{1/p} \leq \|\mathcal{F}^{-1}\rho\|_1 \nu_p(g'), \qquad \forall j \in \mathbb{N}.$$

Hence we obtain (3.9).

The functions f_j are C^∞ such that $f_j(0) = 0$ and satisfy

$$\|f_j\|_{U_p^1(\mathbb{R})} \leq c\,\|f\|_{U_p^1(\mathbb{R})}, \qquad \forall j \in \mathbb{N}. \tag{3.10}$$

To prove (3.10), for all $t > 0$ and all $h \in [-t, t]$ we trivially have

$$|\varphi_j * f'(x+h) - \varphi_j * f'(x)| \leq \int_{\mathbb{R}} |\varphi_j(y)| \sup_{|z|\leq t} |f'(x-y+z) - f'(x-y)| \mathrm{d}y.$$

By the Minkowski inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \sup_{|h|\leq t} |\varphi_j * f'(x+h) - \varphi_j * f'(x)|^p \mathrm{d}x \\
&\qquad \leq \Big(\int_{\mathbb{R}} |\varphi_j(y)| \Big(\int_{\mathbb{R}} \sup_{|z|\leq t} |f'(x-y+z) - f'(x-y)|^p \mathrm{d}x\Big)^{1/p} \mathrm{d}y\Big)^p \\
&\qquad \leq t\|\mathcal{F}^{-1}\rho\|_1^p\, A_p(f')^p, \qquad \text{(see (1.2) for the definition of } A_p).
\end{aligned}$$

Consequently,

$$A_p(f_j') + \|f_j'\|_\infty \leq \|\mathcal{F}^{-1}\rho\|_1 (A_p(f') + \|f'\|_\infty)$$

and we obtain the desired result.

On the other hand, we have

$$\lim_{j\to+\infty} \|f_j - f\|_\infty = 0. \tag{3.11}$$

To prove (3.11), since $\lim_{j\to+\infty} \varphi_j * f(0) = f(0) = 0$, the Lipschitz continuous of f yields

$$\begin{aligned}
|f_j(x) - f(x)| &\leq \|f'\|_\infty \int_{\mathbb{R}} |x-y||\varphi_j(x-y)| \mathrm{d}y \,+\, |\varphi_j * f(0)| \\
&\leq c\,2^{-j}\|f'\|_\infty \,+\, |\varphi_j * f(0)|.
\end{aligned}$$

Then the desired result holds. By the same argument, we obtain

$$\|g_j - g\|_\infty \leq c\,2^{-j}\|g'\|_\infty. \tag{3.12}$$

Now we apply (3.8) to f_j and g_j. Then by (3.9) and (3.10), we obtain

$$\|f_j \circ g_j\|_{F^s_{p,q}(\mathbb{R})} \leq c\,\|f\|_{U_p^1(\mathbb{R})}\Big(\|g\|_p + \|g\|_{BV_p^1(\mathbb{R})}\Big). \tag{3.13}$$

The elementary inequality

$$\|f \circ g - f_j \circ g_j\|_\infty \leq \|f'\|_\infty \|g - g_j\|_\infty + \|f - f_j\|_\infty$$

complemented by (3.11)–(3.12) yields the convergence of the sequence $\{f_j \circ g_j\}_{j\in\mathbb{N}}$ to $f \circ g$ in $L_\infty(\mathbb{R})$. Since

$$|\langle f_j \circ g_j - f \circ g, \psi\rangle| \leq \|f_j \circ g_j - f \circ g\|_\infty \|\psi\|_1, \qquad \forall \psi \in \mathcal{D}(\mathbb{R}),$$

thus we conclude that $\lim_{j\to+\infty} f_j\circ g_j = f\circ g$ in the sense of distributions. Hence, by the Fatou property of $F^s_{p,q}(\mathbb{R})$, see Subsection 2.1, we deduce (3.8).

Step 3: The case $1 < s < 1 + (1/p)$ *and* $n \geq 2$. We use the notation (1.3). Since Triebel-Lizorkin space has the Fubini property (see [**12**, p. 70]), by (3.1) it holds

$$\begin{aligned}\|f\circ g\|_{F^s_{p,q}(\mathbb{R}^n)} &\leq c_1 \sum_{j=1}^n \Big(\int_{\mathbb{R}^{n-1}} \|f\circ g_{x'_j}\|^p_{F^s_{p,q}(\mathbb{R})}\mathrm{d}x'_j\Big)^{1/p}\\ &\leq c_2\,\|f\|_{U^1_p(\mathbb{R})}\sum_{j=1}^n\Big(\int_{\mathbb{R}^{n-1}}\Big(\|g_{x'_j}\|^p_p + \|g_{x'_j}\|^p_{BV^1_p(\mathbb{R})}\Big)\mathrm{d}x'_j\Big)^{1/p}\\ &\leq c_3\,\|f\|_{U^1_p(\mathbb{R})}\Big(\|g\|_p + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)}\Big).\end{aligned}$$

Step 4: The case $0 < s \leq 1$. Due to the monotonicity of the Triebel-Lizorkin scale with respect to the smoothness parameter s, the result holds. Indeed, let $1 < t < 1 + (1/p)$. From Step 3, we have (1.4) with $\|f\circ g\|_{F^t_{p,q}(\mathbb{R}^n)}$ instead of $\|f\circ g\|_{F^s_{p,q}(\mathbb{R}^n)}$. Now we apply the continuous embedding $F^t_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$. This completes the proof. ☐

Remark. In case $n = 1$ and $1 \leq p, q < +\infty$ the inequality (1.4) becomes

$$\|f\circ g\|_{F^s_{p,q}(\mathbb{R})} \leq c\|f\|_{U^1_p(\mathbb{R})}\Big(\|g\|_{F^s_{p,q}(\mathbb{R})} + \|g\|_{BV^1_p(\mathbb{R})}\Big)$$

for all $g \in L_p(\mathbb{R})\cap BV^1_p(\mathbb{R})$, since $F^s_{p,q}(\mathbb{R})\cap BV^1_p(\mathbb{R}) = L_p(\mathbb{R})\cap BV^1_p(\mathbb{R})$ if $s < 1 + (1/p)$. To prove this equality, we have $\dot{B}^{1+(1/p)}_{p,\infty}(\mathbb{R})\cap L_p(\mathbb{R}) = B^{1+(1/p)}_{p,\infty}(\mathbb{R})$ (see [**12**, 2.6.2, p. 95]). Then by (2.4) and by both $B^{1+(1/p)}_{p,\infty}(\mathbb{R}) \hookrightarrow B^s_{p,1}(\mathbb{R})$ and $B^s_{p,1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n)$, it holds $L_p(\mathbb{R})\cap BV^1_p(\mathbb{R}) \hookrightarrow F^s_{p,q}(\mathbb{R})$.

4. Concluding remarks

4.1. Some corollaries

In this section we fix a smooth cut-off function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi(x) = 1$ for $|x| \leq 1$. We put $\varphi_t(x) := \varphi(t^{-1}x)$, $\forall x \in \mathbb{R}$ and for all $t > 0$. Also for brevity we introduce the space $\mathcal{F}^s_{p,q}(\mathbb{R}^n) := F^s_{p,q}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$ endowed with the quasi-norm

$$\|f\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} := \|f\|_{F^s_{p,q}(\mathbb{R}^n)} + \|f\|_\infty.$$

Theorem 1.1 has a consequence for the case of functions f which are only locally in $U^1_p(\mathbb{R})$.

Corollary 4.1. *Let* s, p, q *be real numbers as in Theorem* 1.1. *Then there exists a constant* $c > 0$ *such that the inequality*

$$\|f\circ g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} \leq c\|f\varphi_{\|g\|_\infty}\|_{U^1_p(\mathbb{R})}\Big(\|g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)}\Big) \tag{4.1}$$

holds for all functions $g \in \mathcal{F}^s_{p,q}(\mathbb{R}^n)\cap\mathcal{V}_p(\mathbb{R}^n)$ *and all* $f \in U^{1,\ell oc}_p(\mathbb{R})$ *satisfying* $f(0) = 0$. *Moreover, for all such functions* f, *the composition operator* T_f *takes* $\mathcal{F}^s_{p,q}(\mathbb{R}^n)\cap\mathcal{V}_p(\mathbb{R}^n)$ *to* $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$.

Proof. Since $f \circ g = (f\varphi_{\|g\|_\infty}) \circ g$ and $(f\varphi_t)(0) = 0$, the result follows from Theorem 1.1. $\square$

There is consequence of Theorem 1.1 that we can obtain the equivalence of acting condition and boundedness.

Corollary 4.2. *Let s, p, q be real numbers as in Theorem 1.1. Let f be a function in $U_p^{1,\ell oc}(\mathbb{R})$ satisfying $f(0) = 0$. Then the following assertions are equivalent:*
(i) *T_f satisfies the acting condition $T_f(\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)) \subseteq \mathcal{F}^s_{p,q}(\mathbb{R}^n)$.*
(ii) *T_f maps bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ into bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$.*

Proof. Let $t > 0$. By (4.1), it holds

$$\|f \circ g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} \le c\, t \|f\varphi_t\|_{U_p^1(\mathbb{R})} \tag{4.2}$$

for all $g \in \mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ such that $\|g\|_{\mathcal{F}^s_{p,q}(\mathbb{R}^n)} + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)} \le t$. Now, from (4.2), we conclude that T_f maps bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n) \cap \mathcal{V}_p(\mathbb{R}^n)$ into bounded sets in $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$. $\square$

Remark. If $n/p < s < 1 + (1/p)$, then we can replace $\mathcal{F}^s_{p,q}(\mathbb{R}^n)$ by $F^s_{p,q}(\mathbb{R}^n)$ in Corollaries 4.1–4.2, since $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n)$.

We show that Theorem 1.1 can be extended to the case of the boundedness between Besov spaces and Triebel-Lizorkin spaces.

Corollary 4.3. *Let $1 \le p, q < +\infty$ and $0 < s < 1 + (1/p)$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \le c\, \|f\|_{U_p^1(\mathbb{R})}\, \|g\|_{B^{1+(1/p)}_{p,1}(\mathbb{R}^n)}$$

holds for all functions $g \in B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$ and all $f \in U^1_p(\mathbb{R})$ satisfying $f(0) = 0$. Moreover, for all such functions f, the operator T_f takes $B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$ to $F^s_{p,q}(\mathbb{R}^n)$.

Proof. This is an easy consequence of Theorem 1.1 and the following continuous embedding

$$B^{1+(1/p)}_{p,1}(\mathbb{R}^n) \hookrightarrow \mathcal{V}_p(\mathbb{R}^n). \tag{4.3}$$

To prove (4.3), we use the notation (1.3) and the equivalent norm in Besov space given by

$$\|f\|_p + \sum_{j=1}^n \left(\int_0^1 t^{-sq}\, \|\Delta^2_{te_j} f\|_p^q \,\frac{\mathrm{d}t}{t}\right)^{1/q}, \qquad (0 < s < 2),$$

where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, see [**15**, p. 96].

Let $f \in B^{1+(1/p)}_{p,1}(\mathbb{R}^n)$. Since $\dot{B}^{1+(1/p)}_{p,1}(\mathbb{R}) \cap L_p(\mathbb{R}) = B^{1+(1/p)}_{p,1}(\mathbb{R})$ (in the sense of equivalent norms, see, e.g. [**15**]), then by (2.4), we get

$$\|f\|_{\mathcal{V}_p(\mathbb{R}^n)} \le c \sum_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} \|f_{x'_j}\|^p_{B^{1+(1/p)}_{p,1}(\mathbb{R})} \mathrm{d}x'_j \right)^{1/p}.$$

Using the Minkowski inequality with respect to $L_p(\mathbb{R}^{n-1})$, it follows

$$\int_{\mathbb{R}^{n-1}}\left(\int_0^1 t^{-(1+(1/p))}\,\|\Delta^2_{te_k} f_{x'_j}\|_p\,\frac{\mathrm{d}t}{t}\right)^p \mathrm{d}x'_j \le \left(\int_0^1 t^{-(1+(1/p))}\,\|\Delta^2_{te_k} f\|_p\,\frac{\mathrm{d}t}{t}\right)^p$$

for $j,k\in\{1,\dots,n\}$. Then we obtain the desired result. □

Remark. As in Corollary 4.1 we can see the case when the function f associated to the composition operator T_f belongs locally to $U^1_p(\mathbb{R})$. Indeed, if $1\le p,q<+\infty$ and $0<s<1+(1/p)$, it holds that

$$\|f\circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \le c\,\|f\varphi_{\|g\|_\infty}\|_{U^1_p(\mathbb{R})}\,\|g\|_{B^{1+(1/p)}_{p,1}(\mathbb{R}^n)}$$

for all $f\in U^{1,\ell oc}_p(\mathbb{R})$ such that $f(0)=0$ and all $g\in B^{1+(1/p)}_{p,1}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$.

4.2. Sharpness of estimate

For simplicity we define

$$\|g\| := \|g\|_{F^s_{p,q}(\mathbb{R}^n)} + \|g\|_{\mathcal{V}_p(\mathbb{R}^n)}.$$

According to Corollary 4.1, there is a substantial class of *nonlinear* functions f for which there exist constants $c_f=c(f)>0$ such that

$$\|\,f\circ g\,\|_{F^s_{p,q}(\mathbb{R}^n)} \le c_f\,\|g\|, \qquad \forall g\in F^s_{p,q}(\mathbb{R}^n)\cap\mathcal{V}_p(\mathbb{R}^n).$$

In this form the inequality is *optimal* if we avoid *linear* functions in the following sense.

Proposition 4.4. *Let $\Omega\colon[0,+\infty)\to[0,+\infty)$ be a continuous function satisfying*

$$\lim_{t\to+\infty} t^{1/p}\,\Omega(t)=0. \tag{4.4}$$

If f is a function such that the inequality

$$\|f\circ g\|_{F^s_{p,q}(\mathbb{R}^n)} \le \Omega(\|g\|) \tag{4.5}$$

holds for all $g\in F^s_{p,q}(\mathbb{R}^n)\cap\mathcal{V}_p(\mathbb{R}^n)$, then f is an affine function (linear, if we assume that $f(0)=0$).

Proof. Let us define a smooth function $\varphi\in\mathcal{D}(\mathbb{R}^n)$ such that $\varphi(x)=1$ on the cube $Q:=[-1,1]^n$ and $\varphi(x)=0$ if $x\notin 2Q$. We put $\Delta^2_h:=\Delta_h\circ\Delta_h$ and

$$g_a(x) := ax_1\varphi(x)\,, \qquad (x=(x_1,x')\in\mathbb{R}\times\mathbb{R}^{n-1},\ a>0).$$

We have $\|g_a\|\sim a$ and

$$\Delta^2_h(f\circ g_a)(x)=\Delta^2_{ah_1}f(ax_1), \qquad (\forall x\in \tfrac{1}{2(\sqrt{n})}Q,\ \forall h\in\tfrac{1}{4(\sqrt{n})}Q,\ \forall a>0).$$

On the other hand, for all $h\in\frac{1}{4(\sqrt{n})}Q$ (i.e. $|h|\le 1/4$), we have

$$\begin{aligned}\|\Delta^2_h(f\circ g_a)\|_p &\ge \Big(\int_{x\in(1/(2\sqrt{n}))Q}|\Delta^2_h(f\circ g_a)(x)|^p\mathrm{d}x\Big)^{1/p}\\ &\ge c\,a^{-1/p}\Big(\int_{-a/(2\sqrt{n})}^{a/(2\sqrt{n})}|\Delta^2_{ah_1}f(y)|^p\mathrm{d}y\Big)^{1/p}.\end{aligned}$$

By the above inequality, the embedding $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n)$ and the assumption (4.5), we obtain

$$\begin{aligned}\Big(\int_{-a/(2\sqrt{n})}^{a/(2\sqrt{n})} |\Delta^2_{ah_1} f(y)|^p \,\mathrm{d}y\Big)^{1/p} &\le c_1\, |h|^s\, a^{1/p}\, \Omega(\|g_a\|)\\ &\le c_2\, a^{1/p}\, \Omega(\|g_a\|), \qquad (\forall h : |h| \le 1/4).\end{aligned}$$

By setting $u := ah_1$, we deduce that

$$\Big(\int_{-a/(2\sqrt{n})}^{a/(2\sqrt{n})} |\Delta^2_u f(y)|^p \,\mathrm{d}y\Big)^{1/p} \le c_1\, a^{1/p}\, \Omega(c_2 a), \quad \forall a > 0,\ \forall u : |u| \le a.$$

By applying the assumption (4.4) on Ω and taking a to $+\infty$, we obtain

$$\int_{-\infty}^{+\infty} |f(y+2u) - 2f(y+u) + f(y)|^p \,\mathrm{d}y = 0, \qquad \forall u \in \mathbb{R}.$$

Hence $f(y+2u) - 2f(y+u) + f(y) = 0$ a.e., $\forall y, u \in \mathbb{R}$. Then

$$f'(y+2u) - f'(y+u) = 0,\ \textit{i.e.},$$

it implies $f'(u) = f'(0)$ $(\forall u \in \mathbb{R})$. We deduce that f' is a constant. □

References

1. Adams D. and Frazier M., *BMO and smooth truncation in Sobolev spaces*, Studia Math. **89** (1988), 241–260.

2. ______, *Composition operators on potential spaces*, Proc. Amer. Math. Soc. **114** (1992), 155–165.

3. Allaoui S. E., *Remarques sur le calcul symbolique dans certains espaces de Besov à valeurs vectorielles*, Ann. Math. Blaise Pascal **16(2)** (2009), 399–429.

4. Bourdaud G. and Kateb M. E. D., *Fonctions qui opèrent sur les espaces de Besov*, Math. Ann. **303** (1995) 653–675.

5. Bourdaud G., Lanza de Cristoforis M. and Sickel W., *Superposition operators and functions of bounded p-variation*, Rev. Mat. Iberoamer. **22** (2006), 455–485.

6. Brezis H. and Mironescu P., *Gagliardo-Nirenberg, compositions and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), 387–404.

7. DeVore R. and Lorentz G. G., *Constructive Approximation*, Springer, Berlin, 1993.

8. Franke J., *On the spaces $F^s_{p,q}(\mathbb{R})$ of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains*, Math. Nachr. **125** (1986), 29–68.

9. Maz'ya V. and Shaposnikova T., *An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces*, J. Evol. Equ. **2** (2002), 113–125.

10. Moussai M., *The composition in multidimensional Triebel-Lizorkin spaces*, Math. Nachr. **284(2-3)** (2011), 317–331.

11. Peetre J., *New thoughts on Besov spaces*, Duke Univ. Math. Series I, Durham, N. C., 1976.

12. Runst T. and Sickel W., *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, de Gruyter, Berlin, 1996.

13. Triebel H., *Interpolation Theory, Function Spaces, Differential Operators*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

14. ______, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

15. ______, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

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