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VARIATIONS ON BROWDER'S THEOREM

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Abstract. In this note we introduce and study the new spectral properties (Bb), (Bab) and (Baw) as continuation of [**7, 8, 12**] which are variants of the classical Browder's theorem.

1. Introduction and terminology

This paper is a continuation of previous papers of the first author and Berkani [**7, 8**] and the paper [**12**], where the generalization of Weyl's theorem and Browder's theorem is studied. The purpose of this paper is to introduce and study the new properties (Bb), (Bab) and (Baw) (see later for definitions) in connection with known Weyl type theorems and properties ([**3, 5, 7, 8, 12, 13**]), which play roles analogous to Browder's theorem and Weyl's theorem, respectively.

To introduce all these concepts, we begin with some preliminary definitions and results. Let $L(X)$ denote the Banach algebra of all bounded linear operators acting on a complex infinite-dimensional Banach space X. For $T \in L(X)$, let T^*, $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_a(T)$ denote the dual, the null space, the range, the spectrum and the approximate point spectrum of T, respectively If $R(T)$ is closed and $\alpha(T) := \dim N(T) < \infty$ (resp. $\beta(T) := \operatorname{codim} R(T) < \infty$), then T is called an *upper* (resp. a *lower*) *semi-Fredholm* operator. If T is either an upper or a lower semi-Fredholm operator, then T is called a *semi-Fredholm* operator, and the index of T is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm* operator. If T is Fredholm operator of index zero, then T is said to be a *Weyl* operator. The *Weyl spectrum* of T is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$ and the *Weyl essential approximate point spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T-\lambda I \text{ is not an upper semi-Fredholm with } \operatorname{ind}(T-\lambda I) \leq 0\}$.

Following [**10**], we say that *Weyl's theorem* holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, $\operatorname{iso} A$ is the set of all isolated points of A. According to Rakočević [**17**], an operator $T \in L(X)$ is said to satisfy *a-Weyl's theorem* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : 0 < \alpha(T-\lambda I) < \infty\}$.

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It is known [**17**] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

For $T \in L(X)$ and a nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n, the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an *upper* (resp. a *lower*) *semi-B-Fredholm* operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [**6**]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a *B-Fredholm* operator, see [**4**]. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a *B-Weyl* operator if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not a B-Weyl operator$\}$.

Following [**5**], an operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem* if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T) = \{\lambda \in \operatorname{iso}\sigma(T) : \alpha(T - \lambda I) > 0\}$ is the set of all isolated eigenvalues of T. It is proven in [**5**, Theorem 3.9] that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general.

Recall that the *ascent* $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$ and the *descent* $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ with $\inf\emptyset = \infty$. Let $\Pi_a(T)$ denote the set of all left poles of T defined by $\Pi_a(T) = \{\lambda \in \mathbb{C} : a(T-\lambda I) < \infty$ and $R((T-\lambda I)^{a(T-\lambda I)+1})$ is closed$\}$; and let $\Pi_a^0(T)$ denote the set of all left poles of T of finite rank, that is $\Pi_a^0(T) = \{\lambda \in \Pi_a(T) : \alpha(T-\lambda I) < \infty\}$. According to [**11**], we say that *a-Browder's theorem* holds for $T \in L(X)$ if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T-\lambda I) < \infty\}$. According to [**14**], a complex number λ is a *pole* of the resolvent of T if and only if $0 < \max\,(a(T-\lambda I), \delta(T-\lambda I)) < \infty$. Moreover, if this is true, then $a(T - \lambda I) = \delta(T - \lambda I)$. Also according to [**14**], the space $R((T - \lambda I)^{a(T-\lambda I)+1})$ is closed for each $\lambda \in \Pi(T)$. Hence we have always $\Pi(T) \subset \Pi_a(T)$ and $\Pi^0(T) \subset \Pi_a^0(T)$. We say that *Browder's theorem* holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$, and *generalized Browder's theorem* holds for $T \in L(X)$ if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. It is proven in [**1**, Theorem 2.1] that generalized Browder's theorem is equivalent to Browder's theorem.

An approximate point spectrum variant of Weyl's theorem was introduced by Rakočević [**16**], property (w). Recall that $T \in L(X)$ possesses *property (w)* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. It is proven in [**16**, Corollary 2.3] that property (w) implies Weyl's theorem, but not conversely.

Following [**12**], an operator $T \in L(X)$ is said to possess *property (Bw)* if $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. It is shown [**12**, Theorem 2.4] that an operator possessing property (Bw) satisfies generalized Browder's theorem. According to [**8**], an operator $T \in L(X)$ is said to possess *property* (gaw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \alpha(T - \lambda I) > 0\}$ and is said to possess *property*

(gab) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$. It is proven in [**8**, Theorem 3.5] that property (gaw) implies property (gab) but not conversely. The two last properties are extensions to the context of B-Fredholm theory, of properties (aw) and (ab), respectively, see [**8**]. Recall [**8**] that an operator $T \in L(X)$ is said to possess *property* (aw) if $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ and is said to possess *property* (ab) if $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$.

An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \longrightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$, is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has this property at every $\lambda \in \mathbb{C}$ (see [**15**]). Trivially, every operator T has the SVEP at $\lambda \in \operatorname{iso} \sigma(T)$.

2. Property (Bb)

In this section we investigate a new variant of Browder's theorem. We introduce the property (Bb) which is intermediate between property (Bw) and Browder's theorem. We also give characterizations of operators possessing property (Bb). Before that we start by some remarks about property (Bw).

Remark 2.1.
1. The property (Bw) is not intermediate between Weyl's theorem and generalized Weyl's theorem (resp. a-Weyl's theorem). Indeed, the operator U defined below as in Example 2.5 satisfies a-Weyl's theorem and as $E(U) = \{0, 1\}$, then U satisfies also generalized Weyl's theorem, but it does not possess property (Bw). Now let $T = 0 \oplus S$ be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where S is defined on $\ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, \ldots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \ldots)$. Then $\sigma(T) = \sigma_{BW}(T) = \{0\}$ and $E(T) = \{0\}$. So $\sigma(T) \setminus \sigma_{BW}(T) \neq E(T)$, i.e. T does not satisfy generalized Weyl's theorem. But since $E^0(T) = \emptyset$, then $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$, i.e. T possesses property (Bw). On the other hand, the operator $T = R \oplus S$ where R is the unilateral right shift operator defined on $\ell^2(\mathbb{N})$ and S is defined on $\ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, x_4, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots)$. Then $\sigma(T) = \sigma_{BW}(T) = D(0, 1)$ which is the closed unit disc in $\mathbb{C}$, $\sigma_a(T) = C(0, 1) \cup \{0\}$ where $C(0, 1)$ is the unit circle of $\mathbb{C}$ and $E^0(T) = \Pi_a^0(T) = \emptyset$. This implies that $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$, i.e. T possesses property (Bw), but it does not satisfy a-Weyl's theorem because $\sigma_a(T) = \sigma_{SF_+^-}(T) = C(0, 1) \cup \{0\}$ and $E_a^0(T) = \{0\}$, so that $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \neq E_a^0(T)$.
2. The property (Bw) is not transmitted from an operator to its dual. To see this, if we consider the operator S defined as in part 1), then S possesses property (Bw) since $\sigma(S) = \sigma_{BW}(S) = \{0\}$ and $E^0(S) = \emptyset$. But its adjoint which is defined on $\ell^2(\mathbb{N})$ by $S^*(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$ does not possess this property, since $\sigma(S^*) = \sigma_{BW}(S^*) = \{0\}$ and $E^0(S^*) = \{0\}$.

It is signaled in [**12**] (precisely after Definition 2.11) that if $T \in L(X)$ is an operator possessing property (Bw) and satisfying the condition $\operatorname{iso}\sigma(T) = \emptyset$, then

T satisfies Weyl's theorem. But the following theorem gives a stronger version of this remark.

Theorem 2.2. *Let $T \in L(X)$. T possesses property (Bw) if and only if T satisfies Weyl's theorem and $\sigma_{BW}(T) = \sigma_W(T)$.*

Proof. Suppose that T possesses property (Bw), that is $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$. Let $\lambda \in \sigma(T) \setminus \sigma_W(T)$, as $\sigma(T) \setminus \sigma_W(T) \subset \sigma(T) \setminus \sigma_{BW}(T)$ then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Thus $\lambda \in E^0(T)$ and $\sigma(T) \setminus \sigma_W(T) \subset E^0(T)$. Now let us consider $\lambda \in E^0(T)$. As $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$, then $T - \lambda I$ is a B-Weyl operator. Since $\alpha(T - \lambda I) < \infty$, by virtue of [**7**, Lemma 2.2], we deduce that $T - \lambda I$ is a Weyl operator. It follows that $\lambda \in \sigma(T) \setminus \sigma_W(T)$, and hence $\sigma(T) \setminus \sigma_W(T) = E^0(T)$, i.e. T satisfies Weyl's theorem. Then we have $\sigma_{BW}(T) = \sigma(T) \setminus E^0(T)$ and $\sigma_W(T) = \sigma(T) \setminus E^0(T)$. So $\sigma_{BW}(T) = \sigma_W(T)$.

Conversely, the condition $\sigma_{BW}(T) = \sigma_W(T)$ entails that $\sigma(T) \setminus \sigma_{BW}(T) = \sigma(T) \setminus \sigma_W(T)$. Weyl's theorem for T implies that $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$ and T possesses property (Bw). □

Definition 2.3. A bounded linear operator $T \in L(X)$ is said to possess property (Bb) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$.

The property (Bb) is not intermediate between Browder's theorem and a-Browder's theorem. Indeed, let R and L denote the unilateral right shift operator and the unilateral left shift operator, respectively on the Hilbert space $\ell^2(\mathbb{N})$ and we consider the operator T defined by $T = L \oplus R \oplus R$. Then $\alpha(T) = 1$, $\beta(T) = 2$ and so $0 \notin \sigma_{SF_+^-}(T)$. Since $a(T) = \infty$, then T does not have the SVEP at 0. Hence T does not satisfy a-Browder's theorem. Since $\sigma(T) = \sigma_{BW}(T) = D(0,1)$ and $\Pi^0(T) = \emptyset$, then T possesses property (Bb). On the other hand, it is easily seen that the operator T defined by $T(x_1, x_2, x_3, \ldots) = (0, \frac{1}{2}x_1, 0, 0, \ldots)$ satisfies a-Browder's theorem. But it does not possess property (Bb), since $\sigma(T) = \{0\}$ and $\sigma_{BW}(T) = \Pi^0(T) = \emptyset$.

However, we have the following characterizations of operators possessing property (Bb).

Theorem 2.4. *Let $T \in L(X)$. Then the following assertions are equivalent:*

(i) T possesses property (Bb).

(ii) T satisfies Browder's theorem and $\Pi(T) = \Pi^0(T)$.

(iii) T satisfies Browder's theorem and $\sigma_{BW}(T) = \sigma_W(T)$

Proof. (i)$\Longrightarrow$(ii) Assume that T possesses property (Bb), that is $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$ and let $\lambda \notin \sigma_{BW}(T)$ be arbitrary. If $\lambda \in \sigma(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$. Consequently, $\lambda \in \mathrm{iso}\,\sigma(T)$ which implies that T has the SVEP at λ. If $\lambda \notin \sigma(T)$, then obviously T has the SVEP at λ. In the two cases, we have T has the SVEP at λ, and this is equivalent [**2**, Proposition 2.2] to the saying that T satisfies generalized Browder's theorem and then Browder's theorem. Thus $\Pi(T) = \Pi^0(T)$.

(ii)$\Longrightarrow$ (iii) Assume that T satisfies Browder's theorem and $\Pi(T) = \Pi^0(T)$. Since Browder's theorem is equivalent to generalized Browder's theorem, then $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T) = \sigma(T) \setminus \Pi^0(T) = \sigma_W(T)$.

(iii)$\Longrightarrow$(i) Obvious. □

The following example shows that in general Weyl's theorem or Browder's theorem do not imply property (Bw) or property (Bb), respectively.

Example 2.5. Let $U \in L(\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, x_3, \ldots) = (0, x_2, x_3, \ldots)$, $\forall (x) = (x_i) \in \ell^2(\mathbb{N})$. Then $\sigma_a(U) = \sigma(U) = \{0, 1\}$, $\sigma_{SF_+^-}(U) = \sigma_W(U) = \{1\}$ and $E_a^0(U) = E^0(U) = \{0\}$. Thus $\sigma_a(U) \setminus \sigma_{SF_+^-}(U) = E_a^0(U)$ and $\sigma(U) \setminus \sigma_W(U) = E^0(U)$, i.e. U satisfies a-Weyl's theorem and Weyl's theorem. On the other hand, $\Pi(U) = \{0, 1\}$ and $\Pi^0(U) = \Pi_a^0(U) = \{0\}$, and consequently $\sigma_a(U) \setminus \sigma_{SF_+^-}(U) = \Pi_a^0(U)$ and $\sigma(U) \setminus \sigma_W(U) = \Pi^0(U)$, so that U satisfies a-Browder's theorem and Browder's theorem. Moreover, $\sigma_{BW}(U) = \emptyset$. Hence $\sigma(U) \setminus \sigma_{BW}(U) \neq E^0(U)$ and $\sigma(U) \setminus \sigma_{BW}(U) \neq \Pi^0(U)$, i.e. U does not possess either property (Bw) no property (Bb). Here $\Pi(U) \neq \Pi^0(U)$.

From Theorem 2.2 and Theorem 2.4 we deduce that property (Bw) implies property (Bb). But the converse is not true in general as shown by the following example.

Example 2.6. Let $T \in L(\ell^2(\mathbb{N}))$ be defined by $T(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \ldots)$. Then T possesses (Bb) because $\sigma(T) = \sigma_{BW}(T) = \{0\}$ and $\Pi^0(T) = \emptyset$, while T does not possess property (Bw) because $E^0(T) = \{0\}$. Note that $\Pi(T) = \emptyset$.

Moreover, we give conditions for the equivalence of property (Bw) and property (Bb) in the next theorem.

Theorem 2.7. *Let $T \in L(X)$. Then the following assertions are equivalent:*

(i) T possesses property (Bw).

(ii) T possesses property (Bb) and $E^0(T) = \Pi^0(T)$.

(iii) T possesses property (Bb) and $E^0(T) = \Pi(T)$.

In particular, if T is polaroid (i.e. $\operatorname{iso}\sigma(T) = \Pi(T)$), then the properties (Bw) and (Bb) are equivalent.

Proof. (i) $\Longrightarrow$ (ii) Assume that T possesses property (Bw). Then from Theorem 2.2, T satisfies Weyl's theorem, which implies from [**3**, Corollary 5] that $E^0(T) = \Pi^0(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$, i.e. T possesses property (Bb) and $E^0(T) = \Pi^0(T)$.

(ii) $\Longrightarrow$ (iii) Follows directly from Theorem 2.4.

(iii) $\Longrightarrow$ (i) Assume that T possesses property (Bb) and $E^0(T) = \Pi(T)$. Again by Theorem 2.4, $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$ and as $E^0(T) = \Pi(T)$, then $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$ and T possesses property (Bw).

In the special case when T is polaroid, the condition $E^0(T) = \Pi^0(T)$ is always satisfied. Therefore the two properties (Bw) and (Bb) are equivalent. $\square$

3. Properties (Baw) and (Bab)

In this section we investigate a new variant of property (aw) (resp. property (ab)). We introduce the property (Baw) which is intermediate between property (Bw) and property (aw). We also introduce the property (Bab) which is intermediate between property (Bb) and property (ab). Furthermore, we shows that property (Bab) is a week version of property (Baw).

Definition 3.1. A bounded linear operator $T \in L(X)$ is said to possess property (Baw) if $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$, and is said to possess property (Bab) if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$.

Theorem 3.2. *Let $T \in L(X)$. Then T possesses property (Baw) if and only if T possesses property (Bab) and $E_a^0(T) = \Pi_a^0(T)$. In particular, if T is a-polaroid (i.e. $\operatorname{iso}\sigma_a(T) = \Pi_a(T)$), then the properties (Baw) and (Bab) are equivalent.*

Proof. Suppose that T possesses property (Baw), that is $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$, then $\lambda \in E_a^0(T)$ and so $\lambda \in \operatorname{iso}\sigma_a(T)$. As $\lambda \notin \sigma_{BW}(T)$, in particular, $T - \lambda I$ is an upper semi-B-Fredholm operator, then from [**5**, Theorem 2.8], we have $\lambda \in \Pi_a(T)$. Since $\alpha(T - \lambda I)$ is finite, $\lambda \in \Pi_a^0(T)$. Therefore $\sigma(T) \setminus \sigma_{BW}(T) \subset \Pi_a^0(T)$. Now if $\lambda \in \Pi_a^0(T)$, as $\Pi_a^0(T) \subset E_a^0(T)$ is always true, then $\lambda \in E_a^0(T) = \sigma(T) \setminus \sigma_{BW}(T)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$, i.e. T possesses property (Bab) and $E_a^0(T) = \Pi_a^0(T)$. The converse is trivial.

Moreover, if T is an a-polaroid, then $E_a^0(T) = \Pi_a^0(T)$, and hence, in this case the two properties (Baw) and (Bab) are equivalent. □

In the next theorem, we give a characterization of operators possessing property (Baw).

Theorem 3.3. *Let $T \in L(X)$. T possesses property (Baw) if and only if T possesses property (aw) and $\sigma_{BW}(T) = \sigma_W(T)$.*

Proof. Suppose that T possesses property (Baw) and let $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$. Therefore $\sigma(T) \setminus \sigma_W(T) \subset E_a^0(T)$. Now if $\lambda \in E_a^0(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. This implies that $\lambda \notin \sigma_{BW}(T)$, and since $\alpha(T - \lambda I)$ is finite, then as it had been already mentioned, we have $\lambda \notin \sigma_W(T)$, so that $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Hence $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ and T possesses property (aw). Then we have $\sigma_{BW}(T) = \sigma(T) \setminus E_a^0(T)$ and $\sigma_W(T) = \sigma(T) \setminus E_a^0(T)$. So $\sigma_{BW}(T) = \sigma_W(T)$.

Conversely, suppose that T possesses property (aw) and $\sigma_{BW}(T) = \sigma_W(T)$. Then $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ and $\sigma_{BW}(T) = \sigma_W(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$ and T possesses property (Baw). □

Remark 3.4.

1. From Theorem 3.3, if $T \in L(X)$ possesses property (Baw), then T possesses property (aw). However, the converse is not true in general: for example, the operator U defined as in Example 2.5 possesses property (aw) because $\sigma(U) \setminus \sigma_W(U) = E_a^0(U) = \{0\}$, but it does not possess property (Baw) because $\sigma(U) \setminus \sigma_{BW}(U) = \{0, 1\}$.

2. Generally, the two properties (gaw) and (Baw) are independent. For this, it is easily seen that the operator $T = 0 \oplus S$ defined as in Remark 2.1 possesses property (Baw), but it does not possess property (gaw) and the operator defined as in Example 2.5 possesses property (gaw), but it does not possess property (Baw).

3. The property (Baw) as well as property (Bw), do not pass from an operator to its dual. Indeed, the operator S defined as in part 2) of Remark 2.1 possesses property (Baw) since $E_a^0(S) = \emptyset$. But its adjoint S^* does not possess this property since $E_a^0(S^*) = \{0\}$. Similarly, property (Bab) is not transmitted from an operator to its dual. To see this, we consider the operator T defined by $T(x_1, x_2, x_3, \ldots) = (\epsilon x_1, 0, x_2, x_3, \ldots)$ for fixed $0 < \epsilon < 1$ on the Hilbert space $\ell^2(\mathbb{N})$. Then $\sigma(T) = \sigma(T^*) = D(0,1)$, $\sigma_{BW}(T) = \sigma_{BW}(T^*) = D(0,1)$ and $\Pi_a^0(T) = \emptyset$. This implies that T possesses property (Bab), but since $\Pi_a^0(T^*) = \{\epsilon\}$, then T^* does not possess property (Bab).

Corollary 3.5. *Let $T \in L(X)$. T possesses property (Baw) if and only if T possesses property (Bw) and $E^0(T) = E_a^0(T)$.*

Proof. Suppose that T possesses property (Baw), then by Theorem 3.3, T possesses property (aw) which implies by virtue of [**9**, Theorem 2.5] that $E^0(T) = E_a^0(T)$. Since $\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$, then $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$ and T possesses property (Bw). Conversely, suppose that T possesses property (Bw) and $E^0(T) = E_a^0(T)$. Then $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T) = E_a^0(T)$ and hence T possesses property (Baw). □

From Theorem 3.2 and Corollary 3.5, we have if $T \in L(X)$ possesses property (Baw), then T possesses property (Bab) and property (Bw). But the converses do not hold in general as shown by the following example. Let $T = R \oplus S$ be defined as in Remark 2.1. Then $\sigma(T) = \sigma_{BW}(T) = D(0,1)$, $\sigma_a(T) = C(0,1) \cup \{0\}$ and $E^0(T) = \Pi_a^0(T) = \emptyset$. This implies that $\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$, i.e. T possesses property (Bw) and property (Bab). But it does not possess property (Baw) because $E_a^0(T) = \{0\}$, so that $\sigma(T) \setminus \sigma_{BW}(T) \neq E_a^0(T)$.

Now we give characterizations of operators possessing property (Bab) in the next theorem.

Theorem 3.6. *Let $T \in L(X)$. Then the following assertions are equivalent:*

(i) T possesses property (Bab).

(ii) T possesses property (ab) and $\sigma_{BW}(T) = \sigma_W(T)$.

(iii) T possesses property (ab) and $\Pi(T) = \Pi_a^0(T)$.

Proof. (i) $\Longleftrightarrow$ (iii) Suppose that T possesses property (Bab). If $\lambda \in \sigma(T) \setminus \sigma_W(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$. Thus $\sigma(T) \setminus \sigma_W(T) \subset \Pi_a^0(T)$. If $\lambda \in \Pi_a^0(T)$, then $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$ and $T - \lambda I$ is a B-Fredholm operator with $\operatorname{ind}(T - \lambda I) = 0$. As $a(T - \lambda I) < \infty$, then $a(T - \lambda I) = \delta(T - \lambda I) < \infty$ and $\lambda \in \Pi^0(T)$. Therefore $\alpha(T - \lambda I) = \beta(T - \lambda I) < \infty$. Consequently, $\lambda \notin \sigma_W(T)$ and $\sigma(T) \setminus \sigma_W(T) \supset \Pi_a^0(T)$. Hence $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$ and T possesses property (ab). Moreover, we have that $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$, i.e. T satisfies Browder's theorem and then generalized Browder's theorem. Thus $\Pi(T) = \Pi_a^0(T)$. Conversely, suppose that T possesses property (ab) and $\Pi(T) = \Pi_a^0(T)$. Then from [**8**, Theorem 2.4], T satisfies generalized Browder's theorem $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$, and as $\Pi(T) = \Pi_a^0(T)$, then $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$ and T possesses property (Bab).

(i) $\Longleftrightarrow$ (ii) Suppose that T possesses property (Bab), then T possesses property (ab). Thus $\sigma_{BW}(T) = \sigma(T) \setminus \Pi_a^0(T)$ and $\sigma_W(T) = \sigma(T) \setminus \Pi_a^0(T)$. So

$\sigma_{BW}(T) = \sigma_W(T)$. Conversely, suppose that T possesses property (ab) and $\sigma_{BW}(T) = \sigma_W(T)$. Then $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$ and $\sigma_{BW}(T) = \sigma_W(T)$. Thus $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$ and T possesses property (Bab). □

Remark 3.7.
1. From Theorem 3.6, if $T \in L(X)$ possesses property (Bab), then T possesses property (ab). But the converse is not true in general as shown by the following example. Let T the operator defined by $T(x_1, x_2, x_3, \ldots) = (0, \frac{1}{2}x_1, 0, 0, \ldots)$ on the Hilbert space $\ell^2(\mathbb{N})$. Then $\sigma(T) = \{0\}$, $\Pi_a^0(T) = \emptyset$, $\sigma_W(T) = \{0\}$. So T possesses property (ab). But it does not possess property (Bab), since $\sigma_{BW}(T) = \emptyset$. Note that $\Pi(T) = \Pi_a(T) = \{0\}$.
2. The property (Bab) is not intermediate between property (gab) and property (ab). Indeed, the operator defined as in the first part of this remark possesses property (gab), but it does not possess property (Bab). On the other hand, if we consider the operator $T = 0 \oplus R$ defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where R is the unilateral right shift operator, then T possesses property (Bab) because $\sigma(T) = \sigma_{BW}(T) = D(0,1)$ and $\Pi_a^0(T) = \emptyset$, but it does not possess property (gab) because $\Pi_a(T) = \{0\}$.

Corollary 3.8. *Let $T \in L(X)$. Then the following assertions are equivalent:*
(i) T possesses property (Bab).
(ii) T possesses property (Bb) and $\Pi^0(T) = \Pi_a^0(T)$.
(iii) T possesses property (Bb) and $\Pi(T) = \Pi_a^0(T)$.

Proof. (i) $\Longleftrightarrow$ (ii) Suppose that T possesses property (Bab), that is $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$. From Theorem 3.6, we deduce that T satisfies Browder's theorem and $\sigma_{BW}(T) = \sigma_W(T)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$, i.e. T possesses property (Bb) and $\Pi^0(T) = \Pi_a^0(T)$. Conversely, suppose that T possesses property (Bb) and $\Pi^0(T) = \Pi_a^0(T)$. Then $\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$ and $\Pi^0(T) = \Pi_a^0(T)$. So $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$ and T possesses property (Bab).
(ii) $\Longleftrightarrow$ (iii) Follows directly from Theorem 2.4. □

From Corollary 3.8, if $T \in L(X)$ possesses property (Bab), then T possesses property (Bb). However, the converse is not true in general as shown in the following example.

Example 3.9. Let T be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus U$, where R is the unilateral right shift operator on $\ell^2(\mathbb{N})$ and U is defined as in Example 2.5. Then $\sigma(T) = \sigma_{BW}(T) = D(0,1)$, $\Pi_a^0(T) = \{0\}$ and $\Pi(T) = \Pi^0(T) = \emptyset$. This shows that T possesses property (Bb), but it does not possess property (Bab).

4. Summary of results

In this last part we give a summary of the results obtained in this paper. We use the abbreviations (Bw), (Baw), (gaw), (aw), (w), W, gW and aW to signify that an operator $T \in L(X)$ obeys property (Bw), property (Baw), property (gaw), property (aw), property (w), Weyl's theorem, generalized Weyl's theorem and a-Weyl's theorem, respectively. Similarly, the abbreviations (Bb), (Bab), (gab), (ab), aB, B and gB have analogous meaning with respect to the properties introduced in this paper or to the properties introduced in [**8**] or to Browder's theorems.

The following table summarizes the meaning of various theorems and properties.

(Bw)	$\sigma(T)\setminus\sigma_{BW}(T)=E^0(T)$	(Bb)	$\sigma(T)\setminus\sigma_{BW}(T)=\Pi^0(T)$
(Baw)	$\sigma(T)\setminus\sigma_{BW}(T)=E^0_a(T)$	(Bab)	$\sigma(T)\setminus\sigma_{BW}(T)=\Pi^0_a(T)$
(gaw)	$\sigma(T)\setminus\sigma_{BW}(T)=E_a(T)$	(gab)	$\sigma(T)\setminus\sigma_{BW}(T)=\Pi_a(T)$
(aw)	$\sigma(T)\setminus\sigma_{W}(T)=E^0_a(T)$	(ab)	$\sigma(T)\setminus\sigma_{W}(T)=\Pi^0_a(T)$
(w)	$\sigma_a(T)\setminus\sigma_{SF_+^-}(T)=E^0(T)$	aB	$\sigma_a(T)\setminus\sigma_{SF_+^-}(T)=\Pi^0_a(T)$
W	$\sigma(T)\setminus\sigma_{W}(T)=E^0(T)$	B	$\sigma(T)\setminus\sigma_{W}(T)=\Pi^0(T)$
gW	$\sigma(T)\setminus\sigma_{BW}(T)=E(T)$	gB	$\sigma(T)\setminus\sigma_{BW}(T)=\Pi(T)$
aW	$\sigma_a(T)\setminus\sigma_{SF_+^-}(T)=E^0_a(T)$		

In the following diagram arrows signify implications between Weyl's theorems, Browder's theorems, property (w), property (Bw), property (Bb), property (Baw) and property (Bab). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$\begin{tikzcd}
 & (w) \arrow[d, "{[\mathbf{16}]}"] & (gaw) \arrow[r, "{[\mathbf{9}]}"] \arrow[d, "{[\mathbf{8}]}"] & gW & & gW \arrow[d, "{[\mathbf{5}]}"] \\
aW \arrow[r, "{[\mathbf{17}]}"] \arrow[d, "{[\mathbf{5}]}"] & W \arrow[d, "{[\mathbf{3}]}"] & (aw) \arrow[l, "{[\mathbf{9}]}"'] \arrow[d, "{[\mathbf{8}]}"] & (Baw) \arrow[l, "3.3"'] \arrow[r, "3.5"] \arrow[d, "3.2"] & (Bw) \arrow[r, "2.2"] \arrow[d, "2.7"] & W \arrow[d, "{[\mathbf{3}]}"] \\
aB \arrow[r, "{[\mathbf{11}]}"'] & B & (ab) \arrow[l, "{[\mathbf{8}]}"] & (Bab) \arrow[l, "3.6"] \arrow[r, "3.8"'] & (Bb) \arrow[r, "2.4"'] & B \arrow[d, Leftrightarrow, "{[\mathbf{1}]}"] \\
 & & (gab) \arrow[u, "{[\mathbf{8}]}"] & (gaw) \arrow[l, "{[\mathbf{8}]}"] & & gB
\end{tikzcd}$$

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