FIXED POINTS OF CONTRACTIONS AND CYCLIC CONTRACTIONS ON C^*-ALGEBRA-VALUED b-METRIC SPACES

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ABSTRACT. In this paper, we discuss and improve some recent results about contractive and cyclic mappings established in the framework of C^*-algebra-valued b-metric spaces. Our proofs are much shorter than the ones in existing literature. Also, we give two examples that support our approach.

1. INTRODUCTION AND PRELIMINARIES

In the last decades many researchers in nonlinear analysis worked with cone metric spaces, cone metric spaces over Banach algebras, C^*-algebra-valued metric spaces, C^*-algebra-valued b-metric spaces and other. Also, many papers in the existing literature from fixed point theory dealt with the so-called generalized metric spaces. However, it is already well-known that almost all results in generalized metric spaces are equivalent to the corresponding ones in standard metric spaces (except in the case of cone metric spaces over Banach algebra). In this paper we consider this phenomenon for self-mappings and cyclic mappings in the framework of complete C^*-algebra-valued b-metric spaces. Otherwise, more details on self-mappings and cyclic type mappings in the framework of metric or b-metric spaces are given in [10, 15, 16].

Based on the notion and properties of C^*-algebra (see [14]) several authors introduced and considered the concepts of C^*-algebra-valued metric spaces as well as C^*-algebra-valued b-metric spaces and gave some fixed point theorems for self-mappings with contractive conditions on such spaces; see [1, 2, 9, 11, 12, 13]. Also, for important facts on b-metric spaces, we refer to [3, 6].

Firstly, we begin with the basic concepts in C^*-algebras. A real or a complex linear space $\mathcal{A}$ is an algebra if vector multiplication is defined for every pair of elements of $\mathcal{A}$ such that $\mathcal{A}$ is a ring with respect to vector addition and vector multiplication and for every scalar β and every pair of elements $u, v \in \mathcal{A}$, we have $\beta(uv) = (\beta u)v = u(\beta v)$. If $\mathcal{A}$ is endowed with a submultiplicative norm $\| \cdot \|$, that is, $\|uv\| \le \|u\| \|v\|$ for all $u, v \in \mathcal{A}$, then $(\mathcal{A}, \| \cdot \|)$ is a normed algebra. A complete normed algebra is called Banach algebra. An involution on the algebra $\mathcal{A}$ is a conjugate linear mapping $* : \mathcal{A} \to \mathcal{A}$ such that

(1) $u^{**} = u$;
(2) $(uv)^* = v^* u^*$

for all $u, v \in \mathcal{A}$. The pair $(\mathcal{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra $\mathcal{A}$ is a $*$-algebra $\mathcal{A}$ with a complete submultiplicative norm such that $\|u^*\| = \|u\|$ for all $u \in \mathcal{A}$. Then, a C^*-algebra is a Banach $*$-algebra such that $\|u^* u\| = \|u\|^2$. Examples of C^*-algebras are the set $\mathbb{C}$ of complex numbers, the set $L(H)$ of all bounded linear operators on a Hilbert space H, and the set $M_n(\mathbb{C})$ of $n \times n$-matrices. If a normed algebra $\mathcal{A}$ admits a unit I, that is, there exists an element $I \in \mathcal{A}$ such that $Iu = uI = u$ for all $u \in \mathcal{A}$, and $\|I\| = 1$, we say that $\mathcal{A}$ is a unital normed algebra. A complete unital normed algebra $\mathcal{A}$ is called a unital Banach algebra. Throughout this paper, $\mathcal{A}$ will denote a unital C^*-algebra with a unit I. For the basic properties and results in the setting of C^*-algebras, the interested reader is referred to [4, 5, 14] and the references therein.

The paper is organized as follows. In Section 2, we recollect some basic notations, definitions and results in the framework of b-metric spaces. In Section 3, we establish some results about fixed points in the setting of C^*-algebra-valued b-metric spaces. Two appropriate examples are included. In Section 4, we provide some results on existence and uniqueness for cyclic mappings in the setting of b-metric spaces and deduce results on fixed point for cyclic mappings in the setting of C^*-algebra-valued b-metric spaces. Our proofs are much shorter than the ones in the existing literature, in particular than those appearing in the paper [11].

2. Fixed points in the setting of b-metric spaces

In this section, we collect some basic notations, definitions, and results concerning b-metric spaces.

Definition 2.1. Let X be a nonempty set. A mapping $D : X^2 \to [0, \infty)$ is called a b-metric if there exists a real number $b \ge 1$ such that, for every $u, v, z \in X$, we have

(1) $D(u, v) = 0$ if and only if $u = v$;
(2) $D(u, v) = D(v, u)$;
(3) $D(u, v) \le b(D(u, z) + D(z, v))$.

The pair (X, D, b) is called a b-metric space.

Every metric space is a b-metric space with $b = 1$. The notions of convergence, closedness and completeness in b-metric space are given in the same way as in metric spaces. Let X be a nonempty set, $u_0 \in X$ and let $T : X \to X$ be a mapping. The sequence $\{u_n\}$ defined by $u_n = T^n u_0$ for all $n \in \mathbb{N}$ is called the Picard sequence (generated by T) starting at u_0. Briefly, we give some known results on fixed points in the context of b-metric spaces, with short proofs, that will be used in the sequel. First of all, we prove the following known result, but with new and much shorter proof.

Lemma 2.2 (Banach type theorem). *Let (X, D, b) be a complete b-metric space, and let $T : X \to X$ be a given mapping. Assume that there exists some $\lambda \in [0, 1)$ such that*

$$D(Tu, Tv) \leq \lambda D(u, v) \quad \text{for all } u, v \in X. \tag{2.1}$$

Then T has a unique fixed point $z \in X$, and for every $u_0 \in X$, the Picard sequence $\{T^n u_0\}$ converges to z.

Proof. First, if $\lambda \in [0, b^{-1})$ the proof follows according to ([7], Theorem 3.3). Therefore, let $\lambda \in [b^{-1}, 1)$. It is clear that (2.1) implies

$$D(T^n u, T^n v) \leq \lambda^n D(u, v),$$

for all $n \in \mathbb{N}$ and $u, v \in X$. Since $\lambda^n \to 0$ as $n \to \infty$ we get that there exists $k \in \mathbb{N}$ such that $\lambda^k < b^{-1}$. Now, again according to ([7], Theorem 3.3), we obtain that T^k has a unique fixed point, say z. Consequently, z is a unique fixed point of T and for every $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to z. The proof is complete. $\square$

Remark 2.3. For more details on the previous Lemma see Theorem 12.2 of [3] and Theorem 2.1 of [6].

Lemma 2.4. *Let (X, D, b) be a complete b-metric space, and let $T : X \to X$ be a given mapping. Assume that there exists some $\lambda \in [0, b^{-1})$ such that*

$$D(Tu, Tv) \leq \lambda \max\{D(u, v), D(u, Tu), D(v, Tv)\} \quad \text{for all } u, v \in X. \tag{2.2}$$

Then T has a unique fixed point z, and for every $u_0 \in X$, the Picard sequence $\{T^n u_0\}$ converges to z.

Proof. Denote $u_n = T^n u_0$. By using (2.2) with $u = u_{n-1}$ and $v = u_n$, we get

$$D(u_n, u_{n+1}) \leq \lambda \max\{D(u_{n-1}, u_n), D(u_n, u_{n+1})\}.$$

If we suppose that $D(u_n, u_{n+1}) \geq D(u_{n-1}, u_n)$, we get $D(u_n, u_{n+1}) \leq \lambda D(u_n, u_{n+1})$, a contradiction since $\lambda < 1$. Hence, $D(u_n, u_{n+1}) < D(u_{n-1}, u_n)$ and we get

$$D(u_n, u_{n+1}) \leq \lambda D(u_{n-1}, u_n) \quad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

From (2.3), we deduce that $\{u_n\}$ is a Cauchy sequence and hence converges to some $z \in X$. We claim that z is a fixed point of T. We have

$$D(z, Tz) \leq bD(z, u_{n+1}) + bD(Tu_n, Tz)$$
$$\leq bD(z, u_{n+1}) + b\lambda \max\{D(u_n, z), D(u_n, Tu_n), D(z, Tz)\}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the previous relation, we obtain $D(z, Tz) \leq b\lambda D(z, Tz)$ that implies $z = Tz$. The uniqueness of the fixed point follows by (2.2). $\qquad\square$

From Lemma 2.4, we obtain the following results.

Lemma 2.5 (Kannan type theorem)**.** *Let* (X, D, b) *be a complete b-metric space, and let* $T : X \to X$ *be a given mapping. Assume that there exists some* $\lambda \in [0, \frac{1}{2b})$ *such that*

$$D\left(Tu, Tv\right) \leq \lambda(D(u, Tu) + D(v, Tv)) \quad \text{for all } u, v \in X.$$

Then T *has a unique fixed point* z *and for every* $u_0 \in X$ *the Picard sequence* $\{T^n u_0\}$ *converges to* z.

Lemma 2.6 (Chatterjea type theorem)**.** *Let* (X, D, b) *be a complete b-metric space, and let* $T : X \to X$ *be a given mapping. Assume that there exists some* $\lambda \in [0, \frac{1}{b(1+b)})$ *and a nonnegative real number* L *such that*

$$D\left(Tu, Tv\right) \leq \lambda D(u, Tv) + LD(v, Tu) \quad \text{for all } u, v \in X. \tag{2.4}$$

Then T *has a unique fixed point* z *and for every* $u_0 \in X$ *the Picard sequence* $\{T^n u_0\}$ *converges to* z.

Proof. Let $\{u_n\}$ be the sequence of Picard starting at $u_0 \in X$. By using (2.4) with $u = u_{n-1}$ and $v = u_n$, we see that

$$D(u_n, u_{n+1}) \leq \frac{b\lambda}{1 - b\lambda} D(u_{n-1}, u_n) \quad \text{for all } n \in \mathbb{N}.$$

From the previous inequality, since $b\lambda(1 - b\lambda)^{-1} < b^{-1}$, we deduce that $\{u_n\}$ is a Cauchy sequence and hence converges to some $z \in X$. Clearly, z is the unique fixed point of T and for every $u_0 \in X$ the sequence $\{T^n u_0\}$ converges to z. $\qquad\square$

3. Fixed points in the setting of C^*-algebra-valued b-metric spaces

Let $\mathcal{A}$ be a C^*-algebra. An element $a \in \mathcal{A}$ is called positive if $a = a^*$ and in this case the spectrum $\sigma(a)$ of a is a subset of nonnegative real numbers. The set of positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}_+$. We define an order relation $\preceq$ by using $\mathcal{A}_+$, where $a \preceq b$ if $a = b$ or $b - a$ is a positive element. We use the notation $\theta \preceq a$ to denote that a is a positive element, where θ is the zero element in $\mathcal{A}$. Now, we recall some properties of the elements of $\mathcal{A}_+$.

(j) The set $\mathcal{A}_+ = \{a^*a : a \in \mathcal{A}\}$ is a closed cone in $\mathcal{A}$;
(jj) if $\theta \preceq a \preceq b$, then $\|a\| \leq \|b\|$;
(jjj) if $\theta \preceq a \preceq b$, then $\theta \preceq \lambda^* a\lambda \preceq \lambda^* b\lambda$ for all $\lambda \in \mathcal{A}$;
(jv) if $a, b \in \mathcal{A}_+$ and $ab = ba$; then $\theta \preceq ab$.

The concept of C^*-algebra-valued b-metric space was introduced by Ma and Jiang [12] as follows.

Definition 3.1. Let X be a nonempty set. A mapping $d : X^2 \to \mathcal{A}$ is called a C^*-algebra-valued b-metric on X if there exists $b \in \mathcal{A}$, with $I \preceq b$ and $ab = ba$ for all $a \in \mathcal{A}$, such that the following conditions hold:

(1) $\theta \preceq d(u, v)$ for all $u, v \in X$;
(2) $\theta = d(u, v)$ if and only if $u = v$;
(3) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(4) $d(u, v) \preceq b(d(u, z) + d(z, v))$ for all $u, v, z \in X$.

Then $(X, \mathcal{A}, d, b)$ is called a C^*-algebra-valued b-metric space.

The following remark is used to obtain fixed point results in C^*-algebra-valued b-metric spaces.

Remark 3.2. Every C^*-algebra-valued b-metric on a set X induces on X a b-metric D with constant $\|b\|$, where $D : X^2 \to [0, \infty)$ is defined by $D(u, v) = \|d(u, v)\|$ for all $u, v \in X$. To verify that D is a b-metric, it is sufficient to show that the triangle inequality holds. By using (jj), we get

$$\begin{aligned} D(u, v) = \|d(u, v)\| &\leq \|b(d(u, z) + d(z, v))\| \\ &\leq \|b\|(\|d(u, z)\| + \|d(z, v)\|) \\ &= \|b\|(D(u, z) + D(z, v)). \end{aligned}$$

Our first contribution in the framework of C^*-algebra-valued b-metric spaces is the following theorem.

Theorem 3.3. *Let $(X, \mathcal{A}, d)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \to X$ be a given mapping. Assume that there exists $\lambda \in \mathcal{A}$ with $\|\lambda\| < 1$ such that*

$$d(Tu, Tv) \preceq \lambda^* d(u, v) \lambda, \quad \text{for all } u, v \in X. \tag{3.1}$$

Then T has a unique fixed point in X.

Proof. Firstly, by the properties (jj) and (jjj), the condition (3.1) implies

$$\|d(Tu, Tv)\| \leq \|\lambda^*\| \|d(u, v)\| \|\lambda\| = \|\lambda\|^2 \|d(x, y)\|,$$

that is, by Remark 3.2,

$$D(Tu, Tv) \leq \|\lambda\|^2 D(u, v), \quad \text{for all } u, v \in X.$$

The proof of Theorem 3.3 follows by Lemma 2.2. □

Theorem 3.3 generalizes Corollary 40 of [11]. Now, we improve Theorem 41 of [11].

Theorem 3.4. *Let $(X, \mathcal{A}, d)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \to X$ be a surjective mapping. Assume that there exists an invertible element $\lambda \in \mathcal{A}$ with $\|\lambda^{-1}\| < 1$ such that*

$$d(Tu, Tv) \succeq \lambda^* d(u, v) \lambda \quad \text{for any } u, v \in X. \tag{3.2}$$

Then T has a unique fixed point in X.

Proof. By using the same reasoning as in [11], we get that T is injective. In this case the condition (3.2) becomes

$$\lambda^* d(u, v) \lambda = \lambda^* d\left(T^{-1} Tu, T^{-1} Tv\right) \lambda \preceq d(Tu, Tv).$$

Thus by (jjj), we have

$$d\left(T^{-1}Tu, T^{-1}Tv\right) \preceq (\lambda^*)^{-1} d\left(Tu, Tv\right)\lambda^{-1} = \left(\lambda^{-1}\right)^* d\left(Tu, Tv\right)\lambda^{-1}.$$

Further, by Remark 3.2 and the injectivity of T, we get

$$D\left(T^{-1}u, T^{-1}v\right) \leq \left\|\lambda^{-1}\right\|^2 D\left(u, v\right) \quad \text{for all } u, v \in X.$$

According to Lemma 2.2, the mapping T^{-1} has a unique fixed point, and hence there is a unique fixed point of T. The theorem is proved. □

From Lemma 2.4, we derive the following result.

Theorem 3.5. *Let $(X, \mathcal{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \to X$ be a given mapping. Assume that there exists $\lambda \in \mathcal{A}$ with $\|\lambda\| < \frac{1}{\sqrt{3\|b\|}}$ such that*

$$d\left(Tu, Tv\right) \preceq \lambda^* \left[d(u, v) + d\left(u, Tu\right) + d\left(v, Tv\right)\right]\lambda, \quad \text{for all } u, v \in X. \tag{3.3}$$

Then T has a unique fixed point in X.

Proof. From (3.3), by using (jj) and (jjj), we obtain

$$\begin{aligned}
\|d\left(Tu, Tv\right)\| &\leq \left\|\lambda^* \left[d(u, v) + d\left(u, Tu\right) + d\left(v, Tv\right)\right]\lambda\right\| \\
&\leq \|\lambda^*\|(\|d(u, v)\| + \|d\left(u, Tu\right)\| + \|d\left(v, Tv\right)\|)\|\lambda\| \\
&= \|\lambda^*\|\|\lambda\|(\|d(u, v)\| + \|d\left(u, Tu\right)\| + \|d\left(v, Tv\right)\|).
\end{aligned}$$

Now, by using Remark 3.2, we get

$$D(Tu, Tv) \leq 3\|\lambda\|^2 \max\{D(u, v), D(u, Tu), D(v, Tv)\} \quad \text{for all } u, v \in X.$$

Thus Lemma 2.4 ensures that T has a unique fixed point in X. □

Similarly, from Lemmas 2.5 and 2.6, we obtain the following results.

Theorem 3.6. *Let $(X, \mathcal{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \to X$ be a given mapping. Assume that there exists $\lambda \in \mathcal{A}_+$ with $\|\lambda\| < \frac{1}{2\|b\|}$ and $\lambda a = a\lambda$ for all $a \in \mathcal{A}_+$ such that*

$$d\left(Tu, Tv\right) \preceq \lambda\left(d\left(u, Tu\right) + d\left(v, Tv\right)\right), \quad \text{for all } u, v \in X.$$

Then T has a unique fixed point in X.

Theorem 3.7. *Let $(X, \mathcal{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, and let $T : X \to X$ be a given mapping. Assume that there exist $\lambda, L \in \mathcal{A}_+$ with $\|\lambda\| < \frac{1}{\|b\|(1+\|b\|)}$ and $\lambda a = a\lambda$, $La = aL$ for all $a \in \mathcal{A}_+$ such that*

$$d\left(Tu, Tv\right) \preceq \lambda d\left(u, Tv\right) + L d\left(v, Tu\right), \quad \text{for all } u, v \in X.$$

Then T has a unique fixed point in X.

The following Examples 3.8 and 3.9 support Theorems 3.6 and 3.7, respectively.

Example 3.8. Let $\mathcal{A} = M_{2 \times 2}(\mathbb{R})$ be endowed with the norm $\|A\| = \max_{i,j} |a_{ij}|$, where a_{ij} are the entries of the matrix $A \in M_{2 \times 2}(\mathbb{R})$, and the involution given by $A^* = \left(\overline{A}\right)^T = A^T$. Clearly, each matrix of the type $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ belongs to $\mathcal{A}_+$ if $\alpha, \beta \geq 0$. This implies $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \preceq \begin{bmatrix} \delta & 0 \\ 0 & \gamma \end{bmatrix}$ if and only if $\alpha \leq \delta$ and $\beta \leq \gamma$.

Let $X = [-1, 1]$, $b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and

$$d(u, v) = \begin{bmatrix} |u - v|^2 & 0 \\ 0 & |u - v|^2 \end{bmatrix} = |u - v|^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |u - v|^2 \, 1_{\mathcal{A}}. \tag{3.4}$$

Then $(X, \mathcal{A}, d, b)$ is a C^*-algebra-valued b-metric space. Define a mapping $T : X \to X$ by $Tu = -\frac{u}{5}$. For all $u, v \in X$, we have

$$d(Tu, Tv) = \frac{1}{25}(u - v)^2 \, 1_{\mathcal{A}} \text{ as well as}$$

$$d(u, Tu) + d(v, Tv) = \left[\left(u + \frac{u}{5}\right)^2 + \left(v + \frac{v}{5}\right)^2 \right] 1_{\mathcal{A}}$$

$$= \left[\frac{36u^2}{25} + \frac{36v^2}{25} \right] 1_{\mathcal{A}}.$$

Putting $\lambda = \frac{2}{25} 1_{\mathcal{A}} = \begin{bmatrix} \frac{2}{25} & 0 \\ 0 & \frac{2}{25} \end{bmatrix}$, we get

$$d(Tu, Tv) \preceq \lambda \left(d(u, Tu) + d(v, Tv) \right)$$

$$\Leftrightarrow \frac{1}{25}(u - v)^2 \leq \frac{2}{25} \left(\frac{36u^2}{25} + \frac{36v^2}{25} \right)$$

$$\Leftrightarrow 0 \leq 47u^2 + 47v^2 + 50uv,$$

and the last inequality is true for all $u, v \in X$.

Since $\|\lambda\| = \frac{2}{25} \|1_{\mathcal{A}}\| < \frac{1}{4\|1_{\mathcal{A}}\|} = \frac{1}{2 \cdot 2\|1_{\mathcal{A}}\|} = \frac{1}{2\|b\|}$, we obtain that T satisfies the contractive condition of Theorem 3.6 and hence T has a unique fixed point in X, that is, $F(T) = \{0\}$.

Example 3.9. Let $\mathcal{A} = M_{2 \times 2}(\mathbb{R})$ be endowed with the norm and the involution considered in Example 3.8. Let $X = [-1, 1]$, $b = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, $1 < k < \frac{\sqrt{13} - 1}{2}$, and $d : X \times X \to \mathcal{A}$ be given as in (3.4). Also, define a mapping $T : X \to X$ by $Tu = -\frac{u}{4}$ for all $u \in X$. Putting $\lambda = \frac{1}{3} 1_{\mathcal{A}}$, we obtain that

$$d(Tu, Tv) \preceq \lambda d(u, Tv) + L d(v, Tu) \tag{3.5}$$

for all $u, v \in X$. Indeed, if we choose $3^{-1}1_{\mathcal{A}} \preceq L$, (3.5) follows by

$$\frac{1}{16}|u-v|^2 \le \frac{1}{3}\left(\left(u+\frac{v}{4}\right)^2 + \left(v+\frac{u}{4}\right)^2\right)$$

$$= \frac{1}{3 \cdot 16}\left(17u^2 + 17v^2 + 16uv\right)$$

$$\Leftrightarrow 0 \le 7u^2 + 7v^2 + 11uv,$$

and the last inequality is true for all $u, v \in X$.

Now, $\|\lambda\| = \frac{1}{3} < \frac{1}{\|b\|(1+\|b\|)} = \frac{1}{k(1+k)} \Leftrightarrow 0 < k < \frac{\sqrt{13}-1}{2}$, which is true since $1 < k < \frac{\sqrt{13}-1}{2}$. Thus all the conditions of Theorem 3.7 are satisfied. This means that T has a unique fixed point. Here, it is $z = 0$.

4. FIXED POINTS OF CYCLIC MAPPINGS

Let X be a non-empty set and A, B two subsets of X. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $TA \subset B$ and $TB \subset A$. First, we recall some fixed point results for cyclic mappings in the setting of b-metric spaces, which one can easily get from the corresponding non-cyclic versions.

Theorem 4.1. *Let (X, D, b) be a complete b-metric space, and let A and B be two nonempty closed subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping and that there exists some $\lambda \in [0, 1)$ such that*

$$D(Tu, Tv) \le \lambda D(u, v), \quad \text{for all } u \in A, v \in B.$$

Then T has a unique fixed point in $A \cap B$.

Proof. Let $k \in \mathbb{N}$ be such that $\lambda^{2k} < b^{-1}$. Note that $S = T^{2k+1}$ is a cyclic mapping on $A \cup B$ such that

$$D(Su, Sv) \le \lambda^{2k}D(u, v), \quad \text{for all } u \in A, v \in B.$$

Now, fix $u_0 \in A$ and consider the Picard sequence $\{u_n\}$ generated by S, starting at u_0. As for each $n \in \mathbb{N}$ the elements u_{n-1} and u_n belong one to the set A and the other to the set B, by the previous inequality, we get

$$D(u_n, u_{n+1}) \le \lambda^{2k}D(u_{n-1}, u_n), \quad \text{for all } n \in \mathbb{N}.$$

This condition ensures that $\{u_n\}$ is a Cauchy sequence and so there exists some $z \in X$ such that $u_n \to z$ as $n \to \infty$. Since $u_{2n} \in A$ and $u_{2n+1} \in B$ for all $n \in \mathbb{N}$, we get $z \in A \cap B \ne \emptyset$. To conclude, note that $T : A \cap B \to A \cap B$ satisfies the condition of Lemma 2.2 and since $A \cap B$ is complete, we deduce that T has a unique fixed point in $A \cap B$. $\square$

Theorem 4.2. *Let (X, D, b) be a complete b-metric space, and let A and B be two nonempty closed subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping and that there exists some $\lambda \in [0, b^{-1})$ such that*

$$D(Tu, Tv) \le \lambda \max\{D(u, v), D(u, Tu), D(v, Tv)\} \quad \text{for all } u \in A, v \in B.$$

Then T has a unique fixed point in $A \cap B$.

Proof. As in the proof of Lemma 2.4, we get that every Picard sequence starting at a point $u_0 \in A$ is a Cauchy sequence. Consequently, there exists some $z \in X$ which is the limit of such sequence. Obviously $z \in A \cap B \neq \emptyset$. Now, one can apply Lemma 2.4 to $T : A \cap B \to A \cap B$ for concluding that T has a unique fixed point in $A \cap B$.   □

From Theorem 4.2, we obtain the following result.

Theorem 4.3. *Let (X, D, b) be a complete b-metric space, and let A and B be two nonempty closed subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping and that there exists some $\lambda \in [0, \frac{1}{2b})$ such that*

$$D\,(Tu, Tv) \leq \lambda(D(u, Tu) + D(v, Tv)) \quad \text{for all } u \in A, v \in B.$$

Then T has a unique fixed point in $A \cap B$.

Similarly, we establish the cyclic version of Lemma 2.6.

Theorem 4.4. *Let (X, D, b) be a complete b-metric space, and let A and B be two nonempty closed subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic mapping and that there exists some $\lambda \in [0, \frac{1}{b(1+b)})$ and a nonnegative real number L such that*

$$D\,(Tu, Tv) \leq \lambda D(u, Tv) + LD(v, Tu) \quad \text{for all } u \in A, v \in B.$$

Then T has a unique fixed point in $A \cap B$.

From Theorem 4.1, we obtain the following cyclic version of Theorem 3.3 in the setting of C^*-algebra-valued b-metric spaces.

Theorem 4.5. *Let $(X, \mathcal{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let A and B be two nonempty closed subsets of X, and let $T : A \cup B \to A \cup B$ be a cyclic mapping. Assume that there exists $\lambda \in \mathcal{A}$ with $\|\lambda\| < 1$ such that*

$$d(Tu, Tv) \preceq \lambda^* d(u, v)\lambda, \quad \text{for all } u \in A, v \in B.$$

Then T has a unique fixed point in $A \cap B$.

Note that Theorem 4.5 generalizes Theorem 38 of [11]. Now, we give the cyclic versions of Theorems 3.6 and 3.7 which are simple consequences of Theorems 4.3 and 4.4, respectively.

Theorem 4.6. *Let $(X, \mathcal{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let A and B be two nonempty closed subsets of X, and let $T : A \cup B \to A \cup B$ be a cyclic mapping. Assume that there exists $\lambda \in \mathcal{A}_+$ with $\|\lambda\| < \frac{1}{2\|b\|}$ and $\lambda a = a\lambda$ for all $a \in \mathcal{A}_+$ such that*

$$d\,(Tu, Tv) \preceq \lambda(d\,(u, Tu) + d\,(v, Tv)), \quad \text{for any } u \in A, \ v \in B,$$

Then T has a unique fixed point in $A \cap B$.

Theorem 4.7. *Let $(X, \mathcal{A}, d, b)$ be a complete C^*-algebra-valued b-metric space, let A and B be two nonempty closed subsets of X, and let $T : A \cup B \to A \cup B$ be*

a cyclic mapping. Assume that there exist $\lambda, L \in \mathcal{A}_+$ *with* $\|\lambda\| < \frac{1}{\|b\|(1+\|b\|)}$ *and* $\lambda a = a\lambda$, $La = aL$ *for all* $a \in \mathcal{A}_+$ *such that*

$$d(Tu, Tv) \preceq \lambda d(u, Tv) + L d(v, Tu), \quad \text{for any } u \in A, \ v \in B.$$

Then T *has a unique fixed point in* $A \cap B$.

Theorem 4.7 generalizes Theorem 44 of [11]. Next we give the cyclic version of Theorem 3.5.

Theorem 4.8. *Let* $(X, \mathcal{A}, d, b)$ *be a complete* C^*-*algebra-valued* b-*metric space, let* A *and* B *be two nonempty closed subsets of* X, *and let* $T : A \cup B \to A \cup B$ *be a cyclic mapping. Assume that there exists* $\lambda \in \mathcal{A}$ *with* $\|\lambda\| < \frac{1}{\sqrt{3\|b\|}}$ *such that*

$$d(Tu, Tv) \preceq \lambda^* [d(u, v) + d(u, Tu) + d(v, Tv)] \lambda, \quad \text{for all } u \in A, v \in B.$$

Then T *has a unique fixed point in* $A \cap B$.

Remark 4.9. Putting $A = B = X$ in each of Theorems 4.5, 4.6, 4.7 and 4.8, we obtain Theorems 3.3, 3.6, 3.7 and 3.5, respectively. This shows that each true cyclic type extension is in fact a generalization of usual non-cyclic type assertion.

Remark 4.10. By using the same reasoning as in [15, 16], it follows easily that Theorem 4.5 and Theorem 3.3 are equivalent results.

The following result is new and complements Theorem 42 of [11].

Theorem 4.11. *Theorem 42 of* [11]*, that is, Theorem 4.6 and Theorem 3.6 are equivalent.*

Proof. It is sufficient to prove that Theorem 42 of [11] is a consequence of Theorem 3.6. As in the proof of Theorem 4.2, we firstly get that $A \cap B \neq \emptyset$. Further, since $(A \cap B, \mathcal{A}, d, b)$ is now a complete C^*-algebra-valued b-metric space, the result follows by Theorem 3.6. □

The next result improves and complements Theorem 44 of [11]. Its proof is omitted.

Theorem 4.12. *Theorem 4.7 and Theorem 3.7 are equivalent.*

Remark 4.13. It is not hard to verify that Examples 3.8 and 3.9 satisfy all the conditions of Theorems 42 and 44 of [11], respectively. Also, it is easy to check the contractive conditions (75) and (96) of [11]. All this shows that cyclic results in many cases are not generalizations of ordinary fixed point results. In particular, Examples 39, 43 and 45 of [11] can be treated, and the existence of a fixed point proved, in a much easier way without using results of the paper [11].

5. Conclusion

If a certain (non-cyclic) fixed point result in the framework of a complete C^*-algebra-valued metric space (respectively, C^*-algebra-valued b-metric space) is known, in order to obtain the respective cyclic-type fixed point result in the same framework, it is enough to prove that the respective cyclic contractive condition

implies that $A \cap B \neq \emptyset$. Indeed, in this case $(A \cap B, \mathcal{A}, d)$ is a complete C^*-algebra-valued metric space (respectively, C^*-algebra-valued b-metric space) and the restriction of T to $A \cap B$ satisfies the given standard condition. In other words, if some ordinary fixed point result in the framework of a complete C^*-algebra-valued metric space (respectively, C^*-algebra-valued b-metric space) has a true cyclic-type extension, then these results are equivalent.

It is worth noticing, that Theorems 4.5, 4.6 and 4.7, are also true, if we suppose that only one of A, B is closed ([8], Remarks 1 and 3). However, in this case the cyclic and non-cyclic versions of this assertion are not equivalent anymore. Further, the approach in this paper shows that many results in the framework of C^*-algebra-valued b-metric spaces are immediate consequences of the corresponding ones in standard b-metric spaces. This follows immediately since each C^*-algebra-valued b-metric space is in fact a cone b-metric space over normal cone with the normal constant equal to 1. Hence, a cyclic result is either not true or it is equivalent to the corresponding ordinary fixed point result. In this sense the results in [11] are superfluous.

Finally we pose an open question as follows.

Problem 5.1. Prove or disprove the following claim: The normal cone $\mathcal{A}_h = \{u \in \mathcal{A} : u = u^*\}$ in each C^*-algebra $\mathcal{A}$ is solid.

References

1. H. H. Alsulami, R. P. Agarwal, E. Karapinar, and F. Khojasteh, *A short note on C^*-valued contraction mappings* J. Inequal. Appl. **(2016)**, 2016:50, 3 pp.
2. S. Batul and T. Kamran, *C^*-valued contractive type mappings*, Fixed Point Theory Appl. **(2015) 2015**:142, 9 pp.
3. S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostraviensis **1** (1993), 5–11.
4. K. R. Davidson, *C^*-Algebras by Example*, Fields Institute Monographs, Vol. 6, American Mathematical Society, 1996.
5. R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, NY, USA, 1972.
6. N. V. Dung and V. T. Le Hang, *On relaxations of contraction constants and Caristi's theorem in b-metric spaces*, J. Fixed Point Theory Appl. **18** (2016), 267–284.
7. M. Jovanović, Z. Kadelburg, and S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl. **2010**, Article ID 978121, 15 pp.
8. Z. Kadelburg, S. Radenović, and J. Vujaković, *A note on the paper "Fixed point theorems for cyclic weak contractions in compact metric spaces"*, Fixed Point Theory Appl. **2016**, 2016:46, 6 pp.
9. T. Kamran, M. Postolache, A. Ghiura, S. Batul, and R. Ali, *The Banach contraction principle in C^*-algebra-valued b-metric spaces with application*, Fixed Point Theory Appl. **2016**, 2016:10, 7 pp.
10. W. A. Kirk, P. S. Srinavasan, and P. Veeramani, *Fixed points for mapping satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.

11. C. Klin-eam and P. Kaskasem, *Fixed point theorems for cyclic contractions in C^*-algebra-valued b-metric spaces*, J. Funct. Spaces **2016** (2016), Article ID 7827040, 16 pp.

12. Z. Ma and L. Jiang, *C^*-algebra-valued b-metric spaces and related fixed point theorems*, Fixed Point Theory Appl. **2015**, 2015:222, 12 pp.

13. Z. Ma, L. Jiang and H. Sun, *C^*-algebra-valued metric spaces and related fixed point theorems*, Fixed Point Theory Appl. **2014**, 2014:206, 11 pp.

14. G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, 1990.

15. S. Radenović, *A note on fixed point theory for cyclic φ-contractions*, Fixed Point Theory Appl. **2015**, 2015:189, 9 pp.

16. S. Radenović, T. Došenović, T. Aleksić-Lampert, and Z. Golubović, *A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations*, Appl. Math. Comput. **273** (2016), 155–164.

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