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THE AHSP IS INHERITED BY E-SUMMANDS

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ABSTRACT. In this short note we prove that the Approximate Hyperplane Series property (AHSp) is hereditary to E-summands via characterizing the E-projections.

1. Introduction and basic definitions

A projection on a Banach space X is a continuous, linear and idempotent map $P : X \to X$. Its dual operator $P^* : X^* \to X^*$ is also a projection. The complementary projection of P is defined as $I - P$, which is also a projection. Every non-zero projection has norm greater than or equal to 1. An M-projection is a projection of norm M and an (M, N)-projection is a projection of norm M whose complementary projection has norm N. A particular case of $(1, 1)$-projections are the E-projections. A projection on X is said to be an E-projection if there exists a 2-dimensional Banach space $E := (\mathbb{R}^2, \|\cdot\|_E)$ such that $\{(1, 0), (0, 1)\}$ is a normalized 1-unconditional basis and $\|x\| = \|(\|P(x)\|, \|(I - P)(x)\|)\|_E$ for each $x \in X$. All ℓ_p-projections, for $1 \leq p \leq \infty$, are E-projections but the converse is not true.

The AHSp was originally studied in 2008. What follows is an equivalent formulation.

Definition 1.1. [1, Remark 3.2] Let X be a Banach space. We say that X satisfies the AHSp if for every $\varepsilon > 0$ there exist $\gamma_X(\varepsilon) > 0$ and $\eta_X(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0} \gamma_X(\varepsilon) = 0$ such that for every sequence $(x_k)_{k\in\mathbb{N}} \subset \mathsf{S}_X$ and every convex

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series $\sum_{k=1}^{\infty} \alpha_k$ with

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta_X(\varepsilon),$$

there are a subset $A \subseteq \mathbb{N}$ with $\sum_{k \in A} \alpha_k > 1 - \gamma_X(\varepsilon)$, an element $x^* \in \mathsf{S}_{X^*}$, and $(z_k)_{k\in A} \subseteq (x^*)^{-1}(1) \cap \mathsf{B}_X$ such that $\|z_k - x_k\| < \varepsilon$ for all $k \in A$.

The AHSp appears in the characterization of the Bishop-Phelps-Bollobás property for operators (BPBp) when the first space of the pair is fixed to ℓ_1. We refer the reader to [1, 2, 3] for a wider perspective on the AHSp and the BPBp. In [2] it was shown that the AHSp is stable under finite ℓ_p-sums for $p \in [1, \infty]$. In particular, the AHSp is hereditary to ℓ_p-complemented subspaces (see [2, Proposition 2.1]). It was also shown (see [2, Remark 2.2]) that the AHSp is not hereditary to general closed subspaces. Here we will show that the AHSp is inherited by E-summands.

2. Main results

To achieve our main result we need some lemmas.

Lemma 2.1. *Let X be a Banach space and $P : X \to X$ a projection on X. The following conditions are equivalent:*

(1) *P is an E-projection.*
(2) $\|m^*\| \|m\| + \|n^*\| \|n\| \leq \|m^* + n^*\| \|m + n\|$ *for all* $m \in P(X)$, $n \in \ker(P)$, $m^* \in P^*(X^*)$, $n^* \in \ker(P^*)$.

Proof. (1)⇒(2) Observe that

$$\begin{aligned} \|m^*\| \|m\| + \|n^*\| \|n\| &= \langle (\|m\|, \|n\|), (\|m^*\|, \|n^*\|) \rangle \\ &\leq \|(\|m^*\|, \|n^*\|)\|_{E^*} \|(\|m\|, \|n\|)\|_E \\ &= \|m^* + n^*\| \|m + n\| \ . \end{aligned}$$

(2)⇒(1) For an arbitrary $(a, b) \in \mathbb{R}^2$ denote

$$\begin{aligned} \|(a, b)\|_E \ &:= \ \sup\{\|m + n\| : m \in P(X), n \in \ker(P) \\ &\qquad \|m\| = |a|, \|n\| = |b|\}. \end{aligned}$$

Evidently, we have $\|(a, b)\|_E = \|(|a|, |b|)\|_E$ for every $(a, b) \in \mathbb{R}^2$ and $\|(1, 0)\|_E = \|(0, 1)\|_E = 1$. Thus, $\{(1, 0), (0, 1)\}$ is a normalized 1-unconditional basis. It remains to show that $\|x\| = \|(\|P(x)\|, \|(I - P)(x)\|)\|_E$ for all $x \in X$. The inequality $\|x\| \leq \|(\|P(x)\|, \|(I - P)(x)\|)\|_E$ follows directly from the definition of $\|(a, b)\|_E$. Fix an arbitrary $\varepsilon > 0$. Choose $m \in P(X)$ and $n \in \ker(P)$ with $\|m\| = \|P(x)\|$, $\|n\| = \|(I-P)(x)\|$ and $\|(\|P(x)\|, \|(I - P)(x)\|)\|_E - \varepsilon \leq \|m+n\|$. Next, select a supporting functional $y^* \in \mathsf{S}_{X^*}$ at $m + n$, that is, $y^*(m + n) =$

$\|m+n\|$. By hypothesis we have that

$$\begin{aligned} \|x\| &\geq \|P^*(y^*)\|\|P(x)\| + \|(I-P)^*(y^*)\|\|(I-P)(x)\| \\ &= \|P^*(y^*)\|\|m\| + \|(I-P)^*(y^*)\|\|n\| \\ &\geq P^*(y^*)(m) + (I-P)^*(n) \\ &= y^*(m+n) \\ &= \|m+n\| \\ &\geq \|(\|P(x)\|, \|(I-P)(x)\|)\|_E - \varepsilon. \end{aligned}$$

□

We say that a functional $x^* \in X^*$ attains its norm at $x \in X$ whenever $x^*(x) = \|x^*\|\|x\|$.

Lemma 2.2. *Let $P : X \to X$ be an E-projection on a Banach space X and $m \in P(X)$, $n \in \ker(P)$, $m^* \in P^*(X^*)$, $n^* \in \ker(P^*)$. If $m^* + n^*$ attains its norm at $m + n$, then m^* and n^* attain their norm at m and n respectively.*

Proof. In virtue of Lemma 2.1, we have that

$$\begin{aligned} \|m^* + n^*\| \, \|m+n\| &= (m^* + n^*)\,(m+n) \\ &= m^*\,(m) + n^*\,(n) \\ &\leq \|m^*\| \, \|m\| + \|n^*\| \, \|n\| \\ &\leq \|m^* + n^*\| \, \|m+n\|, \end{aligned}$$

which implies that $m^*(m) = \|m^*\|\|m\|$ and $n^*(n) = \|n^*\|\|n\|$. □

Theorem 2.3. *Let X be a Banach space. If X has the AHSp, then every E-summand subspace M of X also has the AHSp.*

Proof. We will show that M satisfies the AHSp with $\gamma_M(\varepsilon) := \gamma_X(\varepsilon/2)$ and $\eta_M(\varepsilon) := \eta_X(\varepsilon/2)$ for all $\varepsilon > 0$. So, fix an arbitrary $0 < \varepsilon < 1$ and consider $(x_k)_{k\in\mathbb{N}} \subset \mathsf{S}_M$ and a convex series $\sum_{k=1}^{\infty} \alpha_k$ satisfying

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta_M(\varepsilon) = 1 - \eta_X\left(\frac{\varepsilon}{2}\right).$$

Since X enjoys the AHSp, there exist $A \subseteq \mathbb{N}$, $x^* \in \mathsf{S}_{X^*}$, and $(z_k)_{k\in A} \subset (x^*)^{-1}(1) \cap \mathsf{B}_X$ such that

$$\sum_{k\in A} \alpha_k > 1 - \gamma_X\left(\frac{\varepsilon}{2}\right) = 1 - \gamma_M(\varepsilon) \text{ and } \|z_k - x_k\| < \frac{\varepsilon}{2} \text{ for all } k \in A.$$

For every $k \in A$ we can write $z_k = m_k + n_k$ with $m_k \in M$ and $n_k \in N$, where N denotes the E-summand complement of M in X. Suppose that $m_k = 0$ for some $k \in A$. Then

$$1 = \|x_k\| \leq \|n_k - x_k\| = \|z_k - x_k\| < \frac{\varepsilon}{2},$$

which contradicts our assumption on ε. Observe also that for every $k \in A$ we have

$$\begin{aligned}\left\|x_k - \frac{m_k}{\|m_k\|}\right\| &\leq \|x_k - m_k\| + \left\|m_k - \frac{m_k}{\|m_k\|}\right\| \\ &\leq \|x_k - m_k\| + |1 - \|m_k\|| \\ &= \|x_k - m_k\| + |\|x_k\| - \|m_k\|| \\ &\leq 2\|x_k - m_k\| \\ &\leq 2\|x_k - z_k\| \\ &< \varepsilon.\end{aligned}$$

Now, since $X^* = M^* \oplus N^*$, we can write $x^* = m^* + n^*$ where $m^* \in M^*$ and $n^* \in N^*$. Suppose now that $m^* = 0$. Then for every $k \in A$ we have

$$1 = x^*(z_k) - x^*(x_k) \leq \|z_k - x_k\| < \frac{\varepsilon}{2},$$

which is impossible. Finally, Lemma 2.2 applies to ensure that $m^*(m_k) = \|m^*\| \|m_k\|$ for every $k \in A$. As a consequence, $\frac{m^*}{\|m^*\|} \in \mathsf{S}_{M^*}$ and $\left\{\frac{m_k}{\|m_k\|} : k \in A\right\} \subseteq \mathsf{S}_M$ satisfy the conditions of Definition 1.1.

□

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