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THE DETERMINANT FORMULAE FOR THE RODRIGUES GENERAL PROBLEM WHEN THE EIGENVALUES HAVE DOUBLE MULTIPLICITY

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Abstract. We discuss the general Rodrigues problem and we give explicit determinant formulae for the coefficients when the eigenvalues of the matrix have double multiplicity (Theorem 5). When $n = 4$ explicit formulae and effective computations for the exponential map on the Lie group $\mathbf{SO(4)}$ are presented.

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1. Introduction

The exponential map $\exp : gl(n, K) \to GL(n, K)$, where $K = \mathbb{C}$ or $K = \mathbb{R}$, and $GL(n, K)$ denotes the Lie group of the invertible $n \times n$ matrices having the entrees in K, is defined by (9). According to the well-known Hamilton-Cayley Theorem, it follows that every power $X^k, k \geq n$, of the matrix $X \in M_n(K)$, is a linear combination of $X^0 = I_n, X^1, \ldots, X^{n-1}$, hence $\exp(X)$ can be written as in (10), where the coefficients $a_0(X), \ldots, a_{n-1}(X)$ are uniquely defined and depend on X . Inspired by the classical Rodrigues formula (11) for the special orthogonal group $\mathbf{SO(3)}$, we call these numbers the Rodrigues coefficients of the exponential map with respect to the matrix $X \in M_n(K)$.

Considering an analytic function $f(z)$ defined on an open disk containing the spectrum of the matrix $X \in M_n(K)$ and replace z by X we obtain the matrix function $\tilde{f}(X)$ defined as power series (see Section 2). Similarly, we obtain the reduced formula (12), where the coefficients are called the Rodrigues coefficients of $\tilde{f}$ with respect to the matrix X.

In this paper we discuss the Rodrigues problem, that is the problem of determining the Rodrigues coefficients. Section 2 is dealing with a breaf review of matrix functions. We recall the definition of a matrix function by using the Jordan canonical form, by the Hermite's interpolation polynomial, by the Cauchy integral formula, and by a series. The connection betweeen these definitions is given in Theorem 1. Section 3 is devoted to the study of the Rodrigues general problem in terms of the spectrum of the matrix X and it discuss the case when the eigenvalues of the matrix X are pairwise distinct (Theorem 2). Section 4 illustrates the importance of the Hermite interpolation polynomial in solving the Rodrigues general problem when the eigenvalues have double multiplicity (Theorem 4). The determinant formulae for the Rodrigues coefficients, in this case, are presented in Theorem 5 of Section 5. The last section contains the explicit formulae when $n = 4$ and effective computations for the exponential map on the Lie group $\mathbf{SO(4)}$.

2. Short introduction on matrix functions

The concept of matrix function plays an important role in many domains of mathematics with numerous applications in science and engineering, especially in control theory and in the theory of the differential equations in which $\exp(tA)$ has an important role.

A matrix function can be defined in different ways, the following four being the most useful for the developments in this presentation.

1. Using the Jordan canonical form

Let f be defined on a neighborhood of the spectrum of $A \in M_n(\mathbb{C})$. If A has the Jordan canonical form J, then

$$f(A) = Xf(J)X^{-1} = X\mathrm{diag}(f(J_k(\lambda_k)))X^{-1} \tag{1}$$

where

$$f(J_k) = f(J_k(\lambda_k)) = \begin{pmatrix} f(\lambda_k) & f'(\lambda_k) & \dots & \frac{f^{(n_k-1)}(\lambda_k)}{(n_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k) \end{pmatrix} \tag{2}$$

The right member of the relation (2) is independent of the choice of X and J.

2. Using Hermite interpolation

Let f be defined on the spectrum of $A \in M_n(\mathbb{C})$. Then

$$f(A) = r(A), \tag{3}$$

where r is the Hermite interpolation polynomial that satisfies the interpolation conditions

$$r^{(j)}(\lambda_i) = f^{(j)}(\lambda_j), i = 1, \ldots, s, j = 0, \ldots, n_i - 1,$$

and $\lambda_1, \ldots, \lambda_s$ are the distinct eigenvalues of A with the multiplicities $n_1, \ldots n_s$. The standard implicit form of this polynomial is

$$r(t) = \sum_{i=1}^{s} \left[\left(\sum_{j=0}^{n_i-1} \frac{1}{j!} \Phi_i^{(j)}(\lambda_i)(t-\lambda_i)^j \right) \prod_{j \neq i} (t-\lambda_j)^{n_j} \right] \quad (4)$$

where $\Phi_i(t) = f(t)/\prod_{j\neq i}(t-\lambda_j)^{n_j}$. Formula (3) is connected to the well-known Schwerdtfeger formula for matrix functions [9].

If the eigenvalues of the matrix X are pairwise distinct, then the Hermite polynomial r is reduced to Lagrange interpolation polynomial with conditions $r(\lambda_i) = f(\lambda_i), i = 1, \ldots, n$,

$$r(t) = \sum_{i=1}^{n} f(\lambda_i) l_i(t), \quad (5)$$

where l_i are the Lagrange fundamental polynomials defined by

$$l_i(t) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{t-\lambda_j}{\lambda_i - \lambda_j}, i = 1, \ldots, n. \quad (6)$$

3. Using Cauchy's integral formula

Let $\Omega \subset \mathbb{C}$ be a domain and $f : \Omega \to \mathbb{C}$ a analytic function. Let $A \in M_n(\mathbb{C})$ be diagonalizable so that all eigenvalues of A are in Ω. We define $f(A) \in M_n(\mathbb{C})$ through

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z)(zI_n - A)^{-1} dz, \quad (7)$$

where $(zI_n - A)^{-1}$ is the resolvent of A in z and $\Gamma \subset \Omega$ is a simple closed curve around the spectrum $\sigma(A)$, oriented in the opposite trigonometric direction.

4. Matrix functions defined as power series

If the function f is given by $f(z) = \sum_{k=0}^{\infty} c_k z^k$ on an open disk containing the eigenvalues of A, then

$$f(A) = \sum_{k=0}^{\infty} c_k A^k, \quad (8)$$

where we suppose the convergence of the series. The exponential map, considered in various contexts, is the most important example for this definition (see Section 3). Another important example is given by

$$Cay(A) = I_n + 2A + 2A^2 + \cdots ,$$

which defines in some situations the Cayley transform $Cay(A) = (I_n + A)(I_n - A)^{-1}$ (for details see the paper [3]).

The following important result is well-known (see for instance [12]).

Theorem 1. *Let be $A \in M_n(\mathbb{C})$ and let f be an analytical function defined on a domain containing the spectrum of A. Denote*

1. *$f_J(A)$ the matrix $f(A)$ defined with the Jordan canonical form;*
2. *$f_H(A)$ the matrix $f(A)$ defined with the Hermite's interpolation polynomial;*
3. *$f_C(A)$ the matrix $f(A)$ defined with the Cauchy's integral formula.*

Then $f_J(A) = f_H(A) = f_C(A)$.

3. The Rodrigues general problem

The exponential map $\exp : gl(n, \mathbb{C}) = M_n(\mathbb{C}) \to \mathbf{GL}(n, \mathbb{C})$ is defined

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{9}$$

According to the well-known Hamilton-Cayley theorem, it follows that every power X^k, $k \geq n$, is a linear combination of powers X^0, X^1,..., X^{n-1}, hence we can write

$$\exp(X) = \sum_{k=0}^{n-1} a_k(X) X^k, \tag{10}$$

where the real coefficients $a_0(X), \ldots, a_{n-1}(X)$ are uniquely defined and depend on the matrix X. From this formula, it follows that $\exp(X)$ is a polynomial of X with coefficients functions of X.

The problem to find a such formula for $\exp(X)$ is reduced to the problem to determine the coefficients $a_0(X), \ldots, a_{n-1}(X)$. We will call this problem, the *Rodrigues problem*, and the numbers $a_0(X), \ldots, a_{n-1}(X)$ *Rodrigues coefficients* of the exponential map with respect to the matrix $X \in M_n(\mathbb{C})$.

The origin of this problem is the classical Rodriques formula obtained in 1840 for the special orthogonal group $\mathbf{SO(3)}$:

$$\exp(X) = I_3 + \frac{\sin\theta}{\theta}X + \frac{1-\cos\theta}{\theta^2}X^2, \tag{11}$$

where $\sqrt{2}\theta = \|X\|$ and we denoted by $\|X\|$ Frobenius norm of the matrix X. From the many arguments pointing out the importance of this formula we mention here the study of the rigid body rotation in $\mathbb{R}^3$, and the parametrization of the rotations in $\mathbb{R}^3$.

Now, for the general construction, we consider an analytic function f, such that the induced matrix series $\tilde{f}(X)$ is convergent in an open subset of $M_n(\mathbb{C})$. Then, via Hamilton-Cayley-Frobenius Theorem we can write a reduced form for matrix $\tilde{f}(X)$:

$$\tilde{f}(X) = \sum_{k=0}^{n-1} a_k^{(f)}(X)X^k. \tag{12}$$

We call the above relation, the *Rodrigues formula* with respect to f. The numbers $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are the *Rodrigues coefficients* of the map $\tilde{f}$ with respect to the matrix $X \in M_n(\mathbb{C})$. Clearly, the real coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are uniquely defined, they depend on the matrix X, and $\tilde{f}(X)$ is a polynomial of X.

The following result concerning the solution to the general Rodrigues problem with respect to the function f for simple multiplicity is obtained by D. Andrica and O.-L. Chender [2]. The special case $n = 2$ is also discussed in the recent book [15].

Theorem 2. *Assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix X are pairwise distinct. Then we have :*

1) The Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are given by the formulas

$$a_k^{(f)}(X) = \frac{V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)}{V_n(\lambda_1, \ldots, \lambda_n)}, k = 0, \ldots, n-1, \tag{13}$$

where $V_n(\lambda_1, \ldots, \lambda_n)$ is the Vandermonde determinant of order n, and $V_{n,k}^{(f)}(\lambda_1, \ldots, \lambda_n)$ is the determinant of order n obtained from $V_n(\lambda_1, \ldots, \lambda_n)$ by replacing the line $k+1$ by $f(\lambda_1), \ldots, f(\lambda_n)$.

2) The Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are linear combinations of $f(\lambda_1), \ldots, f(\lambda_n)$ having the coefficients rational functions of $\lambda_1, \ldots, \lambda_n$.

There are two ways to prove this result. A direct proof is given in the paper [2] (see also [6] and [7]) and it uses the so-called "trace method" to obtain from relation (9) a linear system with the unknowns $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$.

The second proof is based on the following important result (see [12]):

Theorem 3. *For the polynomials $p, q \in \mathbb{C}[z]$ and $A \in M_n(\mathbb{C})$ we have $p(A) = q(A)$ if and only if p and q take the same values on the spectrum of A.*

From this result and relation (9) it follows that the polynomial $\tilde{f}$ is exactly the Lagrange interpolation polynomial satisfying the conditions $\tilde{f}(\lambda_j) = f(\lambda_j), j = 1, \ldots, n$. Therefore, the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ in (10) are the coefficients of the Lagrange polynomial under the above interpolation conditions.

4. The solution of the Rodrigues problem when the eigenvalues have double multiplicity

In this subsection, we assume that the function f is defined on the spectrum of the matrix $X \in M_{2s}(\mathbb{C})$ and the distinct eigenvalues $\lambda_1, \ldots, \lambda_s$ of X have double multiplicity, that is $n_1 = \cdots = n_s = 2$. In this case the Hermite interpolation polynomial r satisfies the conditions $r(\lambda_i) = f(\lambda_i), r'(\lambda_i) = f'(\lambda_i), i = 1, \ldots, s$. We obtain that the polynomial $\tilde{f}$ is exactly the Hermite interpolation polynomial r satisfying the above conditions, hence the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ in (10) are the coefficients of r. To get the algebraic form of polynomial r in the general case is a difficult problem [13]. After algebraic computations, we get the following result [10]:

Theorem 4. *For any $k = 0, 1, \ldots, n-1$, we have*

$$a_k^{(f)}(X) =$$

$$(-1)^{k+1} \sum_{i=1}^{s} \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{s} (\lambda_i - \lambda_j)^2} \left\{ \left[f'(\lambda_i) - 2f(\lambda_i) \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{\lambda_i - \lambda_j} \right] \sigma_{i,2s-k-1} \right.$$
$$\left. - \left[f(\lambda_i) \left(1 + 2\lambda_i \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{1}{\lambda_i - \lambda_j} \right) - \lambda_i f'(\lambda_i) \right] \sigma_{i,2s-k-2} \right\} \tag{14}$$

In formula (11), $\sigma_{i,k}(\lambda_1, \ldots, \lambda_s) = s_k(\lambda_1, \lambda_1, \ldots, \widehat{\lambda_i}, \widehat{\lambda_i}, \ldots, \lambda_s, \lambda_s)$ is the symmetric polynomial of order k in $2s-2$ variable $\lambda_1, \lambda_1, \ldots, \widehat{\lambda_i}, \widehat{\lambda_i}, \ldots, \lambda_s, \lambda_s$, of which λ_i is missing, for all $k = 1, \ldots, 2s-2$.

5. The determinant formulae

The formulas (11) can be written in a compact and uniform form by using convenient determinants. The starting point is the formula (13) applicated for the function f defined on the spectrum of the matrix $X \in M_{2s}(\mathbb{C})$, which we assume is analytical. We consider the distinct eigenvalues $\lambda_1, \lambda_1', \lambda_2, \lambda_2', \ldots, \lambda_s, \lambda_s'$ of the matrix X and we let successively $\lambda_1' \to \lambda_1, \lambda_2' \to \lambda_2, \ldots, \lambda_s' \to \lambda_s$. Using the derivative formula of a functional determinant and l'Hôpital's rule we obtain the following result:

Theorem 5. *If the eigenvalues $\lambda_1, \ldots, \lambda_s$ of the matrix X are pairwise distinct, then the Rodrigues coefficients $a_0^{(f)}(X), \ldots, a_{n-1}^{(f)}(X)$ are given by*

$$a_k^{(f)}(X) = \frac{1}{\prod\limits_{1 \le i < j \le s} (\lambda_j - \lambda_i)^4} \det U_{n,k}^{(f,f')}(\lambda_1, \ldots, \lambda_s) \tag{15}$$

where $U_{n,k}^{(f,f')}(\lambda_1, \ldots, \lambda_s)$ is the $n \times n$ matrix defined in $n \times 2$ blocks

$$U_{n,k}^{(f,f')}(\lambda_1, \ldots, \lambda_s) = \left(\left[U_k^{(f,f')}(\lambda_1) \right] \ldots \left[U_k^{(f,f')}(\lambda_s) \right] \right)$$

and the block $U_k^{(f,f')}$ is given by

$$U_k^{(f,f')}(\lambda_j) = \begin{pmatrix} 1 & 0 \\ \lambda_j & 1 \\ \vdots & \vdots \\ f(\lambda_j) & f'(\lambda_j) \\ \vdots & \vdots \\ \lambda_j^{n-1} & (n-1)\lambda_j^{n-2} \end{pmatrix}, j = 1, \ldots, s. \tag{16}$$

The entries $f(\lambda_j), f'(\lambda_j)$ are found on the line $k+1$, and the entries situated on the second column are obtained by derivation in relation to λ_j of the corresponding entries on the first column.

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6. Examples

6.1. The general case $n = 4$

To illustrate the formula (15), we consider $n = 4$, the analytical function f is defined on the spectrum of the matrix $X \in M_4(\mathbb{C})$ with distinct eigenvalues λ_1, λ_2, each with double multiplicity. The blocks (16) defined by the 4×2 matrices which give the Rodrigues coefficients $a_0^{(f)}(X), a_1^{(f)}(X), a_2^{(f)}(X), a_3^{(f)}(X)$ from formula (12) are

$$U_0^{(f,f')}(\lambda_1) = \begin{pmatrix} f(\lambda_1) & f'(\lambda_1) \\ \lambda_1 & 1 \\ \lambda_1^2 & 2\lambda_1 \\ \lambda_1^3 & 3\lambda_1^2 \end{pmatrix}, U_0^{(f,f')}(\lambda_2) = \begin{pmatrix} f(\lambda_2) & f'(\lambda_2) \\ \lambda_2 & 1 \\ \lambda_2^2 & 2\lambda_2 \\ \lambda_2^3 & 3\lambda_2^2 \end{pmatrix},$$

$$U_1^{(f,f')}(\lambda_1) = \begin{pmatrix} 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) \\ \lambda_1^2 & 2\lambda_1 \\ \lambda_1^3 & 3\lambda_1^2 \end{pmatrix}, U_1^{(f,f')}(\lambda_2) = \begin{pmatrix} 1 & 0 \\ f(\lambda_2) & f'(\lambda_2) \\ \lambda_2^2 & 2\lambda_2 \\ \lambda_2^3 & 3\lambda_2^2 \end{pmatrix},$$

$$U_2^{(f,f')}(\lambda_1) = \begin{pmatrix} 1 & 0 \\ \lambda_1 & 1 \\ f(\lambda_1) & f'(\lambda_1) \\ \lambda_1^3 & 3\lambda_1^2 \end{pmatrix}, U_2^{(f,f')}(\lambda_2) = \begin{pmatrix} 1 & 0 \\ \lambda_2 & 1 \\ f(\lambda_2) & f'(\lambda_2) \\ \lambda_2^3 & 3\lambda_2^2 \end{pmatrix},$$

$$U_3^{(f,f')}(\lambda_1) = \begin{pmatrix} 1 & 0 \\ \lambda_1 & 1 \\ \lambda_1^2 & 2\lambda_1 \\ f(\lambda_1) & f'(\lambda_1) \end{pmatrix}, U_3^{(f,f')}(\lambda_2) = \begin{pmatrix} 1 & 0 \\ \lambda_2 & 1 \\ \lambda_2^2 & 2\lambda_2 \\ f(\lambda_2) & f'(\lambda_2) \end{pmatrix},$$

Applying formula (12) we obtain

$$a_0^{(f)}(X) = \frac{1}{(\lambda_2 - \lambda_1)^4} \det\left(\left[U_0^{(f,f')}(\lambda_1)\right]\left[U_0^{(f,f')}(\lambda_2)\right]\right) =$$

$$\frac{1}{(\lambda_2 - \lambda_1)^4} \begin{vmatrix} f(\lambda_1) & f'(\lambda_1) & f(\lambda_2) & f'(\lambda_2) \\ \lambda_1 & 1 & \lambda_2 & 1 \\ \lambda_1^2 & 2\lambda_1 & \lambda_2^2 & 2\lambda_2 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_2^3 & 3\lambda_2^2 \end{vmatrix},$$

$$a_1^{(f)}(X) = \frac{1}{(\lambda_2 - \lambda_1)^4} \det\left(\left[U_1^{(f,f')}(\lambda_1)\right]\left[U_1^{(f,f')}(\lambda_2)\right]\right) =$$

$$\frac{1}{(\lambda_2 - \lambda_1)^4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_2) & f'(\lambda_2) \\ \lambda_1^2 & 2\lambda_1 & \lambda_2^2 & 2\lambda_2 \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_2^3 & 3\lambda_2^2 \end{vmatrix},$$

$$a_2^{(f)}(X) = \frac{1}{(\lambda_2 - \lambda_1)^4} \det\left(\left[U_2^{(f,f')}(\lambda_1)\right]\left[U_2^{(f,f')}(\lambda_2)\right]\right) =$$

$$\frac{1}{(\lambda_2-\lambda_1)^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 & 1 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_2) & f'(\lambda_2) \\ \lambda_1^3 & 3\lambda_1^2 & \lambda_2^3 & 3\lambda_2^2 \end{vmatrix},$$

$$a_3^{(f)}(X) = \frac{1}{(\lambda_2-\lambda_1)^4}\det\left(\left[U_3^{(f,f')}(\lambda_1)\right]\left[U_3^{(f,f')}(\lambda_2)\right]\right) =$$

$$\frac{1}{(\lambda_2-\lambda_1)^4}\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & 1 & \lambda_2 & 1 \\ \lambda_1^2 & 2\lambda_1 & \lambda_2^2 & 2\lambda_2 \\ f(\lambda_1) & f'(\lambda_1) & f(\lambda_2) & f'(\lambda_2) \end{vmatrix}.$$

6.2. The exponential map on $\mathbf{SO}(4)$

It is well-known that the set of all real $n \times n$ orthogonal matrices is a Lie group, denoted by $\mathbf{O}(n)$, with respect to the standard multiplication. The subset of $\mathbf{O}(n)$ consisting of those matrices having the determinant equal to $+1$ is a subgroup, denoted by $\mathbf{SO}(n)$ and called the *special orthogonal group* of the Euclidean space $\mathbb{R}^n$. Due to geometric reasons, the elements of $\mathbf{SO}(n)$ are also called *rotation matrices*. The Lie algebra $\mathfrak{so}(n)$ of $\mathbf{SO}(n)$ consists in all skew-symmetric matrices in $M_n(\mathbb{R})$ and the Lie bracket is the standard matrices commutator $[A, B] = AB - BA$. The exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is defined by the same formula (9) because, by the naturality property, it is the restriction $\exp|_{\mathfrak{so}(n)}$ of the exponential map $\exp : \mathfrak{gl}(n, \mathbb{R}) \to \mathbf{GL}(n, \mathbb{R})$.

It is known that for every compact connected Lie group the exponential map is surjective (see T. Bröcker, T. tom Dieck [8] or D. Andrica, I.N. Casu [1] for the standard proof), that is every compact connected Lie group is exponential (see the monograph of M. Wüstner [18] for details about the exponential groups). Because the group $\mathbf{SO(n)}$ is compact it follows that the exponential map $\exp : \mathfrak{so}(n) \to \mathbf{SO}(n)$ is surjective, which is an important property (see also [14]). Indeed, it implies the existence of a locally inverse function $\log : \mathbf{SO}(n) \to \mathfrak{so}(n)$, and this has interesting applications. In the paper of J.Gallier, D.Xu [11] is mentioned that the functions exp and log for the group $\mathbf{SO(n)}$ can be used in the study of motion interpolation. Also, the surjectivity of the exponential map for the group $\mathbf{SO(n)}$ gives the possibility to describe in a natural way the rotations of the Euclidean space $\mathbb{R}^n$ (see R.-A. Rohan [17]). To describe the image of the exponential map of an arbitrary Lie group is a very difficult open problem (see [4] and [5]).

The matrices in $\mathfrak{so}(n)$ have two essential properties which simplify the computation of the Rodrigues coefficients:

- If n is odd, then they are singular, i.e. they have one eigenvalue equal to 0 (possible with a multiplicity);
- The non-zero eigenvalues are purely imaginary and, of course, conjugated.

The general skew-symmetric matrix $X \in so(4)$ is

$$X = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix},$$

and the corresponding characteristic polynomial is given by

$$p_X(t) = t^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)t^2 + (af - be + cd)^2.$$

Let $\pm i\alpha, \pm i\beta$, be the eigenvalues of the matrix X, where $\alpha,\ \beta \in \mathbb{R}$.

To illustrate the formulas obtained in the previous subsection, we consider the situation $\alpha = \beta \neq 0$. That is the distinct eigenvalues are $\lambda_1 = -i\alpha, \lambda_2 = i\alpha$, each with double multiplicity, hence the characteristic polynomial of matrix X is $p_X(t) = t^4 + 2\alpha^2 t^2 + \alpha^4$.

The Rodrigues coefficients $a_0(X), a_1(X), a_2(X), a_3(X)$ of the exponential map with respect to the matrix X are

$$a_0(X) = \frac{1}{(2i\alpha)^4} \begin{vmatrix} e^{-i\alpha} & e^{-i\alpha} & e^{i\alpha} & e^{i\alpha} \\ -i\alpha & 1 & i\alpha & 1 \\ -\alpha^2 & -2i\alpha & -\alpha^2 & 2i\alpha \\ i\alpha^3 & -3\alpha^2 & -i\alpha^3 & -3\alpha^2 \end{vmatrix} = \frac{1}{2}(\alpha \sin\alpha + 2\cos\alpha),$$

$$a_1(X) = \frac{1}{(2i\alpha)^4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ e^{-i\alpha} & e^{-i\alpha} & e^{i\alpha} & e^{i\alpha} \\ -\alpha^2 & -2i\alpha & -\alpha^2 & 2i\alpha \\ i\alpha^3 & -3\alpha^2 & -i\alpha^3 & -3\alpha^2 \end{vmatrix} = \frac{1}{2\alpha}(3\sin\alpha - \alpha\cos\alpha),$$

$$a_2(X) = \frac{1}{(2i\alpha)^4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ -i\alpha & 1 & i\alpha & 1 \\ e^{-i\alpha} & e^{-i\alpha} & e^{i\alpha} & e^{i\alpha} \\ i\alpha^3 & -3\alpha^2 & -i\alpha^3 & -3\alpha^2 \end{vmatrix} = \frac{\sin\alpha}{2\alpha},$$

$$a_3(X) = \frac{1}{(2i\alpha)^4} \begin{vmatrix} 1 & 0 & 1 & 0 \\ -i\alpha & 1 & i\alpha & 1 \\ -\alpha^2 & -2i\alpha & -\alpha^2 & 2i\alpha \\ e^{-i\alpha} & e^{-i\alpha} & e^{i\alpha} & e^{i\alpha} \end{vmatrix} = \frac{1}{2\alpha^3}(\sin\alpha - \alpha\cos\alpha),$$

where we have used simple elementary transformations with determinants and the well-known trigonometric formulas $2i\sin\alpha = e^{i\alpha} - e^{-i\alpha}$, and $2\sin\alpha = e^{i\alpha} + e^{-i\alpha}$.

The corresponding Rodrigues formula is

$$\exp(X) = \frac{1}{2}(\alpha\sin\alpha + 2\cos\alpha)I_4 + \frac{1}{2\alpha}(3\sin\alpha - \alpha\cos\alpha)X + \frac{\sin\alpha}{2\alpha}X^2 + \frac{1}{2\alpha^3}(\sin\alpha - \alpha\cos\alpha)X^3. \tag{17}$$

Formula (17) has been obtained by D. Andrica and R.-A. Rohan [6] by using the Putzer method [16].

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