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MAHLER COEFFICIENTS OF THE 2-ADIC SHIFT

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Abstract. In this work we provide a concrete expression of Mahler coefficients of some locally constant functions on the group of 2-adic integers. We deduce the Mahler coefficients of the 2-adic shift.

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1. Introduction

In [2] locally scaling transformations were described by means of their Mahler coefficients on the ring of p-adic integers $\mathbb{Z}_p$. It was also proved that every locally scaling transformation is topologically isomorphic to the corresponding p-adic shift. Mahler coefficients of locally scaling transformations on $\mathbb{Z}_p$ were also described in [4]. The techniques used in [4] were mainly based on some properties of Mahler coefficients of characteristic functions and their relation with van der Put coefficients. In this paper we use some new techniques that enable us to establish a concrete expression of Mahler coefficients of some test functions on the ring of 2-adic integers $\mathbb{Z}_2$, then deduce an expression of Mahler coefficients of the 2-adic shift.

We recall some facts about the ring of 2-adic integers $\mathbb{Z}_2$. Every $x \in \mathbb{Z}_2$ has the 2-adic representation $x = \sum\limits_{i=0}^{\infty} x_i 2^i$, where for each nonnegative integer i, $x_i \in \{0,1\}$. The 2-adic valuation $\nu_2(x)$ of any 2-adic integer x is defined as the least nonnegative integer i such that $x_i > 0$. It is known that the 2-adic norm $|x|$ of any 2-adic number x is given by $|x| = 2^{-\nu_2(x)}$.

Each set $x + 2^n\mathbb{Z}_2$, $n \geq 1$, is a clopen ball of radius 2^{-n}. Besides, the set $\mathbb{Z}_2$ is the disjoint union of 2^n balls of radius 2^{-n}.

The natural probability measure μ defined on $\mathbb{Z}_2$ gives measure 2^{-n} to any ball $x + 2^n\mathbb{Z}_2$.

For every nonnegative integer k, the 2-adic shift S^k is defined by the formula

$$S_k(x) = \sum_{i=0}^{\infty} x_{i+k} 2^i, \ \ \forall x = \sum_{i=0}^{\infty} x_i 2^i.$$

Every continuous function $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ can be represented by means of its Mahler expansion

$$f(x) = \sum_{i=0}^{\infty} a_i \binom{x}{i},$$

where

$$\binom{x}{i} = \frac{x(x-1)\dots(x-n+1)}{n!},$$

and the 2-adic integers $(a_i)_i$ are called its Mahler coefficients.

It can be easily seen (see for example [1], [3] and [5]) that

$$a_i = \triangle^i f(0), \ \ \forall i \geq 0,$$

where

$$\triangle^0 f = f, \ \ \triangle^{i+1} f(x) = \triangle^i f(x+1) - \triangle^i f(x), \ \ \forall i \geq 0. \tag{1}$$

2. Main results

Definition 1. *For every nonnegative integer k, we define the locally constant function*

$$\delta_k(x) = \begin{cases} 1, & x \in 2^{k+1}\mathbb{Z}_2; \\ -1, & x \in 2^k + 2^{k+1}\mathbb{Z}_2; \\ 0, & x \in \mathbb{Z}_2 \setminus 2^k\mathbb{Z}_2. \end{cases}$$

Lemma 1. *For every nonnegative integers n and k we have*

$$\delta_k(n) = \frac{1}{2^k} e^{\frac{\pi i n}{2^k}} \sum_{s=0}^{2^k-1} e^{\frac{\pi i s n}{2^{k-1}}}.$$

Proof. Let $n \in \mathbb{Z}_2 \setminus 2^k\mathbb{Z}_2$. Then, there exists some nonnegative integers $t \leq k-1$ and r such that $n = 2^t + 2^{t+1}r$. We have

$$\sum_{s=0}^{2^k-1} e^{\frac{\pi i s n}{2^{k-1}}} = \sum_{l=0}^{2^{t+1}-1} \sum_{s=l2^{k-t-1}}^{(l+1)2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}}$$

$$= \sum_{l=0}^{2^{t+1}-1} \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i (l 2^{k-t-1}+s)n}{2^{k-1}}} = \sum_{l=0}^{2^{t+1}-1} e^{\frac{\pi i l 2^{k-t-1} n}{2^{k-1}}} \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}}$$

$$= \sum_{l=0}^{2^{t+1}-1} e^{\frac{\pi i l 2^{k-t-1} n}{2^{k-1}}} \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}} = \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}} \sum_{l=0}^{2^{t+1}-1} e^{\frac{\pi i l n}{2^{t}}}$$

$$= \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}} \sum_{l=0}^{2^{t+1}-1} e^{\frac{\pi i l (2^t+2^{t+1}r)}{2^{t}}} = \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}} \sum_{l=0}^{2^{t+1}-1} e^{\pi i l} = \sum_{s=0}^{2^{k-t-1}-1} e^{\frac{\pi i s n}{2^{k-1}}} \sum_{l=0}^{2^{t+1}-1} (-1)^l = 0.$$

Take $n \in 2^k + 2^{k+1}\mathbb{Z}_2$. Then, there exists some nonnegative integer r such that $n = 2^k + 2^{k+1}r$. In this case we have

$$\frac{1}{2^k} e^{\frac{\pi i}{2^k} n} \sum_{s=0}^{2^k-1} e^{\frac{\pi i s n}{2^{k-1}}} = \frac{1}{2^k} e^{\frac{\pi i}{2^k}(2^k+2^{k+1}r)} \sum_{s=0}^{2^k-1} e^{\frac{\pi i s (2^k+2^{k+1}r)}{2^{k-1}}} = \frac{e^{\pi i}}{2^k} 2^k = -1.$$

If $n \in 2^{k+1}\mathbb{Z}_2$. Then, there exists some nonnegative integer r such that $n = 2^{k+1}r$. Then, we have

$$\frac{1}{2^k} e^{\frac{\pi i}{2^k} n} \sum_{s=0}^{2^k-1} e^{\frac{\pi i s n}{2^{k-1}}} = \frac{1}{2^k} e^{\frac{\pi i}{2^k} 2^{k+1} r} \sum_{s=0}^{2^k-1} e^{\frac{\pi i s 2^{k+1} r}{2^{k-1}}} = 1.$$

Theorem 2. *For every nonnegative integer k, let $(a_l^k)_l$ be the Mahler coefficients of the function δ_k. Then, for every nonnegative integer l, a_l^k is of the form*

$$a_l^k = (-1)^l \left(\sum_{\substack{0 \le m \le l \\ m \in 2^{k+1}\mathbb{Z}_2}} \binom{l}{m} - \sum_{\substack{0 \le m \le l \\ m \in 2^k + 2^{k+1}\mathbb{Z}_2}} \binom{l}{m} \right),$$

if $k \geq 1$, and

$$a_l^0 = (-1)^l \left(\sum_{\substack{0 \le m \le l \\ m \in 2\mathbb{Z}_2}} \binom{l}{m} + \sum_{\substack{0 \le m \le l \\ m \in 1+2\mathbb{Z}_2}} \binom{l}{m} \right) = (-2)^l.$$

Proof. For all $s \in \{0, \ldots, 2^k - 1\}$, let $x_{k,s}$ denote

$$x_{k,s} = e^{\frac{\pi i}{2^k}} e^{\frac{\pi i s}{2^{k-1}}}.$$

According to Lemma 1, for every nonnegative integer n we have

$$\delta_k(n) = \frac{1}{2^k}\sum_{s=0}^{2^k-1} x_{k,s}^n = \frac{1}{2^k}\sum_{s=0}^{2^k-1}(1 + x_{k,s} - 1)^n$$

$$= \frac{1}{2^k}\sum_{s=0}^{2^k-1}\sum_{l=0}^{n}\binom{n}{l}(x_{k,s}-1)^l = \frac{1}{2^k}\sum_{s=0}^{2^k-1}\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}x_{k,s}^m$$

$$= \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\left(\frac{1}{2^k}\sum_{s=0}^{2^k-1}x_{k,s}^m\right) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\delta_k(m).$$

Since

$$\delta_k(n) = \sum_{l=0}^{n} a_l^k\binom{n}{l},$$

we obtain that for all $l \in \{0, \ldots, n\}$,

$$a_l^k = \sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\delta_k(m).$$

The result follows immediately by applying Definition 1.

Theorem 3. *For every positive integer k, let $(b_l^k)_l$ be the Mahler coefficients of the function $1_{2^k\mathbb{Z}_2}$. Then, for every positive integer l, b_l^k is of the form*

$$b_l^k = (-1)^l\sum_{t=1}^{k-1}\frac{1}{2^t}\left(\sum_{\substack{0\le m\le l\\ m\in 2^{k-t+1}\mathbb{Z}_2}}\binom{l}{m} - \sum_{\substack{0\le m\le l\\ m\in 2^{k-t}+2^{k-t+1}\mathbb{Z}_2}}\binom{l}{m}\right) + \frac{(-2)^l}{2^k}.$$

It is obvious that $b_0^k = 1$.

Proof. It can be easily seen that for every positive integer l, we have

$$1_{2^{l-1}\mathbb{Z}_2} + \delta_{l-1} = 2\cdot 1_{2^l\mathbb{Z}_2}.$$

Applying this formula for $l \in \{1, \ldots, k\}$, we obtain

$$1_{2^k\mathbb{Z}_2} = \frac{1}{2}\delta_{k-1} + \frac{1}{2^2}\delta_{k-2} + \ldots + \frac{1}{2^k}\delta_0 + \frac{1}{2^k}1_{2^0\mathbb{Z}_2} = \sum_{t=1}^{k}\frac{1}{2^t}\delta_{k-t} + \frac{1}{2^k}. \tag{2}$$

Hence, if for every nonnegative integer m, $(a_l^m)_l$ are the Mahler coefficients of the function δ_m, the result for $l \geq 1$ can be immediately obtained from Theorem 2, because according to (2),

$$b_l^k = \sum_{t=1}^{k} \frac{1}{2^t} a_l^{k-t}.$$

The identity $b_0^k = 1$ can also be obtained from Theorem 2. Indeed, according to (2),

$$b_0^k = \sum_{t=1}^{k} \frac{1}{2^t} a_0^{k-t} + \frac{1}{2^k} = \sum_{t=1}^{k} \frac{1}{2^t} 1 + \frac{1}{2^k} = 1.$$

Theorem 4. *Denote by $(A_l^k)_l$, $(b_l^k)_l$ and $(c_l^k)_l$ the Mahler coefficients of the functions S_k, $1_{2^k\mathbb{Z}_2}$ and $1_{2^k-1+2^k\mathbb{Z}_2}$ respectively. Then,*

$$A_{l+1}^k = c_l^k = b_l^k + b_{l+1}^k, \quad \forall l \geq 0.$$

It is obvious that $A_0^k = S_k(0) = 0$.

Proof. We first verify that $\triangle^1 S_k = 1_{2^k-1+2^k\mathbb{Z}_2}$.

The function S_k can be expressed in the following way

$$S_k(x) = \frac{x - i_k(x)}{2^k},$$

where $i_k(x) \in \{0, \ldots, 2^k - 1\}$ is the unique integer satisfying $x \in i_k(x) + 2^k\mathbb{Z}_2$. If $x \in \mathbb{Z}_2 \setminus 2^k - 1 + 2^k\mathbb{Z}_2$, we must have $i_k(x+1) = i_k(x) + 1$, which yields

$$\triangle^1 S_k(x) = S_k(x+1) - S_k(x) = \frac{x + 1 - i_k(x+1)}{2^k} - \frac{x - i_k(x)}{2^k} = 0.$$

For $x \in 2^k - 1 + 2^k\mathbb{Z}_2$, we have $i_k(x+1) = 0$ and $i_k(x) = 2^k - 1$. Hence,

$$\triangle^1 S_k(x) = S_k(x+1) - S_k(x) = \frac{x + 1 - i_k(x+1)}{2^k} - \frac{x - i_k(x)}{2^k} = 1.$$

We conclude that

$$\triangle^{n+1} S_k = \triangle^n 1_{2^k-1+2^k\mathbb{Z}_2}, \quad \forall n \geq 0. \tag{3}$$

Combining (1) and (3), we get

$$A_{l+1}^k = c_l^k, \ \ \forall l \geq 0. \tag{4}$$

For every nonnegative integer n, we have

$$1_{2^k-1+2^k\mathbb{Z}_2}(n) = 1_{2^k\mathbb{Z}_2}(n+1) = \sum_{l=0}^{n+1} b_l^k \binom{n+1}{l}$$

$$= b_0^k + \sum_{l=1}^{n} b_l^k \binom{n+1}{l} + b_{n+1}^k = b_0^k + \sum_{l=1}^{n} b_l^k \binom{n}{l} + \sum_{l=1}^{n} b_l^k \binom{n}{l-1} + b_{n+1}^k$$

$$= b_0^k + \sum_{l=1}^{n} b_l^k \binom{n}{l} + \sum_{l=0}^{n-1} b_{l+1}^k \binom{n}{l} + b_{n+1}^k = \sum_{l=0}^{n} b_l^k \binom{n}{l} + \sum_{l=0}^{n} b_{l+1}^k \binom{n}{l} = \sum_{l=0}^{n} (b_l^k + b_{l+1}^k) \binom{n}{l}.$$

It follows that

$$c_l^k = b_l^k + b_{l+1}^k, \ \ \forall l \geq 0.$$

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