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# ON PRODUCT COMBINATION OF GEOMETRIC EXPRESSIONS

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Abstract. In this research work, we introduce generalized geometric expressions of a certain product combination via Salagean and Ruchewehy differential operators denoted as $B^n_\gamma(\beta)$ and$B^n_\gamma(\alpha,\beta)$, and obtain the conditions for univalence of the resulting product combinations, differential subordinations of the class and our result completely unify earlier ones.

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## 1. Introduction

Let $f(z)$ be the class of the functions of the form $f(0) = 0$ and $f'(0) = 1$ in the unit disk $\{\mathbb{U} = |z| < 1\}$. Let $H(z)$ denote the class of the functions $h(z) = 1 + c_1z + c_2z^2 + \cdots$ referred to as a function of positive real part in $\mathbb{U}$, such that $Reh(z) > 0$ and by $H(\beta)$ if $Reh(z) > \beta$ for some real number $0 \leq \beta < 1$.
A function $f(z)$ satisfying the geometric condition $Rezf^{'}(z)/f(z) > \beta$ is known as a starlike function of order $\beta$, $Ref'(z) > \beta$ which is the class of bounded turning of order $\beta$ and $Re(zf')'/f'(z) > \beta$ which is the class of convex functions of order $\beta$ if , (see [1, 6]).
In [2], Al-Amiri and Reade considered the functions $f(z)$ satisfying the condition

$$Re\left\{(1-\gamma)f'f(z) + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta \tag{1}$$

denoted by $H_\gamma(\beta)$, where $\gamma \geq 0$ and $0 \leq \beta < 1$ univalent in $\mathbb{U}$.
In [8, 9] Mocanu introduced the class of the functions satisfying the conditions

$$Re\left\{(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta. \tag{2}$$

where $\gamma \geq 0$ and $0 \leq \beta < 1$ univalent in $\mathbb{U}$ and study the class with the knowledge of arithmetic mean to obtained the conditions of starlikeness and convexity of the class.
In [7], Darus and Thomas studied the class $B^{\gamma}$ defined as follows:

$$Re\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\} > 0. \tag{3}$$

and obtained many interesting properties of the class.
In [12], Rucheweyh introduced the differential operator $\mathbb{D}^n$, defined as:

$$\mathbb{D}^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z).$$

and the operator was used to generalize the geometric quantities for the function $f(z)$ in the unit disk.
Also Ruchewehy studied the concepts of starlikeness and convexity of functions define with the geometric conditions.

$$\frac{\mathbb{D}^{n+1} f(z)}{\mathbb{D}^n f(z)}.$$

In [13], Salagean introduced the differential operator denoted as $\mathcal{D}^n$ with $\mathcal{D}^0 f(z) = f(z)$ in the unit disk.
The use of operators has a very important role in theory of geometric functions and this has lead to introduction of different classes of analytics functions. (see [3, 4, 14, 15, 16, 17, 18, 19]).
In this work, we introduce the classes $B^n_{\gamma}(\beta)$ and $B^n_{\gamma}(\alpha, \beta)$ using the Salagean differential operator and Ruchewehy differential operator and define the classes as:
**Definition 2.1**
The function $f(z) \in B^n_{\gamma}(\beta)$, if it satisfies the geometric conditions

$$Re\left\{\left(\frac{\mathbb{D}^{n+1} f(z)}{\mathbb{D}^n f(z)}\right)^{1-\gamma}\left(\frac{\mathbb{D}^{n+2} f(z)}{\mathbb{D}^{n+1} f(z)}\right)^{\gamma}\right\} > \beta. \tag{4}$$

**Definition 2.2**
The function $f(z) \in B^n_{\gamma}(\alpha, \beta)$, if it satisfies the geometric conditions

$$Re\left\{\left(\frac{\mathcal{D}^{n+1} f(z)^{\alpha}}{\mathcal{D}^n f(z)^{\alpha}}\right)^{1-\gamma}\left(\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)}\right)^{\gamma}\right\} > \beta. \tag{5}$$

**Remark 1.** *(i) If $n = 0$, $\alpha = 1$ and $\gamma = 0$, we have $B_0^0(\beta)$ and $B_0^0(1, \beta)$which is a starlike function of order $\beta$. (ii) If $n = 0$, $\alpha = 1$ and $\gamma = 1$, we have $B_1^0(\beta)$ and $B_0^0(1, \beta)$ which is convex function of order $\beta$.*

The work in [2] having considered the linear combination of geometric inequalities of classes of analytic functions motivated this research by considering the product combination of geometric expressions defined by two operators and our objectives is to obtain the conditions for univalence of the resulting product combinations and differential subordinations of the classes.

## 2. Preliminary Lemmas

**Lemma 1.** *[11] Let $h(z)$ be analytics in $\mathbb{U}$ with $h(z) \neq 0$ and $h(0) =$. If there exists $z_0 \in \mathbb{U}$, such that $|argh(z_0)| < \beta\frac{\pi}{2}$ for $|z| < |z_0|$, and $|argh(z_0)| = \beta\frac{\pi}{2}$ for $\beta > 0$. Then*

$$\frac{z_0 h'(z_0)}{h(z_0)} = ik\beta,$$

*where*

$$k \leq -\frac{1}{2}\left(b + \frac{1}{b}\right),$$

*when $argh(z_0) = \beta\frac{\pi}{2}$ and*

$$k \geq \frac{1}{2}\left(b + \frac{1}{b}\right),$$

*when $argh(z_0) = -\beta\frac{\pi}{2}$ and where $h^{1/\beta}(z_0) = \pm ia$ for $b > 0$*

**Lemma 2.** *Let $\gamma \in [0, 1]$, $\lambda, \zeta \in \mathbb{C}$ and $h \in A$ be convex univalent, in $\mathbb{U}$, with $Re[\lambda q(z) + \zeta] > 0$, $z \in \mathbb{U}$, $h(0) = 1$ and $h \in A$ with $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $z \in \mathbb{U}$. . If*

$$h(z)^{1-\gamma}\left(h(z) + \frac{zh'(z)}{\lambda h(z) + \zeta}\right)^{\gamma} \prec q(z). \tag{6}$$

*Then $h(z) \prec q(z)$.*

The proof of this lemma follows the same as in [8].

**Lemma 3.** *[5] For the function $f(z)$ and any complex number $\gamma$. If for $D^{n+1}f(z)^{\gamma}/D^n f(z)^{\gamma}$ is independent of $n$, for $z \in \mathbb{U}$, then*

$$\frac{D^{n+1}f(z)^{\gamma}}{D^n f(z)^{\gamma}} = \gamma\frac{D^{n+1}f(z)}{D^n f(z)}$$

## 3. Main Results

**Theorem 4.** *For $\gamma \geq 0$, $0 \leq \beta < 1$ such that $\gamma \leq \beta < 1$. If $f \in B^n_\gamma(\beta)$. Then*

$$Re\left(\frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)}\right) > 0.$$

*Also, if $\beta = 1/2$. Then $Re\left(\frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)}\right) > 1/2$.*

*Proof.* Let $f \in B^n_\gamma(\beta)$, there exists $h \in h(\beta)$ such that

$$\frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)} = h(z) \tag{7}$$

by differentiating (7), we obtain

$$\frac{\mathbb{D}^n f(z)(\mathbb{D}^{n+1}f(z))'}{(\mathbb{D}^n f(z))^2} - \frac{\mathbb{D}^{n+1}f(z)(\mathbb{D}^n f(z))'}{(\mathbb{D}^n f(z))^2} = h^{'}(z) \tag{8}$$

multiply through by z, we have

$$\frac{\mathbb{D}^n f(z)z(\mathbb{D}^{n+1}f(z))'}{(\mathbb{D}^n f(z))^2} - \frac{\mathbb{D}^{n+1}f(z)z(\mathbb{D}^n f(z))'}{(\mathbb{D}^n f(z))^2} = zh^{'}(z) \tag{9}$$

divide through by $h(z)$

$$\frac{\mathbb{D}^{n+2}f(z)}{\mathbb{D}^{n+1}f(z)} - \frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)} = \frac{zh^{'}(z)}{h(z)} \tag{10}$$

$$\frac{\mathbb{D}^{n+2}f(z)}{\mathbb{D}^{n+1}f(z)} = \frac{zh^{'}(z)}{h(z)} + h(z). \tag{11}$$

By lemma 1, since $h(z)$ is analytic in $\mathbb{D}$ such that $h(z) \neq 0$ and $p(0) = 1$. There exists $z_0 \in \mathbb{D}$ such that $|argh(z_0)| < \frac{\beta\pi}{2}$ for $|z| < |z_0|$and $|argh(z)| = \frac{\beta\pi}{2}$.
If $argh(z_0) = \frac{\beta\pi}{2}$, then

$$arg\left\{\left(h(z_0 + \frac{zh^{'}(z_0)}{h(z_0)})^\gamma\right)h(z_0)^{1-\gamma}\right\}$$

$$= \gamma arg\left[h(z_0) + \frac{zh^{'}(z_0)}{h(z_0)}\right] + (1-\gamma)argh(z_0)$$

$$= \gamma arg(ia + ik) + (1-\gamma)\frac{\beta\pi}{2}$$

$$= \gamma \frac{\beta\pi}{2} + (1-\gamma)\frac{\beta\pi}{2}$$
$$= \frac{\beta\pi}{2}$$

where $h(z_0) = ib$ and $k \geq \frac{1}{2}(b + \frac{1}{b})$.
If $argh(z_0) = -\frac{\beta\pi}{2}$, then

$$arg\left\{\left(h(z_0 + \frac{zh^{'}(z_0)}{h(z_0)})^{\gamma}\right)p(z_0)^{1-\gamma}\right\}$$

$$= \gamma arg\left[h(z_0) + \frac{zh^{'}(z_0)}{h(z_0)}\right] + (1-\gamma)argh(z_0)$$
$$= \gamma arg(-ib + ik) + (1-\gamma)\frac{\beta\pi}{2}$$
$$= -\gamma\frac{\beta\pi}{2} - (1-\gamma)\frac{\beta\pi}{2}$$
$$= -\frac{\beta\pi}{2}$$

where $p(z_0) = -ib$ and $k \leq -\frac{1}{2}(b + \frac{1}{b})$. Therefore, this contradicts our assumption and the proof completes.

**Corollary 5.** *Let $\gamma \geq 0$, $0 \leq \beta < 1$ and $n = 0$ such that $\gamma \leq \beta < 1$. If $f(z)$ satisfies*

$$Re\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\} > \beta$$

*. Then*

$$Re\frac{zf'(z)}{f(z)} > 0.$$

**Theorem 6.** *Let $\gamma$, $\beta$ and $\alpha$ be real numbers such that $\gamma \leq \beta < 1$. If $f \in B^n_\gamma(\alpha, \beta)$. Then*

$$Re\left(\frac{\mathcal{D}^{n+1}f(z)^\alpha}{\mathcal{D}^n f(z)^\alpha}\right) > 0.$$

*Also, if $\beta = 1/2$. Then $Re\left(\frac{\mathcal{D}^{n+1}f(z)^\alpha}{\mathcal{D}^n f(z)^\alpha}\right) > 1/2$.*

*Proof.* Let $f \in B^n_\gamma(\beta)$, there exists $p \in P(\beta)$ such that

$$\frac{\mathcal{D}^{n+1} f(z)^\alpha}{\mathcal{D}^n f(z)^\alpha} = p(z) \tag{12}$$

by lemma 3, we have

$$\frac{\mathcal{D}^{n+1} f(z)^\alpha}{\mathcal{D}^n f(z)^\alpha} = \alpha \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \tag{13}$$

by differentiating (13) logarithmically , we obtain

$$\alpha \frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} - \alpha \frac{\mathcal{D}^{n+1} f(z)}{D^n f(z)} = \frac{zh^{'}(z)}{h(z)} \tag{14}$$

$$\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} = \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} + \frac{zh^{'}(z)}{\alpha h(z)} \tag{15}$$

$$\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} = h(z) + \frac{zh^{'}(z)}{\alpha h(z)} \tag{16}$$

Since $h(z)$ is analytic in $\mathbb{U}$ such that $h(z) \neq 0$ and $h(0) = 1$. There exists $z_0 \in \mathbb{U}$ such that $|argh(z_0)| < \frac{\beta\pi}{2}$ for $|z| < |z_0|$and $|argh(z)| = \frac{\beta\pi}{2}$.
The remaining part of the proof follows from Theorem 4.

**Corollary 7.** *Let $\gamma \geq 0$, $0 \leq \beta < 1$ and $n = 0$ such that $\gamma \leq \beta < 1$. If $f(z)$ satisfies*

$$Re \left\{ \left( \frac{zf(z)^{\alpha-1} f'(z)}{f(z)^\alpha} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} > \beta$$

*. Then*

$$Re \frac{zf(z)^{\alpha-1} f'(z)}{f(z)^\alpha} > 0.$$

**Corollary 8.** *Let $\gamma \geq 0$, $0 \leq \beta < 1$, $\alpha = 1$ and $n = 0$ such that $\gamma \leq \beta < 1$. If $f(z)$ satisfies*

$$Re \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\gamma \right\} > \beta$$

*. Then*

$$Re \frac{zf'(z)}{f(z)} > 0.$$

**Theorem 9.** *Let $\gamma \geq 0$, $0 \leq \beta < 1$ such that $\gamma \leq \beta < 1$. If $f(z)$ such that $f \in B^n_\gamma(\beta)$. Then*

$$\frac{\mathbb{D}^{n+1} f(z)}{\mathbb{D}^n f(z)} \prec q(z).$$

*Proof.* Let $f \in B^n_\gamma(\beta)$, so that $p \in P(\beta)$, we have

$$\frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)} = h(z) \tag{17}$$

by differentiating (17), we obtain

$$\frac{\mathbb{D}^n f(z)(\mathbb{D}^{n+1}f(z))'}{(\mathbb{D}^n f(z))^2} - \frac{\mathbb{D}^{n+1}f(z)(\mathbb{D}^n f(z))'}{(\mathbb{D}^n f(z))^2} = h^{'}(z) \tag{18}$$

multiply through by z, we have

$$\frac{\mathbb{D}^n f(z)z(\mathbb{D}^{n+1}f(z))'}{(\mathbb{D}^n f(z))^2} - \frac{\mathbb{D}^{n+1}f(z)z(\mathbb{D}^n f(z))'}{(\mathbb{D}^n f(z))^2} = zh^{'}(z) \tag{19}$$

divide through by $h(z)$

$$\frac{\mathbb{D}^{n+2}f(z)}{\mathbb{D}^{n+1}f(z)} - \frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)} = \frac{zh^{'}(z)}{h(z)} \tag{20}$$

$$\frac{\mathbb{D}^{n+2}f(z)}{\mathbb{D}^{n+1}f(z)} = \frac{zh^{'}(z)}{h(z)} + h(z). \tag{21}$$

By lemma (3), we have that

$$\left(\frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)}\right)^{1-\gamma}\left(\frac{\mathbb{D}^{n+2}f(z)}{\mathbb{D}^{n+1}f(z)}\right)^{\gamma} = p(z)^{1-\gamma}\left(\frac{zh^{'}(z)}{h(z)} + h(z)\right)^{\gamma} \prec q(z). \tag{22}$$

With $\lambda = 1$ and $\zeta = 0$, we have that $h(z) \prec q(z)$.
This implies that $\frac{\mathbb{D}^{n+1}f(z)}{\mathbb{D}^n f(z)} \prec q(z)$.

**Corollary 10.** *Let $\gamma \geq 0$, $0 \leq \beta < 1$ and $n = 0$ be such that $\gamma \leq \beta < 1$. If $f(z)$ satisfies*

$$Re\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\} > \beta$$

*. Then*

$$Re\frac{zf'(z)}{f(z)} \prec q(z).$$

**Theorem 11.** *Let $\gamma \geq 0$, $0 \leq \beta < 1$ such that $\gamma \leq \beta < 1$. If $f(z)$ such that $f \in B^n_\gamma(\alpha, \beta)$. Then*

$$\frac{\mathcal{D}^{n+1}f(z)}{\mathcal{D}^n f(z)} \prec q(z).$$

*Proof.* Let $f \in B^n_\gamma(\beta)$, there exists $p \in h(\beta)$ such that

$$\frac{\mathcal{D}^{n+1} f(z)^\alpha}{\mathcal{D}^n f(z)^\alpha} = h(z) \tag{23}$$

by lemma 2, we have

$$\frac{\mathcal{D}^{n+1} f(z)^\alpha}{\mathcal{D}^n f(z)^\alpha} = \alpha \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \tag{24}$$

by differentiating (24) logarithmically , we obtain

$$\alpha \frac{D^{n+2} f(z)}{D^{n+1} f(z)} - \alpha \frac{D^{n+1} f(z)}{D^n f(z)} = \frac{zh^{'}(z)}{h(z)} \tag{25}$$

$$\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} = \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} + \frac{zh^{'}(z)}{\alpha h(z)} \tag{26}$$

$$\frac{\mathcal{D}^{n+2} f(z)}{\mathcal{D}^{n+1} f(z)} = h(z) + \frac{zh^{'}(z)}{\alpha h(z)} \tag{27}$$

With $\lambda = 0$, we have that $h(z) \prec q(z)$.
This implies that $\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \prec q(z)$.

**Corollary 12.** *Let $\gamma \geq 0$, $0 \leq \beta < 0$ and $n = 0$ such that $\gamma \leq \beta < 1$. If $f(z)$ satisfies*

$$Re \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{\gamma} \right\} > \beta$$

*. Then*

$$Re \frac{zf'(z)}{f(z)} \prec h(z).$$

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