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ON THE TYPE SEQUENCES OF SOME NUMERICAL SEMIGROUPS WITH MULTIPLICITY P PRIME NUMBER

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Abstract. In this article, we will give the structure of the type sequences of the numerical semigroups which multiplicity is the prime number $p, p < 10$ and the conductor is K .

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1. Introduction

Let $\mathbb{N} = \{ a \in \mathbb{Z} \ : \ a \geq 0 \}$ and $\mathbb{Z}$ be integers set. $\phi \neq S \subseteq \mathbb{N}$, S is called a numerical semigroup if it satisfied following conditions

- $0 \in S$,
- $a_1 + a_2 \in S,\ for\ all\ a_1, a_2 \in S$,
- $Card(\mathbb{N} \backslash S) < \infty$. (this condition equivalent to $gcd(S) = 1$ and $gcd(S)$ =greatest common divisor the element of S).

We define the following integers for numerical semigroup S:

$F(S) = \max\{x \in \mathbb{Z} \ : \ x \notin S\}$ is the Frobenius number of S;
$m(S) = \min\{a \in S \ : \ a \neq 0\}$ is the multiplicity of S;
$n(S) = Card\left(\{0, 1, 2, ..., F(S)\} \cap S\right)$ is determine number of S ([1, 5, 9]).

The numerical semigroup S is symmetric if $f(S) - a \in S$ for all $a \in \mathbb{Z}\backslash$S. It is known that every numerical semigroup$S =< k_1, k_2 >$ is symmetric, $f(S) = k_1k_2 - k_1 - k_2$ and $n(S) = \frac{f(S)+1}{2}$ ([1, 12]) .

If S is a numerical semigroup such that $S =< x_1, x_2, ..., x_u >$, then we observe that $S =< x_1, x_2, ..., x_u >= \{s_0 = 0, s_{1,}, s_2, ..., s_{n-1}, s_n = F(S) + 1, \rightarrow ...\}$ where $s_i < s_{i+1}$, $n = n(S)$, and the arrow means that every integer greater than $F(S) + 1$ belongs to S, for $i = 1, 2, ..., n = n(S)$. Here, we say the number $K = K(S) = F(S) + 1$ is conductor of S. Let $S =< x_1, x_2, ..., x_u >$ be a numerical semigroup. Then $e(S) = u$ is called embedding dimension of S. It is known that $e(S) \leq m(S)$. The numerical semigroup is maximal embedding dimension (MED) if $e(S) = m(S)$ ([5, 9]).

We give following definitions for a numerical semigroup S

$S_i = \{s \in S : \ s \geq s_i\}$ for $i \geq 0, s_i \in S$;

$S(i) = \{k \in \mathbb{N} : \ k + S_i \subseteq S\}$.

Here, every the set $S(i)$ is a numerical semigroup and we write the following chain:

$$S_n \subset S_{n-1} \subset ... \subset S_1 \subset S_0 = S = S(0) \subset S(1) \subset ... \subset S(n-1) \subset S(n) = \mathbb{N}.$$

The number $t(S) = Card(S(1) \setminus S)$ is called the type of S. Likewise, we put for $i = 1, 2, ..., n = n(S)$; $t_i(S) = Card(S(i) \setminus S(i-1))$. In this way, it is possible to associate with every numerical semigroup S a numerical sequence $\{t_1, t_2, ..., t_{n(S)}\}$ which is called the type sequence of S. It is known that, $1 \leq t_i(S) \leq t_1(S)$ and $t_1(S) = t(S)$ ([7]).

Let S a numerical semigroup then S has maximal length if $n(S)(t(S) + 1) = F(S) + 1$. Also, S has almost maximal length if $n(S)(t(S) + 1) = F(S) + 2$ (for details see [6,11]).

A numerical semigroup S is Arf if $s_1 + s_2 - s_3 \in S$, for all $s_1, s_2, s_3 \in S$ such that $s_1 \geq s_2 \geq s_3$. It is well known that any Arf numerical semigroup is maximal embedding dimension (MED), but its inverse is not true. For example, the numerical semigroup $S =< 3, 10, 14 >$ is MED but it is not Arf. S is called saturated numerical semigroup if $s + n_1 s_1 + n_2 s_2 + ... + n_k s_k \in S$, where $s_j \in S$ and $n_j \in \mathbb{Z}$ such that $n_1 s_1 + n_2 s_2 + ... + n_k s_k \geq 0$ and $s_j \leq s$ for $j = 1, 2, ..., k$. Also, it is known that a saturated numerical semigroup is Arf. But, an Arf numerical semigroup need not be saturated. For example, the numerical semigroup $S =< 4, 14, 17, 19 >$ is Arf but it is not saturated (for details see [2,3,4]). It is known that if $\left\{t_1, t_2, ..., t_{n(S)}\right\}$ the type sequence of S Arf numerical semigroup, then $t_i = s_i - s_{i-1} - 1$, for $i = 1, 2, ..., n(S)$ ([10]).

2. MAIN RESULTS

Theorem 1. ([4]) *Let S be a numerical semigroup and $d_S(a) = \gcd\{\, x \in S : x \leq a \,\}$. Then the following conditions are equalities:*

(1) S is saturated.

(2) $a + d_S(a) \in S$ for all $a \in S \setminus \{0\}$.

(3) $a + k.d_S(a) \in S$ for all $a \in S \setminus \{0\}$ and $k \in \mathbb{N}$.

Theorem 2. ([8]) *Let S be a numerical semigroup with $m(S) = 2$ and conductor K such that $K \equiv 0(2)$. Then $S =< 2, 2K + 1 >$ is saturated.*

Theorem 3. ([8]) *Let S be a numerical semigroup with $m(S) = 3$ and conductor K. Then, S is saturated if S is one of following numerical semigroups:*

(1) $S =< 3, K + 1, K + 2 >$ for $K \equiv 0(3)$;

(2) $S =< 3, K, K + 2 >$ for $K \equiv 2(3)$.

Theorem 4. ([8]) *Let S be a numerical semigroup with $m(S) = 5$ and conductor K. Then, S is saturated if S is one of following numerical semigroups:*

(1) $S =< 5, K + 1, K + 2, K + 3, K + 4 >$ for $K \equiv 0(5)$;

(2) $S =< 5, K, K + 1, K + 2, K + 4 >$ for $K \equiv 2(5)$;

(3) $S =< 5, K, K + 1, K + 3, K + 4 >$ for $K \equiv 3(5)$;

(4) $S =< 5, K, K + 2, K + 3, K + 4 >$ for $K \equiv 4(5)$.

Theorem 5. ([8]) *Let S be a numerical semigroup with $m(S) = 7$ and conductor K. Then, S is saturated if S is one of following numerical semigroups:*

(1) $S = \langle 7, K, K+1, K+2, K+3, K+4, K+5, K+6 \rangle$ *for* $K \equiv 0(7)$;

(2) $S = \langle 7, K, K+1, K+2, K+3, K+4, K+6 \rangle$ *for* $K \equiv 2(7)$;

(3) $S = \langle 7, K, K+1, K+2, K+3, K+5, K+6 \rangle$ *for* $K \equiv 3(7)$;

(4) $S = \langle 7, K, K+1, K+2, K+4, K+5, K+6 \rangle$ *for* $K \equiv 4(7)$;

(5) $S = \langle 7, K, K+1, K+3, K+4, K+5, K+6 \rangle$ *for* $K \equiv 5(7)$;

(6) $S = \langle 7, K, K+2, K+3, K+4, K+5, K+6 \rangle$ *for* $K \equiv 6(7)$.

Theorem 6. ([7]) *If* S *is symmetric numerical semigroup then the type of* S *is* $t(S) = 1$.

Theorem 7. *Let* S *be a saturated numerical semigroup with* $p < 10$ *multiplicity be prime number and the conductor* $K > p$. *If*

$$K \equiv 0(p)$$

then $t_i = p - 1$ *for* $\forall i, 1 \leq i \leq n(S)$, *where* $\left\{t_1, t_2, ..., t_{n(S)}\right\}$ *is the type sequence of* S.

Proof. Let S be a saturated numerical semigroup with $p < 10$, $p = 2, 3, 5, 7$ and conductor K.

(1) If $p = 2$ for $K \equiv 0(2)$. Then we write $S = \langle 2, 2K+1 \rangle = \{0, 2, 4, 6, ..., 2K, \rightarrow ...\}$. Let $\{t_i : \forall i, 1 \leq i \leq n(S)\}$ be the type sequence of positive integers number. Then, we get this $t_i = 1$ for $\forall i, 1 \leq i \leq n(S)$ from S is symmetric.

(2) If $p = 3$ for $K \equiv 0(3)$. Then we write
$S = \langle 3, K+1, K+2 \rangle = \{0, 3, 6, 9, ..., K-3, K, \rightarrow ...\}$. Let $\{t_i : \forall i, 1 \leq i \leq n(S)\}$ be the type sequence of positive integers number. Then

$S_1 = \{s \in S : s \geq s_1 = 3\} = \{3, 6, 9, ..., K-3, K, \rightarrow ...\}$,

$S(1) = \{x \in \mathbb{N} : x + S_1 \subseteq S\} = \{0, 3, 6, 9, ..., K-6, K-3, K-2, K-1, K, \rightarrow ...\}$,

$t_1(S) = t(S) = Card\,(S(1) \backslash S) = Card\,(\{K-2, K-1\}) = 2$.

$S_2 = \{s \in S : s \geq s_2 = 6\} = \{6, 9, ..., K-3, K, \rightarrow ...\}$,

$S(2) = \{x \in \mathbb{N} : x + S_2 \subseteq S\} = \{0, 3, 6, 9, ..., K-6, K-5, K-4, K-3, K-2, K-1, K, \rightarrow ...\}$,

$t_2(S) = Card\,(S(2) \backslash S(1)) = Card\,(\{K-5, K-4\}) = 2.$

$S_3 = \{s \in S : s \geq s_3 = 9\} = \{9, ..., K-6, K-3, K, \rightarrow ...\}$,

$S(3) = \{x \in \mathbb{N} : x + S_3 \subseteq S\} = \{0, 3, 6, 9, ..., K-8, K-7, K-6, K-5, ..., K-2, K-1, K, \rightarrow ...\}$,

$t_3(S) = Card\,(S(3) \backslash S(2)) = Card\,(\{K-8, K-7\}) = 2.$

$\vdots$

$t_n(S) = Card\,(S(n) \backslash S(n-1)) = Card\,(\{K-(K-2), K-(K-1)\})$
$= Card\,(\{2, 1\}) = 2.$

Thus, we obtain $t_i = 2$ for $\forall i, 1 \leq i \leq n(S)$.

(3) If $p = 5$ for $K \equiv 0(5)$. Then we write $S = \langle 5, K+1, K+2, K+3, K+4 \rangle = \{0, 5, 10, 15, ..., K, \rightarrow ...\}$. Let $\{t_i : \forall i, 1 \leq i \leq n(S)\}$ be the type sequence of positive integers number. Then,

$S_1 = \{s \in S : s \geq s_1 = 5\} = \{5, 10, 15, ..., K-5, K, \rightarrow ...\}$,

$S(1) = \{x \in \mathbb{N} : x + S_1 \subseteq S\} = \{0, 5, 10, 15, ..., K-10, K-5, K-4, K-3, K-2, K-1, K, \rightarrow ...\}$,

$t_1(S) = t(S) = Card\,(S(1) \backslash S) = Card\,(\{K-4, K-3, K-2, K-1\}) = 4.$

$S_2 = \{s \in S : s \geq s_2 = 10\} = \{10, 15, ..., K-5, K, \rightarrow ...\}$,

$S(2) = \{x \in \mathbb{N} : x + S_2 \subseteq S\} = \{0, 5, 10, 15, ..., K-10, K-9, K-8, K-7, K-6, ..., K, \rightarrow ...\}$,

$t_2(S) = Card\,(S(2) \backslash S(1)) = Card\,(\{K-9, K-8, K-7, K-6\}) = 4.$

$\vdots$

$t_n(S) = Card\,(S(n)\backslash S(n-1)) = Card\,(\{K-(K-4), K-(K-3), K-(K-2), K-(K-1)\})$
$= Card\,(\{4,3,2,1\}) = 4$

Thus, we obtain $t_i = 4$ for $\forall i, 1 \leq i \leq n(S)$.

(4) If $p = 7$ for $K \equiv 0(7)$. Then we write $S = \langle 7, K+1, K+2, K+3, K+4, K+5, K+6\rangle = \{0, 7, 14, 21, ..., K, \rightarrow ...\}$. If we make some operations in above. We find that $t_i = 6$, for $\forall i, 1 \leq i \leq n(S)$. Therefore, if $K \equiv 0(p)$then we obtain that $t_i = p-1$, for $\forall i, 1 \leq i \leq n(S)$.

Theorem 8. *Let S be a saturated numerical semigroup with $P < 10$ multiplicity be prime number, $j = 2, 3, ..., p-1$ and the conductor $K > p+j$. If*

$$K \equiv j(p)$$

then the type sequence of S is $\left\{t_1 = p-1, t_2 = p-1, ..., t_{n(S)-1} = p-1, t_{n(S)} = j-1\right\}$.

Proof. Let S be a saturated numerical semigroup with $p < 10$, $p = 2, 3, 5, 7$ and conductor K.

(1) If $p = 3$ and $j = 2$. for $K \equiv j(p)$. Then we write $S = \langle 3, K, K+2\rangle = \{0, 3, 6, 9, ..., K-5, K-2, K, \rightarrow ...\}$. Let $\{t_i : \forall i, 1 \leq i \leq n(S)\}$ be the type sequence of positive integers number.

$S_1 = \{s \in S : s \geq s_1 = 3\} = \{3, 6, 9, ..., K-2, K, \rightarrow ...\}$,

$S(1) = \{x \in \mathbb{N} : x + S_1 \subseteq S\} = \{0, 3, 6, 9, ..., K-8, K-3, K-2, K-1, K, \rightarrow ...\}$,

$t_1(S) = t(S) = Card\,(S(1)\backslash S) = Card\,(\{K-1, K-3\}) = 2.$

$S_2 = \{s \in S : s \geq s_2 = 6\} = \{6, 9, ..., K-5, K-2, K, \rightarrow ...\}$,

$S(2) = \{x \in \mathbb{N} : x + S_2 \subseteq S\} = \{0, 3, 6, 9, ..., K-8, K-6, K-5, K-4, ..., K-2, K-1, K, \rightarrow ...\}$,

$t_2(S) = Card\,(S(2)\backslash S(1)) = Card\,(\{K-6, K-4\}) = 2.$

$\vdots$

for $i = n(S) - 1$

$S_{n(S)-1} = \{s \in S : s \geq s_{n(S)-1} = K-2\} = \{K-2, K, \to ...\}$,

$S(n(S)-1) = \{x \in \mathbb{N} : x + S_{n(S)-1} \subseteq S\} = \{0, 2, 3, 4, ..., K-2, K, \to ...\}$,

$S_{n(S)-2} = \{s \in S : s \geq s_{n(S)-2} = K-5\} = \{K-5, K-2, K, \to ...\}$,

$S(n(S)-2) = \{x \in \mathbb{N} : x + S_{n(S)-2} \subseteq S\} = \{0, 3, 5, 6, 7, ..., K-5, K-2, K, \to ...\}$,

$t_{n(S)-1} = Card\,(S(n(S)-1) \backslash S(n(S)-2)) = Card\,(\{2,4\}) = 2$.

Thus, we obtain $t_i = 2$ for $\forall i, 1 \leq i \leq n(S) - 1$.

Finally, for $i = n(S)$,

$S_{n(S)} = \{s \in S : s \geq s_{n(S)} = K\} = \{K, \to ...\}$ and

$S(n(S)) = \{x \in \mathbb{N} : x + S_{n(S)} \subseteq S\} = \{0, 1, 2, ...\} = \mathbb{N}$. So, we find that $t_{n(S)} = Card\,(S(n(S)) \backslash S(n(S)-1) = Card\,(\{1\}) = 1$.

(2) i) If $p = 5$ and $j = 2$ for $K \equiv j(p)$. Then we write $S = \langle 5, K, K+1, K+2, K+4 \rangle = \{0, 5, 10, 15, ..., K-2, K, \to ...\}$. S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. In this case, we have $t_i = s_i - s_{i-1} - 1 = 4 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-2) - 1 = 1 = j - 1$.

ii) If $p = 5$ and $j = 3$ for $K \equiv j(p)$. Then we write $S = \langle 5, K, K+1, K+3, K+4 \rangle = \{0, 5, 10, 15, ..., K-3, K, \to ...\}$. So, S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. In this case, we have $t_i = s_i - s_{i-1} - 1 = 4 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-3) - 1 = 2 = j - 1$.

iii) If $p = 5$ and $j = 4$ for $K \equiv j(p)$. Then we write $S = \langle 5, K, K+2, K+3, K+4 \rangle = \{0, 5, 10, 15, ..., K-4, K, \to ...\}$. So, S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. In this case, we have $t_i = s_i - s_{i-1} - 1 = 4 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-4) - 1 = 3 = j - 1$.

(3) i) If $p = 7$ and $j = 2$ for $K \equiv j(p)$. Then we write
$S = \langle 7, K, K+1, K+2, K+3, K+4, K+6 \rangle = \{0, 7, 14, 21, ..., K-2, K, \rightarrow ...\}$.
So, S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. So, we have $t_i = s_i - s_{i-1} - 1 = 6 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-2) - 1 = 1 = j - 1$.

ii) If $p = 7$ and $j = 3$ for $K \equiv j(p)$. Then we write
$S = \langle 7, K, K+1, K+2, K+3, K+5, K+6 \rangle = \{0, 7, 14, 21, ..., K-3, K, \rightarrow ...\}$.
So, S is Arf since S is saturated. In this case, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. Thus, we have $t_i = s_i - s_{i-1} - 1 = 6 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-3) - 1 = 2 = j - 1$.

iii) If $p = 7$ and $j = 4$ for $K \equiv j(p)$. Then we write
$S = \langle 7, K, K+1, K+2, K+4, K+5, K+6 \rangle = \{0, 7, 14, 21, ..., K-4, K, \rightarrow ...\}$.
S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. We have $t_i = s_i - s_{i-1} - 1 = 6 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-4) - 1 = 3 = j - 1$.

iv) If $p = 7$ and $j = 5$ for $K \equiv j(p)$. Then we write
$S = \langle 7, K, K+1, K+3, K+4, K+5, K+6 \rangle = \{0, 7, 14, 21, ..., K-5, K, \rightarrow ...\}$.
In this case, S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. So, we have $t_i = s_i - s_{i-1} - 1 = 6 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-5) - 1 = 4 = j - 1$.

v) If $p = 7$ and $j = 6$ for $K \equiv j(p)$. Then we write
$S = \langle 7, K, K+2, K+3, K+4, K+5, K+6 \rangle = \{0, 7, 14, 21, ..., K-6, K, \rightarrow ...\}$.
In this case, S is Arf since S is saturated. Thus, we write that $t_i = s_i - s_{i-1} - 1$, for $\forall i, 1 \leq i \leq n(S)$ from S is Arf. We have $t_i = s_i - s_{i-1} - 1 = 6 = p - 1$; for $\forall i, 1 \leq i \leq n(S)$, and $t_{n(S)} = s_{n(S)} - s_{n(S)-1} - 1 = K - (K-6) - 1 = 5 = j - 1$.

Corollary 9. *If S is a saturated numerical semigroup in the Theorem 7 and the conductor of S is $K = pn(S)$ then S has a maximal length.*

Proof. Let S be as in Theorem 7 and $K = pn(S)$. Then, $t(S) = p - 1$ and, we write that $t(S)n(S) = (p-1)n(S) = pn(S) - n(S) = K - n(S) = F(S) + 1 - n(S)$. Thus, S has maximal length.

Corollary 10. *Let S be as in Theorem 8. If the conductor of S is $K = pn(S) - 1$ then S has almost maximal length.*

Proof. Let S be as in Theorem 8. and $K = pn(S) - 1$. Then, $t(S) = p - 1$ and we obtain that $n(S)(t(S)+1) = n(S)(p-1+1) = pn(S) = K+1 = F(S)+2$. Therefore, S has almost maximal length.

Example 1. *Let's take* $j = 0, K = 10$ *and* $p = 5$ *for* $K \equiv j(p)$. *Then we write* $S = \langle 5, 11, 12, 13, 14\rangle = \{0, 5, 10, \rightarrow ...\}$ *saturated numerical semigroup from Theorem* 4(1). *In this case, we obtain* $m(S) = 5, f(S) = 9, n(S) = 2$ *and* $K = pn(S) = 5.2 = 10$. *Also,* $t(S) = p - 1 = 5 - 1 = 4$ *from Theorem* 7. *Thus,* S *has maximal length, since* $n(S)\,(t(S)+1) = 2\,(4+1) = 10 = K = F(S)+1$.

Example 2. *If we put* $j = 6, p = 7$ *and* $K = K(S) = 27$ *in Theorem* 5(6). *Then we write* $S = \langle 7, 27, 29, 30, 31, 32, 33\rangle = \{0, 7, 14, 21, 27, \rightarrow ...\}$. *Thus we find that* $m(S) = 7, f(S) = 26, n(S) = 4, t(S) = 6$ *and* $K = 27 = 7.4 - 1 = pn(S) - 1$. *Therefore,* S *has almost maximal length since* $n(S)\,(t(S)+1) = 4.\,(6+1) = 28 = 26 + 2 = F(S) + 2$.

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