Acta Universitatis Apulensis No. 71/2022
ISSN: 1582-5329 pp. 1-13
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2022.71.01

(F, φ, ω) - GREGUS TYPE CONTRACTION CONDITION APPROACH TO φ-FIXED POINT RESULTS IN METRIC SPACES

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Abstract. In this paper, we introduce (F, φ, ω) - Gregus type contraction, (F, φ, ω) - weak Gregus type contraction condition mappings and establish results of φ - fixed point for such mappings. Our results generalize some results of [1] and [2]. To support our results we illustrate example with numerical experiment for approximating the φ - fixed point.

2010 *Mathematics Subject Classification*: 47H10, 54H25.

Keywords: φ - fixed point, φ - Picard mapping, weakly φ - Picard mapping, (F, φ, ω) - Gregus type contraction, (F, φ, ω) - weak Gregus type contraction condition.

1. Introduction

1.1 φ - fixed points and (F, φ) - contraction mappings:

Recently, Jleli et al.[1] introduced an interesting concept of φ - fixed points, φ - Picard mappings and weakly φ - Picard mappings as follows:

Let X be a nonempty set, $\varphi : X \to [0, \infty)$ be a given function and $T : X \to X$ be a mapping. We denote the set of all fixed points of T by $F_T := \{x \in X : Tx = x\}$ and denote the set of all zeros of the function φ by $Z_\varphi := \{x \in X : \varphi(x) = 0\}$.

Definition 1. *[1] An element $z \in X$ is said to be a φ* - ***fixed point*** *of the operator T if and only if $z \in F_T \cap Z_\varphi$.*

Definition 2. *[1] The operator T is said*

(1) *to be a φ* - ***Picard mapping*** *if and only if*

(i) $F_T \cap Z_\varphi = \{z\}$,

(ii) $T^n x \to z$ *as* $n \to \infty$, *for each* $x \in X$.

(2) *to be a* ***weakly*** φ - ***Picard mapping*** *if and only if* *(i) T has at least one φ - fixed point,*

(ii) the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is a φ - fixed point T.

Also, Jleli et.al [1] introduced a new type of control function $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:
(F_1) $\max\{a, b\} \leq F(a, b, c)$
(F_2) $F(0, 0, 0) = 0$
(F_3) F is continuous

In this paper we are inserting the fourth condition as follows:
(F_4) $F(0, b, c) \leq F(a, b, c)$.

Throughout this paper, the class of all functions satisfying the conditions $(F_1)-(F_4)$ is denoted by $\mathcal{F}$.

Example 1. *Let $f_1, f_2, f_3 : [0, \infty)^3 \to [0, \infty)$ be defined by*
$f_1(a, b, c) = a + b + c,$
$f_2(a, b, c) = max\{a, b\} + c,$
$f_3(a, b, c) = a + a^2 + b + c,$
for all $a, b, c \in [0, \infty)$. Then $f_1, f_2, f_3 \in \mathcal{F}$.

Definition 3. *[1] Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is a (F, φ) - contraction with respect to the metric d if and only if for each $x, y \in X$ and for some constant $k \in (0, 1)$ such that*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq kF(d(x, y), \varphi(x), \varphi(y)). \tag{1}$$

Theorem 1. *[1] Let (X, d) be a complete metric space and $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. Suppose that the following condition holds:*
(a) φ is lower semi-continuous,
(b) $T : X \to X$ is a (F, φ) - contraction with respect to the metric d.
Then the following assertions hold:
(i) $F_T \subseteq Z_\varphi$,
(ii) T is a φ - Picard mapping,
(iii) if $x \in X$ and for $n \in N$, we have

$$d(T^n x, z) \leq \frac{k^n}{1-k} F(d(Tx, x), \varphi(Tx), \varphi(x)) \text{ where } \{z\} = F_T \cap Z_\varphi.$$

Let Ω be the set of all functions $\omega : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(j_1) ω is nondecreasing function,
(j_2) ω is continuous,
(j_3) $\lim\limits_{n\to\infty} \omega^n(t) = 0, \quad \forall\ t \in (0,\infty)$
(j_4) $\sum_{n=0}^{\infty} \omega^n(t) < \infty, \quad \forall\ t > 0.$

Definition 4. *[2] Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. The mapping $T : X \to X$ is said to be a (F,φ,ω) - contraction with respect to the metric d if and only if*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \omega(F(d(x,y),\varphi(x),\varphi(y))) \qquad \forall x,y \in X.$$

Theorem 2. *[2] Let (X,d) be a metric space, $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. Assume that the following conditions are satisfied:*

(H_1) *φ is lower semi continuous,*

(H_2) *$T : X \to X$ is an (F,φ,ω) - contraction with respect to the metric d.*

Then the following assertion hold:

(i) $F_T \subseteq Z_\varphi$,

(ii) *T is a φ - Picard mapping.*

Lemma 3. *[2] If $\omega \in \Omega$, then $\omega(t) < t \ \ \forall\ t > 0$.*

Remark 1. *[2] From j_1 and Lemma 3, we have $\omega(0) = 0$.*

The aim of the work: The main purpose of this paper is to be introduce the concept of (F,φ,ω) - Gregus type contraction mapping and (F,φ,ω) - weak Gregus type contraction mapping in metric space setting and establish φ - fixed point results. These results are partially extend and generalize the results of Jleli et al.[1] and Kumrod et al.[2]. Also proved and example to illustrate the results presented herein.

2. Main Results

2.1. (F,φ,ω) - Gregus type contraction condition

Definition 5. *Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. We say that the mapping $T : X \to X$ is an (F,φ,ω) - Gregus* *`type contraction condition` with respect to the metric d if and only if for any $x,y \in X$ and some $a \in (0,1]$*

$$F(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \leq \omega\Big(a\ F\big(d(x,y),\varphi(x),\varphi(y)\big) + (1-a)\,max\big\{F\big(d(x,Tx),\varphi(x),\varphi(y)\big), F\big(d(y,Tx),\varphi(x),\varphi(y)\big)\big\}\Big). \tag{2}$$

Now, we give the existence of φ- fixed point result for (F,φ,ω) - Gregus type contraction mapping.

Theorem 4. *Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. Suppose that the following conditions are satisfied:*
$(K1)\,\varphi$ is lower semi-continuous,
$(K2)\,T : X \to X$ is an (F,φ,ω) - Gregus type contraction with respect to the metric d.
Then the following conditions hold:
(i) $F_T \subseteq Z_\varphi$,
(ii) T is a φ - Picard mapping.

Proof. (i) Suppose that $\eta \in F_T$. Taking equation (1) with $x = y = \eta$, we have

$$\begin{aligned} F(0,\varphi(\eta),\varphi(\eta)) &\leq \omega\big(a\ F(0,\varphi(\eta),\varphi(\eta)) + (1-a)\,max\,\{F(0,\varphi(\eta),\varphi(\eta)), F(0,\varphi(\eta),\varphi(\eta))\}\big) \\ &= \omega\big(a\ F(0,\varphi(\eta),\varphi(\eta)) + (1-a)F(0,\varphi(\eta),\varphi(\eta))\big) \\ &= \omega(F(0,\varphi(\eta),\varphi(\eta))). \end{aligned}$$

Using Lemma 3, we obtain that

$$F(0,\varphi(\eta),\varphi(\eta)) = 0. \tag{3}$$

By the property of (F_1), we get

$$\varphi(\eta) \leq F(0,\varphi(\eta),\varphi(\eta)). \tag{4}$$

Using equation (3) and (4), we get $\varphi(\eta) = 0$ and then $\eta \in Z_\varphi$.
Hence condition (i) holds.

(ii) Let x be a arbitrary point in X, then, we have

$$\begin{aligned} &F(d(T^n x, T^{n+1}x),\varphi(T^n x),\varphi(T^{n+1}x)) \\ &\qquad \leq \omega\Big(aF(d(T^{n-1}x,T^n x),\varphi(T^{n-1}x),\varphi(T^n x)) \end{aligned}$$

$$
\begin{aligned}
&+(1-a)\, max \left\{F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x)),\right.\\
&\qquad \left. F(d(T^n x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x))\right\}\Big)\\
&= \omega\Big(a(F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x)), \varphi(T^n x))\\
&\qquad +(1-a)\max\left\{F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x)),\right.\\
&\qquad\qquad \left. F(0, \varphi(T^{n-1}x), \varphi(T^n x))\right\}\Big)
\end{aligned}
$$

Now, from (F_4), we get

$$
\begin{aligned}
&F(d(T^n x, T^{n+1}x), \varphi(T^n x), \varphi(T^{n+1}x))\\
&\quad \le \omega\Big(a(F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x)), \varphi(T^n x))\\
&\qquad\qquad +(1-a)F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x)\Big)\\
&\quad = \omega(F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x))).
\end{aligned}
$$

By induction for each $n \in N$ and using the property (F_1), we obtain that

$$
\begin{aligned}
max\{d(T^n x, T^{n+1}x), \varphi(T^n x)\} &\le F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x))\\
&\le \omega^n(F(d(x, Tx), \varphi(x), \varphi(Tx))).
\end{aligned} \tag{5}
$$

From equation (5), we have

$$
d(T^n x, T^{n+1}x) \le \omega^n(F(d(x, Tx), \varphi(x), \varphi(Tx))). \tag{6}
$$

Now, we prove that $\{T^n x\}$ is a Cauchy sequence. Suppose that $m, n \in$ such that $m > n$, we have

$$
\begin{aligned}
d(T^n x, T^m x) &\le d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \ldots + d(T^{m-1}x, T^m x)\\
&= \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))) + \omega^{n+1}(F(d(Tx, x), \varphi(Tx), \varphi(x)))\\
&\qquad + \ldots + \omega^{m-1}(F(d(Tx, x), \varphi(Tx), \varphi(x)))\\
&= \omega^n(1 + \omega + \ldots)(F(d(Tx, x), \varphi(Tx), \varphi(x)))\\
&= \sum_{i=1}^{m-1} \omega^i(F(d(Tx, x), \varphi(Tx), \varphi(x))) - \sum_{k=1}^{n-1} \omega^k(F(d(Tx, x), \varphi(Tx), \varphi(x))).
\end{aligned}
$$

Since $\omega \in \Omega$, then we get $\lim_{m,n\to\infty} d(T^n x, T^m x) = 0$, its leads to the sequence $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there is some point $z \in X$ such that

$$
\lim_{n\to\infty} d(T^n x, z) = 0. \tag{7}
$$

Finally, we have to prove that z is φ - fixed point of T. From (5), we can write,

$$\varphi(T^n x) \leq \omega^n\Big(F(d(x, Tx), \varphi(x), \varphi(Tx))\Big). \tag{8}$$

On taking limits in (8) and using j_3, we get

$$\lim_{n\to\infty} \varphi(T^n x) = 0. \tag{9}$$

Since φ is lower semi continuous and using (7), then we get

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(T^n x) = 0. \tag{10}$$

Taking $x = T^{n-1}x$ and $y = z$ in (2), we have

$$\begin{aligned}
&F(d(T^n x, Tz), \varphi(T^n x), \varphi(Tz)) \\
&\quad \leq \omega\Big(a\ F(d(T^{n-1}x, z), \varphi(T^{n-1}x), \varphi(z)) \\
&\qquad + (1-a)max\big\{F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(z)), \\
&\qquad\qquad F(d(z, T^n x), \varphi(T^{n-1}x), \varphi(z)\big\}\Big)
\end{aligned}$$

On taking limits as $n \to \infty$ in above inequality, using (7), (8) and (9), (F_2), (F_3) and using Lemma 3, we get

$$F(d(z, Tz), 0, \varphi(Tz)) \leq \omega\big(F(0,0,0)\big) = 0,$$

which imply that

$$d(z, Tz) = 0. \tag{11}$$

Then from equation (10) and (11) that z is φ - fixed point of T.

Uniqueness: Assume that z and z^* are two φ-fixed points of T. Applying equation(2) with $x = z$ and $y = z^*$. Then we obtain

$$\begin{aligned}
&F(d(Tz, Tz^*), \varphi(Tz), \varphi(Tz^*)) \\
&\quad \leq \omega\Big(a\ F(d(z, z^*), \varphi(z), \varphi(z^*)) \\
&\qquad + (1-a)max\big\{F(d(z, Tz), \varphi(z), \varphi(z^*)), F(d(z^*, Tz), \varphi(z), \varphi(z^*)\big\}\Big)
\end{aligned}$$

$$\begin{aligned}
&F(d(z, z^*), 0, 0)) \\
&\quad \leq \omega\Big(a\ F(d(z, z^*), 0, 0)) + (1-a)max\big\{F(0,0,0), F(d(z^*, Tz), 0, 0)\big\}\Big) \\
&\quad = \omega(F(d(z, z^*), 0, 0)) = 0.
\end{aligned}$$

By Lemma 3 and Remark 1, we obtain that $F(d(z, z^*), 0, 0) = 0$ and hence $d(z, z^*) = 0$. This implies that the φ - fixed point of T is unique $(\{z\} = F_T \cap Z_\varphi))$. So T is a φ - Picard mapping.

Theorem 5. *Under the hypothesis of Theorem 4, the following condition also hold T is a weakly φ - Picard operator.*

Proof. From equation (7) and (9)-(11) of Theorem 4, we get T is weakly φ - Picard operator.

Example 2. *Let $X = [0, 1]$ and $d : X \times X \to R$ be defined as $d(x, y) = |x - y|$ for all $x, y \in X$. Assume that $T : X \to X$ is defined as*

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{1-x}{2} & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

The function $\varphi : X \to [0, \infty)$ is define $\varphi(x) = \frac{x}{2}$ for all $x \in X$, the function $F : [0, +\infty)^3 \to [0, +\infty)$ is define by $F(a, b, c) = a + b + c$ and ω be a identity mapping on $^+$. At $a = \frac{3}{8}$.

Cases	LHS value of (2)		RHS value of (2)	
$x, y \in [0, \frac{1}{2}]$		0		Positive
$x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1]$		$\frac{3(1-y)}{4}$		$\frac{a(3y-2x)+(1-a)(3y)}{2}$
$y \in [0, \frac{1}{2}], x \in [\frac{1}{2}, 1]$		$\frac{3(1-x)}{4}$		$\frac{a(3x-y)+(1-a)(4x+y-1)}{2}$
$x, y \in [\frac{1}{2}, 1]$	$x = y$	$\frac{1-x}{2}$	$x = y$	$\frac{2ax+(1-a)(5x-1)}{2}$
	$x < y$	$\frac{3y-5x+2}{4}$	$x < y$	$\frac{a(3x-y)+(1-a)(3y+2x-1)}{2}$
	$x > y$	$\frac{3x-5y+2}{4}$	$x > y$	$\frac{a(3y-x)+(1-a)(3y+2x-1)}{2}$

It is easy to see that $F \in \mathcal{F}, \omega \in \Omega$ and φ is lower semi continuous. Finally, the above table shows that the mapping T satisfies the condition (2).

Now, we extend the contractive condition (2) and prove the second main result.

2.2. (F, φ, ω) - weak Gregus type contraction condition

Definition 6. *Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. We say that the mapping $T : X \to X$ is an (F, φ, ω) - `weak Gregus type contraction condition` with respect to the metric d if and only if*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \le \omega\Big(a\ (F(d(x, y), \varphi(x), \varphi(y))) + (1 - a)\, max\left\{F(M(x, y), \varphi(x), \varphi(y)), F(N(x, y), \varphi(x), \varphi(y))\right\}\Big), \tag{12}$$

where $M(x, y) = max\{d(x, y), d(x, Tx)\}$ and
$N(x, y) = min\{d(x, Ty), d(y, Tx), d(y, Ty)\}$, $\forall\, x, y \in X$ and for some $a \in (0, 1]$.

Now, we give the existence of φ- fixed point result for (F,φ,ω) - weak Gregus type contraction mapping.

Theorem 6. *Let (X,d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. Suppose that the following conditions are satisfied:*
$(K1)$ φ is lower semi-continuous,
$(K2)$ $T : X \to X$ is an (F,φ,ω) - Gregus type contraction with respect to the metric d,
Then the following conditions hold:
(i) $F_T \subseteq Z_\varphi$,
(ii) T is a φ - Picard mapping.

Proof. (i) Suppose that $\eta \in F_T$. Taking equation (12) with $x = y = \eta$, we have

$$\begin{aligned} F(0,\varphi(\eta),\varphi(\eta)) \leq \omega\Big(&a\ (F(0,\varphi(\eta),\varphi(\eta))) \\ &+ (1-a)\, max\left\{F(0,\varphi(\eta),\varphi(\eta)), F(0,\varphi(\eta),\varphi(\eta))\right\}\Big) \\ = \omega(&F(0,\varphi(\eta),\varphi(\eta))). \end{aligned}$$

where $M(x,y) = 0 = N(x,y)$.
Using Lemma 3, we obtain that

$$F(0,\varphi(\eta),\varphi(\eta)) = 0. \tag{13}$$

By the property of (F_1), we have

$$\varphi(\eta) \leq F(0,\varphi(\eta),\varphi(\eta)). \tag{14}$$

Using equation (13) and (14), we get $\varphi(\eta) = 0$ and then $\eta \in Z_\varphi$.
Hence condition (i) holds.
(ii) Let x be arbitrary point in X, then we have

$$\begin{aligned} &F(d(T^n x, T^{n+1}x), \varphi(T^n x), \varphi(T^{n+1}x)) \\ &\quad \leq \omega\Big(a\ F(d(T^{n-1}x, T^n x), \varphi(T^n x), \varphi(T^{n-1}x)) \\ &\qquad +(1-a)\ max\ \{F(M(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x))\}, \\ &\qquad\qquad F(N(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x))\}\Big) \end{aligned}$$

where $M(x,y) = M(T^{n-1}x, T^n x) = d(T^{n-1}x, T^n x)$
and $N(x,y) = N(T^{n-1}x, T^n x)$
$= min\{d(T^{n-1}x, T^{n+1}x), d(T^n x, T^n x), d(T^n x, T^{n+1}x) = 0.$

Then above inequality reduced to

$$\begin{aligned}
&F(d(T^n x, T^{n+1}x), \varphi(T^n x), \varphi(T^{n+1}x)) \\
&\quad \leq \omega\Big(a\ F(d(T^{n-1}x, T^n x), \varphi(T^n x), \varphi(T^{n-1}x)) \\
&\qquad\qquad +(1-a)\ max\ \big\{F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n x))\}, \\
&\qquad\qquad\qquad F(0, \varphi(T^{n-1}x), \varphi(T^n x))\big\} \\
&\quad \leq \omega\big(a\ F(d(T^{n-1}x, T^n x), \varphi(T^n x)), \varphi(T^{n-1})x) \\
&\qquad\qquad +(1-a)F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x), \varphi(T^n)x)\big) \quad \text{(by } F_4\text{)} \\
&\quad = \omega(F(d(T^{n-1}x, T^n x), \varphi(T^n x), \varphi(T^{n-1}x)))
\end{aligned}$$

By induction for each $n \in N$ and using the property (F_1), we obtain that

$$\begin{aligned}
max\{d(T^{n+1}x, T^n x), \varphi(T^{n+1}x)\} &\leq F(d(T^{n+1}x, T^n x), \varphi(T^{n+1}x), \varphi(T^n x)) \\
&\leq \omega(F(d(T^{n-1}x, T^n x), \varphi(T^n x), \varphi(T^{n-1}x))) \quad (15) \\
&\leq \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))).
\end{aligned}$$

From (15), we have

$$d(T^{n+1}x, T^n x) \leq \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))). \quad (16)$$

Now, We prove that $\{T^n x\}$ is a Cauchy sequence. Suppose that $m, n \in$ such that $m > n$, we have

$$\begin{aligned}
d(T^n x, T^m x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \ldots + d(T^{m-1}x, T^m x) \\
&= \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))) + \omega^{n+1}(F(d(Tx, x), \varphi(Tx), \varphi(x))) \\
&\qquad + \ldots + \omega^{m-1}(F(d(Tx, x), \varphi(Tx), \varphi(x))) \\
&= \omega^n(1 + \omega + \ldots)(F(d(Tx, x), \varphi(Tx), \varphi(x))) \\
&= \sum_{i=1}^{m-1} \omega^i(F(d(Tx, x), \varphi(Tx), \varphi(x))) - \sum_{k=1}^{n-1} \omega^k(F(d(Tx, x), \varphi(Tx), \varphi(x))).
\end{aligned}$$

By using (j_3) and (j_4), then we get $\lim_{m,n\to\infty} d(T^n x, T^m x) = 0$, its leads to the sequence $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there is some $z \in X$ such that

$$\lim_{n\to\infty} d(T^n x, z) = 0. \quad (17)$$

Finally, we have to prove that z is φ - fixed point of T. From (5), we can write,

$$\varphi(T^{n+1}x) \leq \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))). \quad (18)$$

On taking limits in (18) and from (j_2), we get

$$\lim_{n\to\infty} \varphi(T^{n+1}x) = 0. \tag{19}$$

Since φ is lower semi continuous then the equation (17) - (19), then, we get

$$\varphi(z) \leq \liminf_{n\to\infty} \varphi(T^{n+1}x) = 0. \tag{20}$$

On taking $x = T^{n-1}x$ and $y = z$ in (12), we get

$$\begin{aligned} &F(d(T^n x, Tz), \varphi(T^n x), \varphi(Tz)) \\ &\quad \leq \omega\Big(a\ (F(d(T^n x, z), \varphi(T^{n-1}x), \varphi(z))) \\ &\qquad\qquad + (1-a)\, max\, \{F(M(T^{n-1}x, z), \varphi(T^{n-1}x), \varphi(z)), \\ &\qquad\qquad\qquad\qquad F(N(T^{n-1}x, z), \varphi(T^{n-1}x), \varphi(z))\}\Big) \end{aligned}$$

On taking limit as $n \to \infty$ in above inequality and using equation (19) - (20), the properties F_2, F_3 and also using Lemma 3. and Remark 1. then, we get

$$F(d(z, Tz), 0, \varphi(Tz)) \leq \omega(F(0, 0, 0)) = 0,$$

which implies that

$$d(z, Tz) = 0. \tag{21}$$

Then from equation (20) and (21) that z is φ - fixed point of T (i.e.,$z \in F_T \cap Z_\varphi$).
Finally, we have to show that T - is a φ - Picard mapping. It is sufficient to show that assume that z and z^* are two φ - fixed points of T. Applying equation (12) with $x = z$ and $y = z^*$. Then we obtain that

$$\begin{aligned} &F(d(Tz, Tz^*), \varphi(Tz), \varphi(Tz^*)) \\ &\quad \leq \omega\Big(a\ (F(d(z, z^*), \varphi(z), \varphi(z^*))) + (1-a)\, max\big\{F(M(z, z^*), \varphi(z), \varphi(z^*)), \\ &\qquad\qquad\qquad\qquad F(N(z, z^*), \varphi(z), \varphi(z^*))\big\}\Big) \end{aligned}$$

where $M(x, y) = d(z, z^*)$ and $N(x, y) = 0$.
By using(F_4) and Lemma 3. and Remark 1., then we get

$$F(d(Tz, Tz^*), 0, 0)) \leq \omega\big(F(d(z, z^*), 0, 0\big) = 0.$$

Hence $d(z, z^*) = 0$. This implies that the φ - fixed point of T is unique ($\{z\} = F_T \cap Z_\varphi$). So T is a φ - Picard mapping.

Theorem 7. *Under the hypothesis of Theorem 6., the following condition also hold T is a weakly φ - Picard operator.*

Proof. From equation (17) and (19)-(21) of Theorem 6., we get T is weakly φ - Picard
operator.

3. Example

Example 3. *Let $X = [0, 1]$ and $d : X \times X \to$ be define by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Suppose that $T : X \to X$ is defined by*

$$T(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{2}, \\ k \log(x+1) & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

where $k \in [0, 1)$, the function $\varphi : X \to [0, \infty)$ is define $\varphi(x) = \frac{x}{2}$ for all $x \in X$, the function $F : [0, +\infty)^3 \to [0, +\infty)$ is define by $F(a, b, c) = a + b + c$ and the function $\omega : [0, +\infty) \to [0, +\infty)$ is define by

$$\omega(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1, \\ 5k \log 6t & \text{if } t > 1. \end{cases}$$

It is easy to see that $F \in \mathcal{F}$, $\omega \in \Omega$ and φ satisfies lower semi-continuous condition.

Now, we have to show that T satisfies condition equation (12).
Case 1: *Suppose that $x, y \in [0 \le x < \frac{1}{2})$, then T holds equation (12) trivially.*
Case 2: *Suppose that $x, y \in [\frac{1}{2} \le x \le 1)$. We assume that $y \le x$. Then we have*

$$\begin{aligned} F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) &= d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \\ &= |k \log(x+1) - k \log(y+1)| + \frac{k \log(x+1)}{2} + \frac{k \log(y+1)}{2} \\ &\le k \log(x+1) \\ &< 5k \log(6) \le \ RHS \text{ of } (12). \end{aligned}$$

Case 3: *Suppose that $x, \in [\frac{1}{2}, 1)$ and $y, \in [0, \frac{1}{2})$. Then we have*

$$\begin{aligned} F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) &= d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \\ &= |k \log(x+1) - 0| + \frac{k \log(x+1)}{2} + 0 \\ &= k \log(x+1) + \frac{k \log(x+1)}{2} \\ &= \frac{3}{2} k \log(x+1) \\ &< 5k \log(6) \le \ RHS \text{ of } (12). \end{aligned}$$

All the hypothesis of Theorem 6., are satisfied and 0 is a φ - fixed point the operator T and also fixed point of T.
We can see from the following table approximating the φ - fixed point of T at two different values of k.

k = 0.4	$x_0 = 0.5$	$x_0 = 0.7$	$x_0 = 0.9$	k = 0.8	$x_0 = 0.5$	$x_0 = 0.7$	$x_0 = 0.9$
x_1	0.0704	0.0921	0.1115	x_1	0.1408	0.1843	0.2230
x_2	0.0118	0.0153	0.0184	x_2	0.0458	0.0588	0.0699
x_3	0.0020	0.0026	0.0031	x_3	0.0155	0.0198	0.0234
x_4	0.0000	0.0000	0.0000	x_4	0.0054	0.0068	0.0080
x_5	0.0000	0.0000	0.0000	x_5	0.0000	0.0000	0.0000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1 and Table 2 iterates of Picard iteration for two different values of k

And also, the convergence behavior of these iterations in shown in Fig. 1

aks2-eps-converted-to.pdf

aks1-eps-converted-to.pdf

Fig.1: left figure for $k = 0.4$ and right figure for $k = 0.8$.

References

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