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SUBCLASSES OF UNIVALENT FUNCTIONS INVOLVING TOUCHARD POLYNOMIALS

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Abstract. Making use of the Touchard polynomials, we introduce and study new subclasses of univalent functions related to conic domain defined by convolution. The main object of this paper is to investigate sufficient conditions for this convolution operator belonging to various subclasses of univalent functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions $\mathfrak{l}$ of the form

$$\mathfrak{l}(\varsigma) = \varsigma + \sum_{\psi=2}^{\infty} a_\psi \, \varsigma^\psi \tag{1}$$

normalized by $\mathfrak{l}(0) = \mathfrak{l}'(0) - 1 = 0$ in the open unit disk $\mathbb{U} = \{\varsigma : \varsigma \in \mathbb{C} \text{ and } |\varsigma| < 1\}$ and its subclass consisting of all univalent functions is denoted by Υ. The classes $\Upsilon^*(\mu)$ and $\vartheta(\mu)$ belongs to Υ are starlike and convex functions of order μ $(0 \le \mu < 1)$ are usually characterized by the quantities $\varsigma\mathfrak{l}'(\varsigma)/\mathfrak{l}(\varsigma)$ and $(1 + \varsigma\mathfrak{l}''(\varsigma))/\mathfrak{l}'(\varsigma)$ lying in a certain domain starlike with respect to 1 in the right-half plane. For more details, see ([31],[35]). In fact $\Upsilon^*(\mu) \subseteq \Upsilon^*(0) \equiv \Upsilon^*$, $\vartheta(\mu) \subseteq \vartheta(0) \equiv \vartheta$ and $\vartheta(\mu) \subset \Upsilon^*(\mu) \subset \Upsilon$ for $0 \le \mu < 1$, but $\mu < 0$ the functions $\Upsilon^*(\mu)$ need not to be univalent. For more details, see ([13],[18],[21],[23],[24],[31],[35],[42]). It is well known fact that $\mathfrak{l} \in \vartheta(\mu) \Leftrightarrow \varsigma\mathfrak{l}' \in \Upsilon^*(\mu)$.

Let $\Im, \wp, \imath \in \mathbb{C}, \imath \neq 0, -1, -2, \cdots$. For the hypergeometric function ${}_2\Omega_1(\Im, \wp; \imath; \varsigma)$, which is defined in $|\varsigma| < 1$ by the series

$${}_2\Omega_1(\Im, \wp; \imath; \varsigma) = \sum_{\psi=0}^{\infty} \frac{(\Im)_\psi (\wp)_\psi}{(\imath)_\psi} \frac{\varsigma^\psi}{\psi!},$$

where $(\lambda)_\psi$ denotes the Pochhammer symbol given by $(\lambda)_\psi = \Gamma(\lambda+\psi)/\Gamma(\lambda) = \lambda(\lambda+1)...(\lambda+\psi-1)$, $(\lambda)_0 = 1$ and $(1)_\psi = \psi!$. The hypergeometric functions play an prominent role in many fields of geometric function theory. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the Bessel functions ([8],[9],[10],[32],[39]), Struve functions ([29],[30],[46]), Gaussian hypergeometric functions ([41],[43],[45]) and Lommel functions ([44],[47]). Merkes and Scott [26] proved a result involving starlike hypergeometric functions. In 1984, Carlson and Shaffer investigated certain classes of starlike, convex and prestarlike functions of order ρ by applying a linear operator defined by a certain convolution. In 1986, Ruscheweyh and Singh [37] presented a study of starlikeness of certain hypergeometric functions by using the method of continued fractions. Recently, Dizok and Srivastava ([15],[16]) and Kim and Srivastava [22] introduce and studied a class of analytic functions associated with the generalized hypergeometric functions. For a detailed study and recent development on hypergeometric functions, the reader may see ([2],[6],[25],[34]) and the references listed therein. For parameters $\rho_j \in \mathbb{C}\,(j = 1, 2, 3, ..., p)$ and $\mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^-\,(j = 1, 2, 3, ..., q; \mathbb{Z}_0^- := \{0, -1, -2, ...\})$, function ${}_p\Omega_q$ is given by

$$\begin{aligned} {}_p\Omega_q(\varsigma) &= {}_p\Omega_q(\rho_1, ..., \rho_p, \mu_1, ..., \mu_q; \varsigma) \\ &:= \sum_{\psi=0}^{\infty} \frac{(\rho_1)_\psi ... (\rho_\psi)_n}{(\mu_1)_\psi ... (\mu_q)_\psi} \frac{\varsigma^\psi}{\psi!}, \end{aligned} \tag{2}$$

$$(\, p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \;\; \mathbb{N} := \{1, 2, 3..\}; \;\; p \le q+1 \text{ and } \varsigma \in \mathbb{U}).$$

In [16], the linear operator $\mathcal{H}_q^p(\rho_1, ..., \rho_p; \mu_1, ..., \mu_q) : \mathcal{A} \to \mathcal{A}$ defined by the following Hadamard product:

$$\mathcal{H}_q^p(\rho_1, ..., \rho_p; \mu_1, ..., \mu_q)\mathfrak{l}(\varsigma) = \Xi(\rho_1, ..., \rho_p; \mu_1, ..., \mu_q; \varsigma) * \mathfrak{l}(\varsigma), \tag{3}$$

where

$$\Xi(\rho_1, ..., \rho_p; \mu_1, ..., \mu_q; \varsigma) = \varsigma\, {}_p\Omega_q(\rho_1, ..., \rho_p; \mu_1, ..., \mu_q; \varsigma).$$

By virtue of (1) and (3), it is observed that

$$\begin{aligned} \mathcal{H}_q^p(\rho_1, ..., \rho_p; \mu_1, ..., \mu_q)\mathfrak{l}(\varsigma) &= \varsigma + \sum_{\psi=2}^{\infty} \frac{(\rho_1)_{\psi-1} ... (\rho_p)_{\psi-1}}{(\mu_1)_{\psi-1} ... (\mu_q)_{\psi-1} (1)_{\psi-1}} a_\psi\, \varsigma^\psi \\ &= \varsigma + \sum_{\psi=2}^{\infty} \Upsilon_\psi(\rho_1) a_\psi\, \varsigma^\psi, \end{aligned} \tag{4}$$

where

$$\Upsilon_\psi(\rho_1) = \frac{(\rho_1)_{\psi-1}...(\rho_p)_{\psi-1}}{(\mu_1)_{\psi-1}...(\mu_q)_{\psi-1}(1)_{\psi-1}}.$$

For convenience, we write

$$\mathcal{H}_q^p(\rho_1;\mu_1) := \mathcal{H}(\rho_1,...,\rho_p;\mu_1,...,\mu_q).$$

It should be noted that the linear operator is a generalization of many operators considered earlier.
(1) $\mathcal{H}_1^2(\mu+1,1;1)\mathfrak{l}(\varsigma) := \mathcal{D}^\mu \mathfrak{l}(\varsigma)\,(\mu > -1)$; see ([36]);
(2) $\mathcal{H}_1^2(\Im,1;\imath)\mathfrak{l}(\varsigma) := \mathcal{L}(\Im,\imath)\mathfrak{l}(\varsigma)\,(\Im \in \mathbb{R}, \imath \in \mathbb{R}\backslash\mathbb{Z}_0^-)$; see ([11]);
(3)$\mathcal{H}_1^2(1,1;\psi+1)\mathfrak{l}(\varsigma) := \mathcal{I}_\psi \mathfrak{l}(\varsigma)\,(\psi \in \mathbb{N}_0)$; see ([28]);
(4) $\mathcal{H}_1^2(l,m;n)\mathfrak{l}(\varsigma) := \mathcal{I}_\psi^{l,m}\mathfrak{l}(\varsigma)\,(l,m \in \mathbb{C}, \psi \neq \mathbb{Z}_0^-)$; see ([20]);
(5) $\mathcal{H}_1^2(1,1;\delta+2)\mathfrak{l}(\varsigma) := (\delta+1)/z^\delta \int_0^z t^{(\delta-1)} f(t)dt\,(\delta > -1)$; see ([7]);
(6) $\mathcal{H}_1^2(\delta+1,\imath;\Im)\mathfrak{l}(\varsigma) := \mathcal{I}^\delta(\Im,\imath)\,(\Im,\imath \in \mathbb{R}\backslash\mathbb{Z}_0^-, \delta > -1)$; see ([14]).

El-Ashwah et al. [17] defined the linear extended multiplier Dizok-Srivastava operator $\Psi_{\lambda,t}^{s,p,q}$ as follows:

$$\begin{aligned}
&\Psi_{\lambda,t}^{0,p,q}\mathfrak{l}(\varsigma) = \mathfrak{l}(\varsigma),\\
&\Psi_{\lambda,t}^{1,p,q}\mathfrak{l}(\varsigma) = (1-\lambda) + \mathcal{H}_q^p(\rho_1)\mathfrak{l}(\varsigma) + \frac{\lambda}{(t+1)\varsigma^{t-1}}(z^t\mathcal{H}_q^p(\rho_1)\mathfrak{l}(\varsigma))',\\
&\Psi_{\lambda,t}^{2,p,q}\mathfrak{l}(\varsigma) = \Psi_{\lambda,t}^{p,q}(\Psi_{\lambda,t}^{1,p,q}\mathfrak{l}(\varsigma)),\\
&\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma) = \Psi_{\lambda,t}^{p,q}(\Psi_{\lambda,t}^{s-1,p,q}\mathfrak{l}(\varsigma)) \quad (\lambda \geq 0, t \geq 0, s \in \mathbb{N}_0).
\end{aligned} \tag{5}$$

In view of (1) and (5), we get

$$\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma) = \varsigma + \sum_{\psi=2}^{\infty} \varpi_{\psi,s}(\rho_1,\lambda,t)a_\psi\,\varsigma^\psi \quad (s \in \mathbb{N}_0), \tag{6}$$

where

$$\varpi_{\psi,s}(\rho_1,\lambda,t) = \Big[\frac{t+1+\lambda(\psi-1)}{t+1}\Upsilon_\psi(\rho_1)\Big]^s. \tag{7}$$

Using the operator $\Psi_{\lambda,t}^{s,p,q}$ the following interesting classes of analytic functions are defined as follows:

$$\mathcal{UCV}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta) := \left\{\mathfrak{l} : \Re\left(1 + \frac{\varsigma(\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma))''}{(\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma))'} - \zeta\right) > \xi\left|\frac{\varsigma(\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma))''}{(\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma))'}\right|\right\} \tag{8}$$

and

$$\mathcal{SP}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta) := \left\{ \mathfrak{l} : \Re\left(1 + \frac{\varsigma(\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma))'}{\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma)} - \zeta\right) > \xi \left|\frac{\varsigma(\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma))'}{\Psi_{\lambda,t}^{s,p,q}\mathfrak{l}(\varsigma)}\right| \right\}, \quad (9)$$

for $\xi \geq 0, -1 \leq \zeta < 1, \xi + \zeta \geq 1, p \leq q+1, \lambda \geq 0, t \geq 0, s \in \mathbb{N}_0$. Indeed it follows from (8) and (9) that $\mathfrak{l} \in \mathcal{UCV}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta) \Leftrightarrow \varsigma\mathfrak{l}' \in \mathcal{SP}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta)$. Several authors considered various special cases of above mentioned classes by considering some special parmeters, namely $\mathcal{SP}_{0,0}^{1,p,q}(\rho_1,\mu_1;\xi,\zeta) := \mathcal{SP}^{p,q}(\rho_1;\xi,\zeta)$ [4], $\mathcal{SP}_{0,0}^{1,2,1}(A,1;B;\mu_1,\xi,\zeta) := \mathcal{SP}(A,B;\xi,\zeta)(A>0,B>0)$([19],[27]), $\mathcal{SP}_{\lambda,0}^{s,2,1}(2,1;2-\rho;\xi,\zeta){:=}\mathcal{SP}_{\rho,\lambda}^{s}(\xi,\zeta)$ [5] and $\mathcal{SP}_{0,0}^{1,2,1}(\eta+1,1;1;\xi,\zeta){:=}\ \mathcal{SP}(\eta;\xi,\zeta)(\eta>-1)$[38]. We need the following lemmas to prove our main results.

Lemma 1. ([17, p. 324, Corollary 3]) *A function $\mathfrak{l} \in \mathcal{A}$ of the form* (1) *is in $\mathcal{UCV}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta)$, if*

$$\sum_{\psi=2}^{\infty} \psi[\psi(1+\xi) - (\xi+\zeta)]\, \varpi_{\psi,s}(\rho_1,\lambda,t)|a_\psi| \leq 1-\zeta. \quad (10)$$

Lemma 2. ([17, p. 323, Theorem 3]) *A function $\mathfrak{l} \in \mathcal{A}$ of the form* (1) *is in $\mathcal{SP}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta)$, if*

$$\sum_{\psi=2}^{\infty} [\psi(1+\xi) - (\xi+\zeta)]\, \varpi_{\psi,s}(\rho_1,\lambda,t)|a_\psi| \leq 1-\zeta. \quad (11)$$

The t^{th} moment of Poisson distribution with expected value w is given by

$$\mu_t' = e^{-w} \sum_{\psi=0}^{\infty} \frac{\psi^t w^\psi}{\psi!}. \quad (12)$$

Recently, Alshaqui [1] introduce a series with Touchard polynomials coefficients after the second force as following

$$\Phi(\kappa,w,\varsigma) = \varsigma + \sum_{\psi=2}^{\infty} \frac{(\psi-1)^\kappa w^{\psi-1}}{(\psi-1)!} e^{-w} \varsigma^\psi \quad (\kappa \geq 0, w > 0). \quad (13)$$

The Hadamard product (or convolution) of two power series $\mathfrak{j}(\varsigma) = \sum\limits_{\psi=2}^{\infty} u_\psi\, \varsigma^\psi$ and $\mathfrak{i}(\varsigma) = \sum\limits_{\psi=2}^{\infty} v_\psi\, \varsigma^\psi$ is defined as

$$(\mathfrak{j} * \mathfrak{i})(\varsigma) = (\mathfrak{j} * \mathfrak{i})(\varsigma) = \sum_{\psi=2}^{\infty} u_\psi v_\psi \varsigma^\psi.$$

Next, using Hadamard product (or convolution) the linear operator $\mathcal{F}(\kappa, w, \varsigma) : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{F}(\kappa, w, \varsigma)\mathfrak{l} = \Phi(\kappa, w, z) * \mathfrak{l}(\varsigma) = \varsigma + \sum_{\psi=2}^{\infty} \frac{(\psi-1)^{\kappa} w^{\psi-1}}{(\psi-1)!} e^{-w} a_{\psi} \varsigma^{\psi} \quad (\kappa \geq 0, w > 0). \tag{14}$$

Motivated by results on connections between various subclasses of analytic functions by using special functions (refer[3],[33],[40]), we establish mapping properties, inclusion relations for the classes $\mathcal{UCV}_{\lambda,t}^{s,p,q}(\rho_1, \mu_1, \xi, \zeta)$ and $\mathcal{SP}_{\lambda,t}^{s,p,q}(\rho_1, \mu_1, \xi, \zeta)$ by applying convolution operator given by (14).

2. Main Results

For convenience throughout in the sequel, we use the following notations

$$\sum_{\psi=2}^{\infty} \frac{w^{\psi-1}}{(\psi-1)!} = e^{w} - 1 \tag{15}$$

$$\sum_{\psi=2}^{\infty} \frac{w^{\psi-1}}{(\psi-2)!} = we^{w} \tag{16}$$

$$\sum_{\psi=2}^{\infty} \frac{w^{\psi-1}}{(\psi-3)!} = w^2 e^{w}. \tag{17}$$

Theorem 3. *Let $w > 0, t \in \mathbb{N}_0$ then $\Phi(\kappa, w, \varsigma) \in \mathcal{SP}_{\lambda,t}^{s,p,q}(\rho_1, \mu_1, \xi, \zeta)$ if and only if*

$$e^{-w} \varpi_{\psi,s}(\rho_1, \lambda, t) \begin{cases} (1+\xi)we^{w} + (1-\zeta)(e^{w}-1) \leq 1-\zeta, & \kappa = 0, \\ (1+\xi)\mu'_{t+1} + (1-\zeta)\mu'_{t} \leq 1-\zeta, & \kappa \geq 1, \end{cases} \tag{18}$$

where $\varpi_{\psi,s}(\rho_1, \lambda, t)$ is given by (7).

Proof. Since

$$\Phi(\kappa, w; \varsigma) = \varsigma + \sum_{\psi=2}^{\infty} \frac{(\psi-1)^{\kappa} w^{\psi-1}}{(\psi-1)!} e^{-w} \varsigma^{\psi}.$$

In view of Lemma 2, it is sufficient to show that

$$\sum_{\psi=2}^{\infty} \Big[\psi(1+\xi) - (\xi+\zeta)\Big] \varpi_{\psi,s}(\rho_1, \lambda, t) \frac{(\psi-1)^{\kappa} w^{\psi-1}}{(\psi-1)!} e^{-w} \leq 1-\zeta.$$

Now

$$\sum_{\psi=2}^{\infty}\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\varpi_{\psi,s}(\rho_1,\lambda,t)\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}e^{-w}$$

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\sum_{\psi=2}^{\infty}\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\frac{(n-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}$$

Writing $\psi=(\psi-1)+1$, we get

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\sum_{\psi=2}^{\infty}\Big[(1+\xi)(\psi-1)+1-\zeta\Big]\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}$$

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\left[(1+\xi)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa+1}w^{\psi-1}}{(\psi-1)!}+(1-\zeta)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}\right]$$

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\left[(1+\xi)\sum_{\psi=1}^{\infty}\frac{\psi^{\kappa+1}w^{\psi}}{\psi!}+(1-\zeta)\sum_{\psi=1}^{\infty}\frac{\psi^{\kappa}w^{\psi}}{(\psi)!}\right]$$

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\begin{cases}(1+\xi)we^{w}+(1-\zeta)(e^{w}-1), & \kappa=0,\\ (1+\xi)\mu'_{t+1}+(1-\zeta)\mu'_{t}, & \kappa\geq 1\end{cases}$$

which evidently completes the proof.

Theorem 4. *Let $w>0, t\in\mathbb{N}_0$ then $\Phi(\kappa,w,z)\in\mathcal{UCV}^{s,p,q}_{\lambda,t}(\rho_1,\mu_1,\xi,\zeta)$ if and only if*

$$e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\begin{cases}(1+\xi)(w^2+w)e^{w}+(2+\xi-\zeta)we^{w}+(1-\zeta)(e^{w}-1)\leq 1-\zeta, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \kappa=0,\\ (1+\xi)\mu'_{t+2}+(2+\xi-\zeta)\mu'_{t+1}+(1-\xi)\mu'_{t}\leq 1-\zeta, \quad \kappa\geq 1,\end{cases} \tag{19}$$

where $\varpi_{\psi,s}(\rho_1,\lambda,t)$ is given by (7).

Proof. Since

$$\Phi(\kappa,w;\varsigma)=\varsigma+\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}e^{-w}\varsigma^{\psi}.$$

In view of Lemma 1, it is sufficient to show that

$$\sum_{\psi=2}^{\infty}\psi\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\varpi_{\psi,s}(\rho_1,\lambda,t)\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}e^{-w}\leq 1-\zeta.$$

Now

$$\sum_{\psi=2}^{\infty}\psi\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\varpi_{\psi,s}(\rho_1,\lambda,t)\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}e^{-w}$$

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\sum_{\psi=2}^{\infty}\Big[\psi^2(1+\xi)-\psi(\xi+\zeta)\Big]\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}$$

Writing $\psi^2=(\psi-2)(\psi-1)+3(\psi-1)+1$ and $\psi=(\psi-1)+1$, we get

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\sum_{\psi=2}^{\infty}\Big[(1+\xi)(\psi-1)^2+(2+\xi-\gamma)+1-\zeta\Big]\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}$$

$$=e^{-w}\varpi_{\psi,s}(\rho_1,\lambda,t)\Bigg[(1+\xi)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa+2}w^{\psi-1}}{(\psi-1)!}+(2+\xi-\zeta)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa+1}w^{\psi-1}}{(\psi-1)!}$$

$$+(1-\zeta)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}\Bigg]$$

$$=e^{-w}\varpi_{n,s}(\rho_1,\lambda,t)\begin{cases}(1+\xi)(w^2+w)e^{w}+(2+\xi-\zeta)we^{w}+(1-\zeta)(e^{w}-1)\le 1-\zeta, & \kappa=0,\\ (1+\xi)\mu'_{t+2}+(2+\xi-\zeta)\mu'_{t+1}+(1-\xi)\mu'_{t}\le 1-\zeta, & \kappa\ge 1\end{cases}$$

which evidently completes the proof.

3. Inclusion Properties

In this section, by making use of the Lemma 5, we study the action of the Touchard polynomial on the class $\mathcal{UCV}_{\lambda,t}^{s,p,q}(\rho_1,\mu_1,\xi,\zeta)$, if it satisfies the following inequality introduced by Dixit and Pal [12]

$$\left|\frac{\mathfrak{l}'(\varsigma)-1}{(\gamma-\delta)\tau-\delta[\mathfrak{l}'(\varsigma)-1]}\right|<1,\quad(\varsigma\in\mathbb{U}),\tag{20}$$

where $\tau\in\mathbb{C}\backslash\{0\},-1\le\gamma<\delta\le 1$.

Lemma 5. *[12] If* $\mathfrak{l}\in\mathcal{R}^{\tau}(\gamma,\delta)$ *is of the form* (1) *then*

$$|a_\psi|=\frac{(\gamma-\delta)|\tau|}{\psi},\qquad\psi\in N\backslash\{1\}.\tag{21}$$

Theorem 6. *Let* $w > 0, t \in \mathbb{N}_0$ *and* $\mathfrak{l} \in \mathcal{R}^{\tau}(\gamma, \delta)$ *then* $\mathcal{F}(\kappa, w, \varsigma) \in \mathcal{UCV}^{s,p,q}_{\lambda,t}(\rho_1, \mu_1, \xi, \zeta)$ *if and only if*

$$(\gamma-\delta)|\tau|e^{-w}\varpi_{\psi,s}(\rho_1, \lambda, t)\begin{cases}(1+\xi)we^{w} + (1-\zeta)(e^{w}-1) \leq 1-\zeta, & \kappa = 0,\\ (1+\xi)\mu'_{t+1} + (1-\zeta)\mu'_{t} \leq 1-\zeta, & \kappa \geq 1,\end{cases} \tag{22}$$

where $\varpi_{\psi,s}(\rho_1, \lambda, t)$ *is given by* (7).

Proof. By virtue of Lemma 1, it is sufficient to show that

$$\sum_{\psi=2}^{\infty}\psi\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\varpi_{\psi,s}(\rho_1, \lambda, t)\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}|a_{\psi}|e^{-w} \leq 1-\zeta.$$

In view of Lemma 5, we have

$$\begin{aligned}&\sum_{\psi=2}^{\infty}\psi\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\varpi_{\psi,s}(\rho_1, \lambda, t)\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}|a_{\psi}|e^{-w}\\ \leq\, & e^{-w}\varpi_{\psi,s}(\rho_1, \lambda, t)\sum_{\psi=2}^{\infty}\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}(\gamma-\delta)|\tau|\end{aligned}$$

Writing $\psi = (\psi-1)+1$, we get

$$\begin{aligned}=&(\gamma-\delta)|\tau|e^{-w}\varpi_{\psi,s}(\rho_1, \lambda, t)\sum_{n=2}^{\infty}\Big[(1+\xi)(\psi-1)+1-\zeta\Big]\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}\\ =&(\gamma-\delta)|\tau|e^{-w}\varpi_{\psi,s}(\rho_1, \lambda, t)\left[(1+\xi)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa+1}w^{\psi-1}}{(\psi-1)!}+(1-\zeta)\sum_{\psi=2}^{\infty}\frac{(\psi-1)^{\kappa}w^{\psi-1}}{(\psi-1)!}\right]\\ =&(\gamma-\delta)|\tau|e^{-w}\varpi_{\psi,s}(\rho_1, \lambda, t)\left[(1+\xi)\sum_{\psi=1}^{\infty}\frac{\psi^{\kappa+1}w^{\psi}}{\psi!}+(1-\zeta)\sum_{\psi=1}^{\infty}\frac{\psi^{\kappa}w^{\psi}}{(\psi)!}\right]\\ =&(\gamma-\delta)|\tau|e^{-w}\varpi_{\psi,s}(\rho_1, \lambda, t)\begin{cases}(1+\xi)we^{w}+(1-\zeta)(e^{w}-1), & \kappa=0,\\ (1+\xi)\mu'_{t+1}+(1-\zeta)\mu'_{t}, & \kappa \geq 1\end{cases}\end{aligned}$$

which evidently completes the proof.

Theorem 7. *Let* $w > 0, t \in \mathbb{N}_0$ *then* $\mathcal{I}(\kappa, w, \varsigma) = \int_0^{\varsigma}\frac{\mathcal{L}(\kappa,w,r)}{r}dr$ *is in* $\mathcal{UCV}^{s,p,q}_{\lambda,t}(\rho_1, \mu_1, \xi, \zeta)$ *if and only if* (18) is satisfied.

Proof. Since $\mathcal{I}(\kappa, w, \varsigma) = \int_0^{\varsigma}\frac{\mathcal{L}(\kappa,w,r)}{r}dr$, then by Lemma 1 we need only to show that

$$\sum_{\psi=2}^{\infty}\psi\Big[\psi(1+\xi)-(\xi+\zeta)\Big]\varpi_{\psi,s}(\rho_1, \lambda, t)\frac{(\psi-1)^{\kappa}w^{\psi-1}}{\psi(\psi-1)!}e^{-w} \leq 1-\zeta. \tag{23}$$

Proceeding as in Theorem 3 we obtain the required result.

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