SCHEMES DOMINATED BY ALGEBRAIC VARIETIES AND SOME CLASSES OF SCHEME MORPHISMS. I

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The subject of this lecture is dedicated to the memory of Professor Gheorghe Galbura (1916-2007), the age dean of the Romanian mathematical community, the supervisor of my PhD thesis.

Honorary member of the Committee of the International Conference ICTAMI, he passed away few time before the opening of the fifth edition of this Conference.

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1. Introduction

The terminology used in this paper is that of EGA ([11]).

In [5] we dealt with some *universally topological* conditions of finite generation of Noetherian subalgebras of algebras of finite type over a field.

Recall the following

Definition ([2].II, §1, Def.; [5], §3, Def.) - *A ring A is called universally 1-equicodimensional if it is Noetherian and if every integral A-algebra of finite type, having an 1-height maximal ideal, is of dimension 1.*

A scheme X is called universally 1-equicodimensional if there exists a finite open affine covering $(U_i)_{i=1,...,n}$ such that the ring of sections $\Gamma(U_i, \mathcal{O}_X)$ is universally 1-equicodimensional for each $i = 1, ..., n$.

If k is a field, every k-algebra of finite type, respectively every k-scheme of finite type, is universally 1-equicodimensional.

We had the following

Theorem A ([5], §3, Th.3, Cor. 7, Comment) - *Let A be a subalgebra of an algebra of finite type over a field. Then A is finitely generated over k if (and only if) A is universally 1-equicodimensional.*

In particular a k-scheme X dominated by a scheme of finite type over k is of finite type over k if (and only if) X is universally 1-equicodimensional.

The condition of finite generation in Theorem A above does not involve the base field k.

If A is a subalgebra of an algebra A' of finite type over k, the inclusion of k-algebras $A \subseteq A'$ induces a canonical dominant morphism of k-schemes

$$f : X = Spec\, A' \longrightarrow Y = Spec\, A,$$

X being a k-scheme of finite type. To find favorable cases when A is finitely generated over k it is usefully to find suitable conditions on f under which the property of universally 1-equicodimensionality goes down by f.

So in [2].II, §1, Th.1, has been introduced a class of morphisms (of finite type) of schemes $f : X \longrightarrow Y$ having the following (universally) topological property :

(P) *For every integral Y-scheme Y', the canonical morphism $f_{(Y')} : X \times_Y Y' \longrightarrow Y'$ has the property that its restriction to the union of all irreducible components of $X \times_Y Y'$ dominating Y' is surjective,*

or equivalently

For every irreducible (respectively irreducible affine) Y-scheme Y' and each point $y \in Y'$ there exists an irreducible component Z of $X \times_Y Y'$ dominating Y' such that $y \in f_{(Y')}(Z)$.

According to [2].II, Remark 1, let us consider a weaker form (P') of the property (P) above, as follows :

(P') *For every integral Y-scheme Y' of finite type, the canonical morphism $f_{(Y')} : X \times_Y Y' \longrightarrow Y'$ has the property that its restriction to the union of*

all irreducible components of $X \times_Y Y'$ *dominating* Y' *is surjective,*
or equivalently

For every irreducible (respectively irreducible affine) Y*-scheme* Y' *of finite type and each point* $y \in Y'$ *there exists an irreducible component* Z *of* $X \times_Y Y'$ *dominating* Y' *such that* $y \in f_{(Y')}(Z)$.

In [2].II has been proven the following

Theorem B ([2].II, §1, Theorem 1, Remark 1) - *Let* $f : X \longrightarrow Y$ *be a morphism of finite type of Noetherian schemes having the property* (P') *above. If* X *is universally* 1*-equicodimensional then* Y *is also universally* 1*-equicodimensional.*

Therefore in conjunction with Theorem A it follows

Corollary C - *Let* k *be a field. If* $f : X \longrightarrow Y$ *is a morphism of* k*-schemes having the property* (P')*,* X *is of finite type over* k *and* Y *is Noetherian then* Y *is also of finite type over* k*.*

The goal of this lecture is to show that the Noetherianity condition about Y in the last Corollary C *can be removed* if f is assumed to have the *stronger property* (P) and Y is a reduced scheme (see §3, Theorem, bellow).

2. Some examples and properties

The following types of scheme morphisms (not all of finite type) *have the property* (P) :

1) the proper surjective morphisms

2) the integral surjective morphisms

In fact in both cases if $f : X \longrightarrow Y$ is such a morphism, for every irreducible base change $Y' \longrightarrow Y$ there exists at least an irreducible component Z of $X \times_Y Y'$ dominating Y'. Since in both cases the canonical morphism $f_{(Y')} : X \times_Y Y' \longrightarrow Y'$ is closed, the restriction $f_{(Y')}|_Z$ is surjective.

3) the faithfully flat morphisms

In this case for every irreducible base change $Y' \longrightarrow Y$ the canonical morphism $f_{(Y')}$ is still faithfully flat and it is well known that all irreducible components of $X \times_Y Y'$ dominate Y' ([9], §1, Lema 1.17; §3, Lema 3.16).

4) the canonical morphisms of affine schemes $f : X = Spec\, A' \longrightarrow Y = Spec\, A$ *induced by a strongly submersive morphism of rings* $A \longrightarrow A'$

Cf. [2].II, proof of Cor. 6 .

For Example 4 let us recall the following

Definition (*Nagata-Mumford*, [14]) - *Let* $\phi : A \longrightarrow A'$ *be a ring morphism.* ϕ *is called strongly submersive if for every minimal prime ideal* $\mathfrak{p} \subset A$ *and for every valuation subring* $V, A/\mathfrak{p} \subseteq V \subseteq Q(A/\mathfrak{p})$, *there exists a valuation ring* W *dominating* V *and a ring morphism* $\psi' : A' \longrightarrow W$ *such that* $\psi'\phi = i\psi$ *where* $\psi : A \longrightarrow V$ *is the canonical morphism and* $i : V \hookrightarrow W$ *the ring inclusion.*

5) the canonical morphisms of finite type $X = Spec\, A \longrightarrow X/G = Spec\, A^G$, *where* A *is an algebra over an algebraically closed field* k, G *is a linearly reductive algebraic group over* k *acting rationally on* A *and* $A^G \subseteq A$ *is the subalgebra of invariants.*

Cf. [2].II, proofs of Cor. 4 and Cor. 5 (In the proof of [2].II, Cor. 4 above one shows the property (P) for $X = Spec\, A \longrightarrow X/G = Spec\, A^G$ based on some elements of the first part of the proof of *D. Mumford* for the famous Theorem of finite generation of the subalgebra of invariants ([12], Ch. 1, §2, Th. 1.1); we could say that the property (P) in Example 5 above is an interpretation of the first part of the proof of [12], Th. 1.1).

The following type of scheme morphism (not necessarily of finite type) *has the property* (P') :

6) the universally open (see [11], EGA IV, 2.4.2, for definition) *surjective morphisms* $f : X \longrightarrow Y$ *with* X *a Noetherian scheme*

Indeed, if Y' is an irreducible Y-scheme of finite type, then $X' = X \times_Y Y'$ is of finite type over X and so it is Noetherian and has finitely many irreducible components. Since the canonical morphism $f_{(Y')} : X' = X \times_Y Y' \longrightarrow Y'$ is open, it is easy to see that all irreducible components of X' dominate Y'. $f_{(Y')}$ is also surjective and so the property (P') is fulfilled.

Let us remark that in Example 6 if f is a universally open surjective morphism of finite type of schemes then for each irreducible Noetherian Y-scheme Y' all (finitely many) irreducible components of $X \times_Y Y'$ dominate Y' too and $f_{(Y')}$ is surjective.

We intend to continue the discussion on a particular case of universally open surjective morphisms of schemes in the second part of the lecture.

In the following statement one presents some properties of the morphisms (not necessarily of finite type) of arbitrary schemes, having the property (P):

Proposition - *i) The scheme morphism $f : X \longrightarrow Y$ has the property (P) iff the associated canonical morphism $f_{red} : X_{red} \longrightarrow Y_{red}$ has the property (P).*

ii) Let $f : X \longrightarrow Y, g : Y \longrightarrow Z$ be some morphism of schemes.

If f and g have the property (P), then their composition gf has this property too.

If gf has the property (P), then g has also this property.

iii) If $f : X \longrightarrow Y$ is a scheme morphism with the property (P) then for every Y-scheme Y' the canonical morphism $f_{(Y')} : X \times_Y Y' \longrightarrow Y'$ has still the property (P).

In particular for every reduced (locally closed) subscheme $Y' \subseteq Y$, the canonical morphism $f^{-1}(Y')_{red} \longrightarrow Y'$ has the property (P).

iv) A scheme morphism $f : X \longrightarrow Y$ has the property (P) iff there exists an open affine covering $(U_i)_{i \in I}$ of Y such that the canonical morphism $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \longrightarrow U_i$ has the property (P) for each $i \in I$.

v) Every scheme morphism with the property (P) is surjective.

Proof - *i)* In fact, for each scheme morphism $\phi : Y' \longrightarrow Y$ with Y' an integral scheme, ϕ factorizes by the canonical morphism $Y' \longrightarrow Y_{red}$ and $Hom(Y', Y) = Hom(Y', Y_{red})$. By the general properties of tensor product of rings and schemes we have $(X \times_Y Y')_{red} = (X_{red} \times_Y Y')_{red}$ and so the underlying topological spaces of the schemes $X \times_Y Y'$ and $X_{red} \times_Y Y'$ coincide as well as the continuous maps $f_{(Y')}$ and $(f_{red})_{(Y')}$.

ii) Let $Z' \longrightarrow Z$ be a scheme morphism with Z' an integral scheme and $z \in Z'$. Since g has the property (P), there exists an irreducible component Y' of $Y \times_Z Z'$ dominating Z' such that $z \in g_{(Z')}(Y')$. Suppose $z = g_{(Z')}(y)$. Since f has the property (P) there exists an irreducible component X' of $(f_{(Y \times_Z Z')})^{-1}(Y') \subseteq X \times_Y (Y \times_Z Z') = X \times_Y Z'$ such that $y \in f_{(Y \times_Z Z')}(X')$. Then X' dominates Z' and $z \in (gf)_{(Z')}(X')$. If we include the irreducible subset X' in an irreducible component X'' of $X \times_Y Y'$ then X'' also dominates Z' and $z \in (gf)_{(Z')}(X'')$.

Now suppose gf having the property (P) and let $Z' \longrightarrow Z$ be a scheme morphism with Z' an integral scheme and let $z \in Z'$. By the property (P) there exists an irreducible component X' of $X \times_Y Z'$ dominating Z' such that $z \in (gf)_{(Z')}(X')$. Then the topological closure $Y' = \overline{f_{(Y \times_Z Z')}(X')} \subseteq Y \times_Z Z'$ is a closed irreducible subset dominating Z' such that $z \in g_{(Z)}(Y')$. If we include Y' in an irreducible component Y'' of $Y \times_Z Z'$, then Y'' dominates Z' and $z \in g_{(Z)}(Y'')$.

iii) The first part of *iii)* is obvious by the definition of the property (P). For the second part one uses the fact that f_{red} has the property (P) (via *i)*) and $f^{-1}(Y')_{red} = (X_{red} \times_Y Y')_{red}$.

iv) It is obvious.

v) By the definition of the property (P), the assertion *v)* is clear if Y is irreducible. If Y is arbitrary, for each reduced irreducible component Y' of Y the canonical morphism $f^{-1}(Y')_{red} \longrightarrow Y'$ has still the property (P) (via *iii)*) and then it is surjective because Y' is irreducible. It follows that f is surjective. Q.E.D.

3. The main result

We will present the main result of this lecture:

Theorem - *Let $f : X \longrightarrow Y$ be a morphism of schemes over a field k such that X is of finite type over k and f has the property (P). Then Y_{red} is of finite type over k.*

For the proof of this Theorem we need some preparatory facts.

Lemma 1 - *Let $f : X \longrightarrow Y$ be a morphism of schemes with the property (P) such that X has a finite number of irreducible components and Y is affine and integral. If X_d is the union of all irreducible components of X dominating Y endowed with the reduced subscheme structure, then $\Gamma(X_d, \mathcal{O}_{X_d}) \cap K(Y)$ is an integral extension of the ring $\Gamma(Y, \mathcal{O}_Y)$.*

Proof - Let $\mathcal{O} \subset K(Y)$ be a valuation subring containing the subring $\Gamma(Y, \mathcal{O}_Y)$. Let $X' = X \times_Y Spec\,\mathcal{O}$ and X" the disjoint union of all (finitely many) irreducible components of X' dominating $Spec\,\mathcal{O}$ endowed with the reduced subscheme structures and $A = \Gamma(X'', \mathcal{O}_{X''}) \cap K(Y)$. Then we have the natural scheme morphisms $f' : X' \longrightarrow Spec\,\mathcal{O}$, $\phi : X' \longrightarrow X$, $i : X'' \longrightarrow X'$, $f'' : X'' \longrightarrow Spec\,A$ and $\pi : Spec\,A \longrightarrow Spec\,\mathcal{O}$. Since f has the property (P), then $f'i$ is surjective; from the equality $f'i = \pi f''$ it follows π surjective. Let $\mathfrak{p} \subset A$ be a prime ideal lying over the maximal ideal of $\mathcal{O}$. It is clear that we have a dominating inclusion $\mathcal{O} \subseteq A_{\mathfrak{p}} (\subset K(Y))$; it follows that $\mathcal{O} = A_{\mathfrak{p}}$ because $\mathcal{O}$ is a valuation ring and next that $A \subseteq \mathcal{O}$. From the fact that each irreducible component of X' dominates Y it follows $\phi i(X'') \subseteq X_d$ and then $\Gamma(X_d, \mathcal{O}_{X_d}) \cap K(Y) \subseteq \Gamma(X'', \mathcal{O}_{X''}) \cap K(Y) = A \subseteq \mathcal{O}$. Therefore $\Gamma(X_d, \mathcal{O}_{X_d}) \cap K(Y)$ and $\Gamma(Y, \mathcal{O}_Y)$ have the same integral closure in $K(Y)$ ([19], Cap. II, §1, Teorema 1.13). Q.E.D.

Lemma 2 - *Let $f : X \longrightarrow Y$ be a scheme morphism of finite type such that X is Noetherian, Y is integral and f has the property (P). If X_d is the union of all irreducible components of X dominating Y, endowed with the reduced subscheme structure, then the canonical morphism $f^N : X_d^N \longrightarrow Y^N$ between the normalizations of X_d and Y is surjective and Y^N is a Krull scheme.*

In Lemma 2 and in the proof bellow, by the normalization Z^N of a reduced (reducible) scheme Z we will understand the *disjoint union* of the normalizations of all irreducible components of the scheme Z (endowed with the reduced closed subscheme structures).

Recall that a scheme Z is called *Krull* if there exists an open covering $(U_i)_{i \in I}$ of Z such that all rings $\Gamma(U_i, \mathcal{O}_Z)$ are Krull.

Proof - Let us consider the canonical morphisms $p : Y^N \longrightarrow Y$, $q : X \times_Y Y^N \longrightarrow X$, $f' : X \times_Y Y^N \longrightarrow Y^N$, $f^N : X_d^N \longrightarrow Y^N$, $u : X_d^N \longrightarrow X$ and $v : X_d^N \longrightarrow X \times_Y Y^N$. We have $fq = pf'$, $u = qv$ and $f^N = f'v$. It is obvious that the morphisms q, u and v are integral (not necessarily surjective). By the properties of the fiber product of schemes it follows that there exists via q an one-to-one correspondence between the irreducible components of $X \times_Y Y^N$ dominating Y^N and those of X dominating Y. Then the irreducible components of X_d^N are in one-to -one correspondence via v with those of $X \times_Y Y^N$ dominating Y^N. Since v is closed, it is a surjection onto the union of all irreducible components of $X \times_Y Y^N$ dominating Y^N. By the property (P) of the morphism f it results that $f^N = f'v$ is surjective.

Let $(X \times_Y Y^N)^N$ be the disjoint union of the normalizations of all irreducible components of $X \times_Y Y^N$ (endowed with the reduced closed subscheme structures). It is clear that X_d^N is the reduced closed subscheme of $(X \times_Y Y^N)^N$ equal with the union of all irreducible components of $(X \times_Y Y^N)^N$

dominating Y^N. Since the normalization morphism $(X \times_Y Y^N)^N \longrightarrow X \times_Y Y^N$ is integral and surjective it has the property (P) (cf. §2, Ex.2). So the canonical morphism $(X \times_Y Y^N)^N \longrightarrow Y^N$ has still the property (P) as composition of two morphisms with such a property (cf. §2, Prop., ii)). In virtue of Lemma 1 it follows that for each open affine subset $U \subseteq Y^N, \Gamma(U, \mathcal{O}_{Y^N}) = \Gamma((f^N)^{-1}(U), \mathcal{O}_{X_d^N}) \cap K(Y^N)$, because the scheme Y^N is normal. Since the scheme X is Noetherian, it follows that X_d^N is a Krull scheme by *Mori-Nagata* Theorem ([13], Th. 33.10). Let $V \subseteq Y$ be an open affine subset and $V^N = p^{-1}(V)$. Then V^N is an open affine subset of Y^N and $(f^N)^{-1}(V^N) = u^{-1}(f^{-1}(V))$ is quasi-compact because $f^{-1}(V)$ is quasi-compact and u is an affine morphism. Hence there exists a finite covering $(W_i)_{1 \leq i \leq n}$ with open affine irreducible subsets of $(f^N)^{-1}(V^N)$. Then $\Gamma(V^N, \mathcal{O}_{Y^N}) = (\bigcap_{i=1}^{i=n} \Gamma(W_i, \mathcal{O}_{X_d^N})) \cap K(Y^N)$ and $\Gamma(W_i, \mathcal{O}_{X_d^N})$ is a Krull ring for each $i, 1 \leq i \leq n$. It follows that $\Gamma(V^N, \mathcal{O}_{Y^N})$ is a Krull ring. Since V has been arbitrary chosen as open affine subset of Y, by definition it results that Y^N is a Krull scheme. Q.E.D.

Lemma 3 - *Let Z be a reduced scheme over a Noetherian ring k, having finitely many irreducible components $Z_1, ..., Z_n$. Then Z is of finite type over k iff for each $i, 1 \leq i \leq n$, the reduced irreducible component Z_i is of finite type over k.*

Proof - We can reduce the situation to the affine case when $X = Spec\, A$, with A a reduced k-algebra. Let $\mathfrak{p}_1, ..., \mathfrak{p}_n \subset A$ be the the minimal prime ideals of A. Then $Z_i = Spec\, A/\mathfrak{p}_i$ for each $i, 1 \leq i \leq n$ and $\mathfrak{p}_1 \cap ... \cap \mathfrak{p}_n = 0$.

We have to prove that A is finitely generated over k iff $A/\mathfrak{p}_i$ does, for all $i, 1 \leq i \leq n$. This fact follows from the following more general property :

Lemma 3′ - *Let A be an algebra over a Noetherian ring k, $\mathfrak{a}_1, ..., \mathfrak{a}_n \subset A$ some ideals and $\mathfrak{a} = \mathfrak{a}_1 \cap ... \cap \mathfrak{a}_n$. Then $A/\mathfrak{a}$ is finitely generated over k iff for each $i, 1 \leq i \leq n$, $A/\mathfrak{a}_i$ is finitely generated over k.*

Proof - There exists a canonical morphism of k-algebras

$$\phi : A/\mathfrak{a} \longrightarrow A' = A/\mathfrak{a}_1 \times ... \times A/\mathfrak{a}_n$$

defined by $\phi(\hat{a}) = (\hat{a}, ..., \hat{a})$ for each $a \in A$ (By $\hat{a}$ we denoted the class of $a \in A$ in $A/\mathfrak{a}$ or $A/\mathfrak{a}_i$. The ring A' above is the ring product)

ϕ is injective and so $A/\mathfrak{a}$ can be view as a subalgebra of A'.

The ring extension $A/\mathfrak{a} \subseteq A'$ is integral. In fact for each $x = (\hat{a}_1, ..., \hat{a}_n) \in A'$, x satisfies the integral equation in A'

$$(x - (\hat{a}_1, ..., \hat{a}_1))...(x - (\hat{a}_n, ..., \hat{a}_n)) = 0$$

(i.e. the equation $(x - \phi(\hat{a}_1))...(x - \phi(\hat{a}_n)) = 0$).

If $A/\mathfrak{a}$ is of finite type over k, then its quotient algebras $A/\mathfrak{a}_1, ..., A/\mathfrak{a}_n$ are also of finite type over k.

Conversely, if $A/\mathfrak{a}_1, ..., A/\mathfrak{a}_n$ are finitely generated over k then A' is also of finite type over k and $A/\mathfrak{a} \subseteq A'$ is a finite extension. Then it follows by a well known reasoning (using the Noetherianity of the finitely generated k-algebras) that the subalgebra $A/\mathfrak{a}$ is also of finite type over k. Let us recall this argument: the ring A' is finite over a finitely generated k-subalgebra A'' of $A/\mathfrak{a}$, generated by the coefficients of the integral equations of the elements of a finite set of generators of A' over $A/\mathfrak{a}$; since k is Noetherian then A'' is also Noetherian and A' is an A''-module of finite type; it follows that the submodule $A/\mathfrak{a}$ of A' is also an A''-module of finite type; so $A/\mathfrak{a}$ becomes a k-algebra of finite type. Q.E.D.

Now we can present

Proof of Theorem - Since f has the property (P), it is surjective and from the Noetherianity of X it follows that Y has finitely many irreducible components $Y_1, ..., Y_n$ (More general, every k-scheme dominated by a k-scheme of finite type has finitely many irreducible components ([3], Prop. 2 and Prop. 1). By Lemma 3, Y_{red} is of finite type over k if each reduced irreducible component Y_i is of finite type over k. By replacing Y with Y_i and f by $f_{(Y_i)} : X \times_Y Y_i \longrightarrow Y_i$, we can reduce the proof to the particular case when Y is an *integral* scheme. On other hand, via §2, Prop., i), we may assume that X is a *reduced* scheme by replacing f with f_{red}.

If $p : Y^N \longrightarrow Y$ is the normalization morphism, then for each $y \in Y^N$, the residue field extension $k(p(y)) \hookrightarrow k(y)$, induced by the morphism p, is finite. Indeed, by Lemma 2 and with the notations of that Lemma, it follows that there exists $x \in X_d^N$ such that $y = f^N(x)$. X_d^N is a scheme of finite type over k and if $u : X_d^N \longrightarrow X$ is the canonical morphism then $fu = pf^N$ (see the proof of Lemma 2) is a morphism of finite type. Hence the composition of the extensions $k(p(y)) \hookrightarrow k(y) \hookrightarrow k(x)$, induced by the morphisms p and

f^N, is a field extension of finite type. By [8], Th.5.56 (or [17], Ch. 3, 2, (IV)), it follows that $k(p(y)) \hookrightarrow k(y)$ is an extension of finite type. Since p is an integral morphism, it results that this last field extension is algebraic and so it is finite.

By [3], Prop. 2, we have $dim\, Y \leq dim\, X$. To prove the Theorem we will proceed by induction on $dim\, Y$ (Y being an integral scheme).

If $dim\, Y = 0$ then Y is of finite type over k in an obvious way.

Let us suppose that $dim\, Y > 0$ and that the Theorem is true for morphisms with the property (P) of the form $f : X' \longrightarrow Y'$, where X' is of finite type over k and Y' is integral of dimension $< dim\, Y$. Since for each closed irreducible subscheme $Y' \subset Y$, endowed with the reduced subscheme structure, $f_{(Y')} : X \times_Y Y' \longrightarrow Y'$ has the property (P) (cf. §2, Prop., iii)) and $X \times_Y Y'$ is a closed subscheme of X, it follows that $X \times_Y Y'$ is of finite type over k and by the inductive hypothesis Y' is of finite type over k. Therefore each closed irreducible subscheme $Y' \subset Y$, endowed with the reduced subscheme structure, is of finite type over k.

Now we will show that the normalization scheme Y^N has the same property: each closed irreducible subscheme $Z \subset Y^N$, endowed with the reduced subscheme structure, is of finite type over k. Indeed, $p(Z) \subset Y$ and so $p(Z)$ is of finite type over k; the rational field extension $K(p(Z)) \subseteq K(Z)$ is finite, according to the previous remark on the residue field extensions induced by p; if $\overline{Z}$ is the integral closure of $p(Z)$ in $K(Z)$, then $\overline{Z}$ is a scheme of finite type over k and there exists a surjective integral morphism $\overline{Z} \longrightarrow Z$ of k-schemes. Then by a well known fact it follows that Z itself is a scheme of finite type over k.

By Lemma 2, Y^N is a Krull scheme. Then by *Mori-Nishimura* Theorem ([18], Theorem), it follows that Y^N is a Noetherian scheme.

The canonical morphism $f' = f_{(Y^N)} : X \times_Y Y^N \longrightarrow Y^N$ has the property (P) (cf. §2, Prop., iii)) If we prove that $(X \times_Y Y^N)_{red}$ is of finite type over k, then by the going-down of the property of universally 1-equicodimensionality by the morphisms with the property (P),(cf. §1, Theorem B), it follows firstly that Y^N is a universally 1-equicodimensional scheme and next that Y^N is of finite type over k, by §1, Theorem A . Since the canonical morphism $p : Y^N \longrightarrow Y$ is integral, it follows that Y is of finite type over k too.

We could give now a second argument for the fact that Y^N is of finite type over k : if we prove that $(X \times_Y Y^N)_{red}$ is of finite type over k, from the property (P) of f' each point $y \in Y^N$ is in the image by f' of an irreducible component of $(X \times_Y Y^N)_{red}$. Then by a Theorem of $Nagata - Otsuka$ ([16], Th. 3), we have the equality $dim\, \mathcal{O}_{Y^N,y} + tr.deg._k\, k(y) = dim\, Y^N$. Since $\mathcal{O}_{Y^N,y}$ is Noetherian it follows that y has an open neighborhood $U_y \subseteq Y^N$ of finite type over k by a local algebraization result ([3], §2, Th. and §1, Prop. 1). From the surjectivity of f', it follows the quasi-compacity of Y^N and so the fact that Y^N is of finite type over k. As above, using the fact that the normalization morphism is integral, it results that Y is of finite type over k.

Therefore it remains to show that $(X \times_Y Y^N)_{red}$ is of finite type over k.

In fact, the canonical morphism $q : X \times_Y Y^N \longrightarrow X$ is integral and $fq = pf'$ (see the proof of Lemma 2). Then for each $x \in X \times_Y Y^N$ we have $q^* f^* = f'^* p^*$, where $f^* : k(f(q(x))) \hookrightarrow k(q(x))$, $q^* : k(q(x)) \hookrightarrow k(x)$, $p^* : k(p(f'(x))) \hookrightarrow k(f'(x))$ and $f'^* : k(f'(x)) \hookrightarrow k(x)$ are the residue field extensions induced by the morphisms f, q, p, respectively f'. Since f' is a morphism of finite type, f'^* is a field extension of finite type. As it was shown above, p^* is a finite field extension and so $f'^* p^*$ is a finite type extension. From the equality $q^* f^* = f'^* p^*$ it follows that q^* is also a finite type field extension.

Let $X'' \subseteq X \times_Y Y^N$ an irreducible component and $X' = q(X'')$, both endowed with the reduced subscheme structure. As we showed above the rational field extension $K(X') \subseteq K(X'')$ is finite. If we denote by $\overline{X'}$ the integral closure of X' in $K(X'')$, then $\overline{X'}$ is a k-scheme of finite type and there exists a surjective integral morphism $\overline{X'} \longrightarrow X''$. So X'' is a k-scheme of finite type. Therefore all irreducible components of $X \times_Y Y^N$, with reduced subscheme structures, are of finite type over k; then it follows that the reduced scheme $(X \times_Y Y^N)_{red}$ is also a k-scheme of finite type, by Lemma 3. Q.E.D.

4. Some particular cases

Now we can present some particular cases and consequences of the previous Theorem.

A first consequence is a particular case, when the base ring k is a field, of a strong result of *M. Nagata* ([14], Main Theorem, p. 193).

Corollary 1 - *Let A' be an algebra of finite type over a field k and A a strongly submersive subalgebra of A'. Then A_{red} is finitely generated over k.*

Proof - The associated morphism of affine schemes $Spec\,A' \longrightarrow Spec\,A$ has the property (P) (see §2, Ex.4). By the previous Theorem it follows that the k-scheme $(Spec\,A)_{red} = Spec\,(A_{red})$ is of finite type over k. So A_{red} is finitely generated over k. Q.E.D.

Now for giving some more general forms for other consequences we need the following simple

Lemma 4 - *Let A be an algebra over a ring k, such that the associated reduced k-algebra A_{red} is finitely generated and the nilradical ideal $\mathfrak{a} \subset A$ is of finite type. Then A is finitely generated over k.*

In particular a Noetherian algebra A over a ring k is finitely generated iff A_{red} is finitely generated.

Proof - We have $A_{red} = A/\mathfrak{a}$ and there exists $p \in \mathbb{N}$ such that $\mathfrak{a}^p = 0$

Let $\hat{x}_1, ..., \hat{x}_m \in A/\mathfrak{a}$ be a finite system of generators of k-algebra and $a_1, ..., a_n \in \mathfrak{a}$ a finite system of generators of the ideal $\mathfrak{a}$.

Then $x_1, ..., x_m, a_1, ..., a_n$ is a system of generators for the k-algebra A.

In fact, let $x \in A$. Then we have $x = P(x_1, ..., x_m) + f_1a_1 + ... + f_na_n$ with P a polynomial over k and $f_i \in A$. In the same way $f_i = P_i(x_1, ..., x_m) + f_{i1}a_1 + ... + f_{in}a_n$ with P_i a polynomial over k and $f_{ij} \in A$ and then $x = P(x_1, ..., x_m) + \sum_{i=1}^{n} a_iP_i(x_1, ..., x_m) + \sum_{i,j=1}^{n} f_{ij}a_ia_j$. Representing in the same way f_{ij} and next continuing in the same manner, after p steps (because $\mathfrak{a}^p = 0$) we will obtain x as a k-polynomial expression in $x_1, ..., x_m, a_1, ..., a_n$. Q.E.D.

Remark 1 - The conditions of finite generation required in the first part of Lemma 4 are also necessary.

There exist some examples of subalgebras of algebras of finite type over a field k which are not finitely generated but the associated reduced algebras (which are still subalgebras of algebras of finite type over k) are finitely generated (see [1]); via Lemma 4, their nilradical ideals are not ideals of finite type.

Now we will return to the consequences.

The following fact has been established by *J.E. Goodman* and *A. Landman* ([10], Cor. 3.9, p. 279) for (irreducible) algebraic varieties and integral schemes over an algebraically closed field.

Corollary 2 - *Let $f : X \longrightarrow Y$ be a proper surjective morphism of schemes over a field k, with X of finite type over k and such that the canonical morphism $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is injective. Then Y is of finite type over k.*

Proof - Since the scheme morphism $f : X \longrightarrow Y$ is proper and surjective then it has the property (P) (see §2, Ex.1). Via the previous Theorem it follows that Y_{red} is of finite type over k.

For each open affine subset $V \subseteq Y$, by [], §2, Lemma 4, p. 1010 applied to the scheme morphism $f|_{f^{-1}(V)} : f^{-1}(V) \longrightarrow V$, it follows that $\Gamma(V, f_*\mathcal{O}_X)$ is a Noetherian ring which is finite over the subring $\Gamma(V, \mathcal{O}_Y)$. So $\Gamma(V, \mathcal{O}_Y)$ is a Noetherian ring by a Theorem of *Eakin-Nagata* ([15], Theorem; [7], Theorem 2).

So Y is a Noetherian scheme such that Y_{red} is of finite type over k. Then Y is of finite type over k by Lemma 4. Q.E.D.

Corollary 3 - *Let $f : X \longrightarrow Y$ be a faithfully flat morphism of schemes over a field k, with X of finite type over k. Then Y is of finite type over k.*

Proof - Since the morphism $f : X \longrightarrow Y$ is faithfully flat, then it has the property (P) (see §2, Ex. 3). Via the previous Theorem it follows that Y_{red} is of finite type over k.

From the faithfully flatness of f and the Noetherianity of X it results the Noetherianity of Y. So Y is a Noetherian scheme such that Y_{red} is of finite type over k. By Lemma 4 it follows that Y is of finite type over k. Q.E.D.

Remark 2 - Let us recall ([5], §3, Prop. 1) a second and more direct argument to prove Corollary 3 : f is a morphism of finite type of Noetherian schemes with the property (P) and X is universally 1-equicodimensional as a scheme of finite type over a field ; then by §1, Theorem B it follows that Y is universally 1-equicodimensional and next a scheme of finite type over k via §1, Theorem A.

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