THE ESTIMATES FOR THE REMAINDER TERM OF SOME QUADRATURE FORMULAE

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Abstract. A new generalization of Ostrowski′s integral inequality is established. A consequence of the generalization is that we can derive new estimates for remainder term of the midpoint, trapezoid and Simpson formulae. These estimates are improvements of some recently obtained estimates. Applications in numerical integration are also given.

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1. Introduction

In [2] Cerone, Dragomir and Roumeliotis proved the following generalization of Ostrowski′s inequality:

Theorem 1. *[2] Let $f:[a,b]\to R$ be continous on (a,b) and whose first derivative $f':(a,b)\to R$ is bounded on (a,b). Denote $\|f'\|_\infty=\sup\limits_{t\in[a,b]}|f'(t)|<\infty$. Then we have:*

$$\left|\int_a^b f(t)dt-\left[f(x)(1-\lambda)+\frac{f(a)+f(b)}{2}\lambda\right](b-a)\right| \tag{1}$$
$$\leq\left[\frac{1}{4}(b-a)^2(\lambda^2+(1-\lambda)^2)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty$$

for all $\lambda\in[0,1]$ and $a+\lambda\frac{b-a}{2}\leq x\leq b-\lambda\frac{b-a}{2}$.

Using (1), they also obtained estimates for the remainder term of the midpoint, trapezoid and Simpson′s formulae. They also gave applications of the mentioned rezults in numerical integration and for special means.

In [4], N. Ujevic proved the following generalization of Ostrowski′s inequality:

Theorem 2.*[4] Let $I \subset R$ be a open interval and $a, b \in I$, $a < b$. If $f : I \to R$ is a differentiable function such that $\gamma \le f'(t) \le \Gamma$, for all $t \in [a, b]$, for some constants $\Gamma, \gamma \in R$, then we have*

$$\left|f(x)-\frac{\Gamma+\gamma}{2}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \frac{1}{2}(\Gamma-\gamma)(b-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right].$$

In this paper we give some generalizations of Ostrowski′s inequalities and we obtain new estimations for the remainder term of some quadrature formulae.

2. Main results

We denote $[x]$ the integer part of x, $x \in R$.

Theorem 3. *If $f \in C^{n-1}[a, b]$, $f^{(n-1)}$, $n > 1$ is absolutely continuous and there exist real numbers γ, Γ such that $\gamma \le f^{(n)}(t) \le \Gamma$, $t \in [a, b]$, then*

$$\left|2\sum_{k=0}^{n-1}\sum_{j=0}^{m}\frac{(-1)^k}{(k+1)!}(b-a)^k A_{k,k-2j} f^{(k)}(x) - \frac{1}{b-a}\int_a^b f(t)dt \right.$$
$$\left. + (-1)^n(\Gamma+\gamma)\frac{(b-a)^n}{(n+1)!}\sum_{j=0}^{\tilde{m}} A_{n,n-2j}\right|$$
$$\le \frac{\Gamma-\gamma}{2}\cdot\frac{(b-a)^n}{n!}\left\{\sum_{j=0}^{p} A_{n-2,2j}+\left|\sum_{j=0}^{\tilde{p}} A_{n-2,2j+1}\right|\right\}\cdot\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad (2)$$

where

$$A_{k,i} = \frac{1}{2^{k+1-i}(b-a)^i}\binom{k+1}{i}\left(x-\frac{a+b}{2}\right)^i, \; k=\overline{0,n-1}, \; i=\overline{0,k+1},$$

and $m = \left[\frac{k}{2}\right]$, $\tilde{m} = \left[\frac{n}{2}\right]$, $p = \left[\frac{n-1}{2}\right]$, $\tilde{p} = \left[\frac{n-2}{2}\right]$.

Proof. Let $P:[a,b]^2\to R$ a mapping given by

$$P(x,t)=\begin{cases}\dfrac{(t-a)^n}{n!}, & \text{if } t\in[a,\ x)\\[2ex] \dfrac{(t-b)^n}{n!}, & \text{if } t\in[x,b].\end{cases}$$

We observe that successive integration by parts yields the relation:

$$\int_a^b P(x,t)f^{(n)}(t)dt=\sum_{k=0}^{n-1}(-1)^{n-1-k}\left[\frac{(t-a)^n}{n!}\right]^{(n-1-k)}\cdot f^{(k)}(t)\Bigg|_a^x+(-1)^n\int_a^x f(t)dt$$

$$+\sum_{k=0}^{n-1}(-1)^{n-1-k}\left[\frac{(t-b)^n}{n!}\right]^{(n-1-k)}\cdot f^{(k)}(t)\Bigg|_x^b+(-1)^n\int_x^b f(t)dt$$

$$=(-1)^n\int_a^b f(t)dt+\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(x-a)^{k+1}}{(k+1)!}f^{(k)}(x)-\sum_{k=0}^{n-1}(-1)^{n-1-k}\frac{(x-b)^{k+1}}{(k+1)!}f^{(k)}(x)$$

$$=(-1)^{n-1}\left\{-\int_a^b f(t)dt+\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\left[(x-a)^{k+1}-(x-b)^{k+1}\right]f^{(k)}(x)\right\}$$

$$=(-1)^{n-1}\left\{-\int_a^b f(t)dt+\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}\sum_{i=0}^{k+1}\left[1+(-1)^{k-i}\right]\right.$$

$$\left.\cdot\binom{k+1}{i}\cdot\left(x-\frac{a+b}{2}\right)^i\frac{1}{2^{k+1-i}}(b-a)^{k+1-i}f^{(k)}(x)\right\}$$

$$=(-1)^{n-1}\left\{-\int_a^b f(t)dt+\sum_{k=0}^{n-1}\sum_{i=0}^{k+1}\frac{(-1)^k}{(k+1)!}\left[1+(-1)^{k-i}\right](b-a)^{k+1}A_{k,i}f^{(k)}(x)\right\}$$

$$=(-1)^{n-1}(b-a)\left\{-\frac{1}{b-a}\int_a^b f(t)dt\right.$$

$$\left.+\sum_{k=0}^{n-1}\sum_{i=0}^{k+1}\frac{(-1)^k}{(k+1)!}(b-a)^k\left[1+(-1)^{i+1}\right]A_{k,k+1-i}f^{(k)}(x)\right\}$$

$$=(-1)^{n-1}(b-a)\left\{-\frac{1}{b-a}\int_a^b f(t)dt+2\sum_{k=0}^{n-1}\sum_{j=0}^{m}\frac{(-1)^k}{(k+1)!}(b-a)^kA_{k,k-2j}\cdot f^{(k)}(x)\right\}.$$

We have

$$\int_a^b P(x,t)dt=\int_a^x \frac{(t-a)^n}{n!}dt+\int_x^b \frac{(t-b)^n}{n!}dt=\frac{1}{(n+1)!}\left[(x-a)^{n+1}-(x-b)^{n+1}\right]$$
$$=\frac{1}{(n+1)!}\sum_{i=0}^{n+1}\left[1+(-1)^{n-i}\right](b-a)^{n+1}A_{n,i}=\frac{(b-a)^{n+1}}{(n+1)!}\sum_{i=0}^{n+1}\left[1+(-1)^{i+1}\right]A_{n,n+1-i}$$
$$=2\frac{(b-a)^{n+1}}{(n+1)!}\sum_{j=0}^{\tilde{m}}A_{n,n-2j}.$$

Since

$$\int_a^b P(x,t)\left[f^{(n)}(t)-\frac{\Gamma+\gamma}{2}\right]dt=(-1)^{n-1}(b-a)\left\{-\frac{1}{b-a}\int_a^b f(t)dt\right.$$
$$\left.+2\sum_{k=0}^{n-1}\sum_{j=0}^{m}\frac{(-1)^k}{(k+1)!}(b-a)^k\cdot A_{k,k-2j}\cdot f^{(k)}(x)+(-1)^n(\Gamma+\gamma)\frac{(b-a)^n}{(n+1)!}\sum_{j=0}^{\tilde{m}}A_{n,n-2j}\right\}$$

we have

$$\left|2\sum_{k=0}^{n-1}\sum_{j=0}^{m}\frac{(-1)^k}{(k+1)!}(b-a)^k\cdot A_{k,k-2j}\cdot f^{(k)}(x)-\frac{1}{b-a}\int_a^b f(t)dt\right.$$
$$\left.+(-1)^n(\Gamma+\gamma)\frac{(b-a)^n}{(n+1)!}\sum_{j=0}^{\tilde{m}}A_{n,n-2j}\right|\leq\frac{1}{b-a}\int_a^b|P(x,t)|\left|f^{(n)}(t)-\frac{\Gamma+\gamma}{2}\right|dt$$
$$\leq\frac{1}{b-a}\max_{t\in[a,b]}\left|f^{(n)}(t)-\frac{\Gamma+\gamma}{2}\right|\int_a^b|P(x,t)|\,dt.$$

By using the relation

$$\max\left\{(x-a)^{n-1},(b-x)^{n-1}\right\}=\frac{1}{2}\left\{(x-a)^{n-1}+(b-x)^{n-1}+\left|(x-a)^{n-1}-(b-x)^{n-1}\right|\right\}$$
$$=\frac{1}{2}\left\{\sum_{i=0}^{n-1}\left[1+(-1)^i\right]\binom{n-1}{i}\left(\frac{b-a}{2}\right)^{n-1-i}\left(x-\frac{a+b}{2}\right)^i\right.$$
$$\left.+\left|\sum_{i=0}^{n-1}\left[1+(-1)^{i+1}\right]\binom{n-1}{i}\left(\frac{b-a}{2}\right)^{n-1-i}\left(x-\frac{a+b}{2}\right)^i\right|\right\}$$
$$=\frac{(b-a)^{n-1}}{2}\left\{\sum_{i=0}^{n-1}\left[1+(-1)^i\right]A_{n-2,i}+\left|\sum_{i=0}^{n-1}\left[1+(-1)^{i+1}\right]A_{n-2,i}\right|\right\}$$
$$=(b-a)^{n-1}\left\{\sum_{j=0}^{p}A_{n-2,2j}+\left|\sum_{j=0}^{\tilde{p}}A_{n-2,2j+1}\right|\right\}$$

we obtain

$$\int_a^b |P(x,t)|\,dt = \int_a^x \frac{(t-a)^n}{n!}dt + \int_x^b \frac{(b-t)^n}{n!}dt$$
$$\leq (x-a)^{n-1}\int_a^x \frac{t-a}{n!}dt + (b-x)^{n-1}\int_x^b \frac{b-t}{n!}dt$$
$$\leq \frac{1}{n!}\max\left\{(x-a)^{n-1},(b-x)^{n-1}\right\}\cdot\left\{\int_a^x (t-a)dt + \int_x^b (b-t)dt\right\}$$
$$= \frac{1}{n!}\max\left\{(x-a)^{n-1},(b-x)^{n-1}\right\}\cdot(b-a)^2\cdot\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$
$$= \frac{(b-a)^{n+1}}{n!}\cdot\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot\left\{\sum_{j=0}^{p} A_{n-2,2j} + \left|\sum_{j=0}^{\tilde{p}} A_{n-2,2j+1}\right|\right\}.$$

Since $\gamma \leq f^{(n)}(t) \leq \Gamma$, we obtain

$$\left|f^{(n)}(t) - \frac{\Gamma+\gamma}{2}\right| \leq \frac{\Gamma-\gamma}{2} \quad (\forall) \quad t\in[a,b].$$

From the above relations we get the desired inequality (3).

From Theorem 3 we obtain a new quadrature formula and we give a estimation of the remainder term.

Theorem 4. *If $f \in C^{n-1}[a,b]$, $f^{(n-1)}, n > 1$ is absolutely continuous and there exist real numbers γ, Γ such that $\gamma \leq f^{(n)}(t) \leq \Gamma$, $t \in [a,b]$ then*

$$\int_a^b f(t)dt = 2\sum_{k=0}^{n-1}\sum_{j=0}^{m}\frac{(-1)^k}{(k+1)!}(b-a)^{k+1}A_{k,k-2j}f^{(k)}(x)$$
$$+ (-1)^n(\Gamma+\gamma)\frac{(b-a)^{n+1}}{(n+1)!}\sum_{j=0}^{\tilde{m}} A_{n,n-2j} + \mathcal{R}_n[f]$$

and the remainder term $\mathcal{R}_n[f]$ satisfies the estimation

$$|\mathcal{R}_n[f]| \leq \frac{\Gamma-\gamma}{2}\cdot\frac{(b-a)^{n+1}}{n!}\left\{\sum_{j=0}^{p} A_{n-2,2j} + \left|\sum_{j=0}^{\tilde{p}} A_{n-2,2j+1}\right|\right\}\cdot\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right],$$

where

$$A_{k,i} = \frac{1}{2^{k+1-i}(b-a)^i}\begin{pmatrix} k+1 \\ i\end{pmatrix}\left(x-\frac{a+b}{2}\right)^i,\ k=\overline{0,n-1},\ i=\overline{0,k+1},$$

and $m=\left[\frac{k}{2}\right]$, $\tilde{m}=\left[\frac{n}{2}\right]$, $p=\left[\frac{n-1}{2}\right]$, $\tilde{p}=\left[\frac{n-2}{2}\right]$.

Remark 1. *If we choose* $x=\frac{a+b}{2}$, $n=2$, *we obtain the following quadrature formula:*

$$\int_a^b f(t)dt=(b-a)f\left(\frac{a+b}{2}\right)+\frac{\Gamma+\gamma}{48}(b-a)^3+\mathcal{R}_2[f]$$

and the remainder therm $\mathcal{R}_2[f]$ *satisfies the estimation*

$$|\mathcal{R}_2[f]|\leq\frac{\Gamma-\gamma}{32}(b-a)^3.$$

Remark 2. *If in Theorem 4, for* $n=2$, *we choose* $x=a$, $x=b$, *respectively, and we sum the results, to obtain the following quadrature formula*

$$\int_a^b f(t)dt=\frac{b-a}{2}[f(a)+f(b)]+\frac{(b-a)^2}{4}[f'(a)-f'(b)]+\frac{1}{12}(\Gamma+\gamma)(b-a)^3+\mathcal{R}_2[f]$$

and the remainder therm $\mathcal{R}_2[f]$ *satisfies the estimation*

$$|\mathcal{R}_2[f]|\leq\frac{1}{8}(\Gamma-\gamma)(b-a)^3.$$

Theorem 5. *If* $f\in C^1[a,b]$, f' *is absolutely continuous and there exist real numbers* γ, Γ *such that* $\gamma\leq f''(t)\leq\Gamma$, $t\in[a,b]$ *then*

$$\begin{aligned}\int_a^b w(t)f(t)dt &= \frac{(b-a)^3}{6}\left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right.\\ &+ \left.\frac{\Gamma+\gamma}{20}\left(\left(\frac{b-a}{2}\right)^2+5\left(x-\frac{a+b}{2}\right)^2\right)\right]+\mathcal{R}[f],\end{aligned}$$

where $w(t)=(b-t)(t-a)$ *and the remainder term* $\mathcal{R}[f]$ *satisfies the estimation*

$$|\mathcal{R}[f]|\leq\frac{\Gamma-\gamma}{8}(b-a)^5\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\cdot\left[\frac{7}{36}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right].\quad(3)$$

Proof. Let $P:[a,b]^2\to R$ a mapping given by

$$P(x,t)=\begin{cases}(b-a)\dfrac{(t-a)^3}{6}-\dfrac{(t-a)^4}{12}, & t\in[a,\ x)\\ (a-b)\dfrac{(t-b)^3}{6}-\dfrac{(t-b)^4}{12}, & t\in[x,b].\end{cases}$$

We observe that successive integration by parts yields the relation:

$$\int_a^b P(x,t)f''(t)dt=\int_a^x\left[(b-a)\frac{(t-a)^3}{6}-\frac{(t-a)^4}{12}\right]f''(t)dt$$
$$+\int_x^b\left[(a-b)\frac{(t-b)^3}{6}-\frac{(t-b)^4}{12}\right]f''(t)dt$$
$$=\int_a^b w(t)f(t)dt+\left\{-\frac{b-a}{2}\left[(x-a)^2+(b-x)^2\right]+\frac{1}{3}\left[(x-a)^3+(b-x)^3\right]\right\}f(x)$$
$$+\left\{\frac{b-a}{6}\left[(x-a)^3-(b-x)^3\right]-\frac{1}{12}\left[(x-a)^4-(b-x)^4\right]\right\}f'(x)$$
$$=\int_a^b w(t)f(t)dt-\frac{(b-a)^3}{6}f(x)+\frac{(b-a)^3}{6}\left(x-\frac{a+b}{2}\right)f'(x).$$

We have

$$\begin{aligned}\int_a^b P(x,t)dt &= \int_a^x\left[(b-a)\frac{(t-a)^3}{6}-\frac{(t-a)^4}{12}\right]dt+\int_x^b\left[(a-b)\frac{(t-b)^3}{6}-\frac{(t-b)^4}{12}\right]dt\\ &= \frac{b-a}{24}\left[(x-a)^4+(b-x)^4\right]-\frac{1}{60}\left[(x-a)^5+(b-x)^5\right]\\ &= \frac{(b-a)^3}{60}\left\{\left(\frac{b-a}{2}\right)^2+5\left(x-\frac{a+b}{2}\right)^2\right\}.\end{aligned}$$

Since

$$\begin{aligned}\int_a^b P(x,t)\left[f''(t)-\frac{\Gamma+\gamma}{2}\right]dt &= \int_a^b w(t)f(t)dt-\frac{(b-a)^3}{6}\left\{f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right.\\ &+\left.\frac{\Gamma+\gamma}{20}\left[\left(\frac{b-a}{2}\right)^2+5\left(x-\frac{a+b}{2}\right)^2\right]\right\},\end{aligned}$$

we have

$$\left|\int_a^b w(t)f(t)dt - \frac{(b-a)^3}{6}\left\{f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right.\right.$$
$$\left.\left. + \frac{\Gamma+\gamma}{20}\left[\left(\frac{b-a}{2}\right)^2 + 5\left(x - \frac{a+b}{2}\right)^2\right]\right\}\right| \le \int_a^b |P(x,t)| \cdot \left|f''(t) - \frac{\Gamma+\gamma}{2}\right| dt$$
$$\le \max_{t\in[a,b]}\left|f''(t) - \frac{\Gamma+\gamma}{2}\right| \cdot \int_a^b |P(x,t)|\, dt.$$

Using the relation

$$\begin{aligned}\max\left\{(x-a)^2,(b-x)^2\right\} &= \frac{1}{2}\left\{(x-a)^2 + (b-x)^2 + \left|(x-a)^2 - (b-x)^2\right|\right\}\\ &= (b-a)^2\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right]\end{aligned}$$

we obtain

$$\begin{aligned}\int_a^b |P(x,t)|dt &= \int_a^x \left|(b-a)\frac{(t-a)^3}{6} - \frac{(t-a)^4}{12}\right| dt + \int_x^b \left|(a-b)\frac{(t-b)^3}{6} - \frac{(t-b)^4}{12}\right| dt\\ &\le \int_a^x (b-a)\frac{(t-a)^3}{6}dt + \int_a^x \frac{(t-a)^4}{12}dt + \int_x^b (b-a)\frac{(b-t)^3}{6}dt\\ &+ \int_x^b \frac{(b-t)^4}{12}dt \le \frac{1}{6}\left\{(x-a)^2\int_a^x (b-a)(t-a)dt + (x-a)^2\int_a^x \frac{(t-a)^2}{2}dt\right.\\ &+ \left.(b-x)^2\int_x^b (b-a)(b-t)dt + (b-x)^2\int_x^b \frac{(b-t)^2}{2}dt\right\}\\ &\le \frac{1}{6}\max\left\{(x-a)^2,(b-x)^2\right\} \cdot \left\{\int_a^x (b-a)(t-a)dt + \int_a^x \frac{(t-a)^2}{2}dt\right.\\ &+ \left.\int_x^b (b-a)(b-t)dt + \int_x^b \frac{(b-t)^2}{2}dt\right\}\\ &= \max\left\{(x-a)^2,(b-x)^2\right\} \cdot \frac{(b-a)^3}{4} \cdot \left[\frac{7}{36} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right]\\ &= \frac{1}{4}(b-a)^5 \cdot \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}\right] \cdot \left[\frac{7}{36} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right].\end{aligned}$$

Since $\gamma \leq f''(t) \leq \Gamma$ we obtain

$$\left| f''(t) - \frac{\Gamma+\gamma}{2} \right| \leq \frac{\Gamma-\gamma}{2} \quad (\forall) \quad t \in [a,b].$$

From the above relations we get the relation (3).

Remark 3. *If we choose* $x = \dfrac{a+b}{2}$*, we obtain the following quadrature formula:*

$$\int_a^b w(t)f(t)dt = \frac{(b-a)^3}{6} f\left(\frac{a+b}{2}\right) + \frac{\Gamma+\gamma}{80}(b-a)^2 + \mathcal{R}[f]$$

and the remainder therm $\mathcal{R}[f]$ *satisfies the estimation*

$$|\mathcal{R}[f]| \leq \frac{7}{1152}(\Gamma-\gamma)(b-a)^5.$$

Remark 4. *If in Theorem 5, for* $n=2$*, we choose* $x=a$*,* $x=b$*, respectively, and we sum the results, to obtain the following quadrature formula*

$$\int_a^b w(t)f(t)dt = \frac{(b-a)^3}{12}\left[f(a)+f(b)+\frac{b-a}{2}\left(f'(a)-f'(b)\right)+\frac{3}{20}\left(\Gamma+\gamma\right)(b-a)^2\right]+\mathcal{R}[f]$$

and the remainder therm $\mathcal{R}[f]$ *satisfies the estimation*

$$|\mathcal{R}[f]| \leq \frac{1}{18}(\Gamma-\gamma)(b-a)^5.$$

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