THE METRIC REGULARITY OF POLYNOMIAL FUNCTIONS IN ONE OR SEVERAL VARIABLES

Alexandru V. Blaga

Abstract. The metric regularity property at set-valued mappings or single-valued functions has several applications in variational calculus. The Fréchet differential at a point or the strict differential at a point assures several features of this property. The condition which assure the metric regularity property are mentioned in [1], [2], [3] and [4]. This article contains conditions from the metric regularity property at the graph points for polynomial functions in one or several variable. The metric regularity at extremal points for Fréchet differentiable functions can be found here.

2000 Mathematics Subject Classification: 49J52, 58C20; 47D15,49J50

*Keywords and phrases:*Banach spaces, Fréchet differential, metric regularity, Lyusternik-Graves theorem, Robinson-Ursescu theorem, strict differential.

1. Preliminaries

The metric regularity property has many equivalent definitions which will be remainded. The theory of metric regularity is an extension of two classical Theorems. Developments in non-smooth analysis in the 1980s and 1990s paved the way for a number of far-reaching extension of these results. The well-known Lyusternik-Graves Theorem assures the property of metric regularity for differentiable functions at each point from the graph, proving through the equality $reg\, F(\bar{x}|\bar{y}) = reg\, DF(\bar{x})$ exactly the modulus of metric regularity [4]. Additionally, the Robinson-Ursescu Theorem [4] assures the metric regularity at the graph points for the multifunctions which have a closed and convex graph. This article shows the metric regularity at the graph points for polynomial functions, while the metric regularity at extremal or at critical points is not assured. In the case of global extreme points the metric regularity property is not fulfilled and on the case of other ex! treme points the same conclusion is reached.

2. Backgroud in metric regularity

Unless otherwise stated, we always assume set-valued maps to be closed-valued, that is, we assume that all sets $F(x)$ are closed, where $F : X \to Y$ is a set-valued function. The sets

$$dom\ F = \{x|F(x) \neq \Phi\} \text{ and } Im\ F = \cup\{F(x) : x \in X\} \tag{1}$$

are respectively called the domain and the image of F. It will be presumed that F is a self application, meaning that the domain of F is nonempty and $F(X) \subset Y$ with strict inclusion. A set-valued map is often identified by its graph

$$Gr\ f = \{(x, y) \in X \times Y|y \in F(x)\}. \tag{2}$$

The inverse of F is defined by

$$F^{-1}(y) = \{x \in X|y \in F(x)\}. \tag{3}$$

From their definition, $dom\ F^{-1} = Im\ f$. Moreover, if $f : X \to \bar{\mathbb{R}}$ and the multifunction

$$x \mapsto \{\alpha \in \mathbb{R}|\alpha \geq f(x)\} \tag{4}$$

is defined, its graph is called epigraph of f and it is defined by

$$epi(f) = \{(x, \alpha) \in X \times \mathbb{R}|\alpha \geq f(x)\}. \tag{5}$$

The symbol $d(\cdot, \cdot)$ is used for distance. We will consider the norm $d_\alpha((x, y), (u, v)) = d(x, u) + \alpha d(y, v)$, with $\alpha > 0$, on the space product $X \times Y$. We have the closed ball of center x and radius r, defined by

$$B(x, r) = \{y|d(x, y) \leq r\}. \tag{6}$$

Definition 1 ([1]). *Let V a subset of the set $X \times Y$. We say that F is metrically regular on V if then there exists $K > 0$ such that*

$$(x, y) \in V \Rightarrow d(x, F^{-1}(y)) \leq Kd(y, f(x)). \tag{7}$$

The smallest K for which (7) holds will be called the norm of metric regularity on F and written $Reg_\vee F$.

Definition 2 ([1]). *F is metrically regular near $(\bar{x}, \bar{y}) \in Gr\ F$ if for some $\epsilon > 0$ it is metrically regular on the set $V = B(\bar{x}, \epsilon) \times B(\bar{y}, \epsilon)$. The lower bound of such K in this case will be called the norm of metric regularity of F near $(\bar{x}, \bar{y})$ and written $Reg\ F(\bar{x}, \bar{y})$.*

The following proposition assured the equivalence of Definition 1 and Definition 2.

Proposition 1 ([1]) *A set-valued map is metrically regular near $(\bar{x}, \bar{y}) \in Gr\ F$ if and only if is metrically regular on the set*

$$V = \{(x, y) \in B(\bar{x}, \epsilon) \times B(\bar{y}, \epsilon) : d(y, F(x)) \leq \epsilon\}$$

for some $\epsilon > 0$.

Definition 3 ([1]). *We say that F covers on V at a linear rate if there is a $K > 0$ such that*

$$\begin{aligned} &(x, y) \in V,\ v \in F(x) \text{ and } d(v, y) < \epsilon \Rightarrow (\exists) u : d(u, x) \leq Kt \\ &\text{and } y \in F(u). \end{aligned} \tag{8}$$

The lower bound of those K for which (8) *holds will called the norm of covering of F on V, and its inverse, the constant of covering, written $Sur_{\vee} F$.*

Definition 4 ([1]). *Let $W \subset X \times Y$. We say that F is pseudo-Lipschitz on W if there is a $K > 0$ such that*

$$(x, y) \in W \text{ and } y \in F(u) \Rightarrow d(y, F(x)) \leq Kd(x, u). \tag{9}$$

The smallest K for (9) *holds is called the pseudo-Lipschitz norm of F on W.*

The following proposition assured the equivalence of Definition 3 and Definition 4.

Proposition 2 ([1]) *The following statements are equivalent:*
(a). F is regular on V;
(b). F covers on V at a linear rate;
(c). F^{-1} is pseudo-Lipschitz on $W = \{(x, y) | (x, y) \in V\}$.
Moreover, the norm of regularity and covering of F on V and the pseudo-Lipschitz norm of F^{-1} on W are equal, and $Reg_{\vee}\ F \cdot Sur_{\vee}\ F = 1$.

The graph of a function allows a characterization of its metric regularity. Only the definition will be mentioned here.

Definition 5 ([1]). *We say that F is graph-regular on V with norm not greather then K if*

$$d(x, F^{-1}(y)) \leq d_k((x,y), GrF) \tag{10}$$

for all $(x,y) \in V$.

Another characterization of the metric regularity can be made with the help of the strict slope of a function at a point (Theorem 2 from [1] and Theorem 3 from [1], pag. 516 to be seen).

3. Main results

For single-valued mappings F that are nonlinear but differentiable, the notion of metric regularity if not the term itself goes back to a basic theorem in analysis, which is associated with the work of Lyusternik and graves [5]. Here we denote by $DF(\bar{x})$ the derivative mapping in $L(X,Y)$ that is associated with F at x, where $L(X,Y)$ denote the space of continuous linear mappings $F : X \to Y$; here, X and Y are Banach spaces.

Theorem 1 (Lyusternik-Graves, [4]). *For any continuously Fréchet differentiable mapping $F : X \to Y$ and any $(\bar{x}, \bar{y}) \in gph\, F$ one has*

$$reg\, F(\bar{x}|\bar{y}) = reg\, DF(\bar{x}). \tag{11}$$

Thus F is metrically regular at $\bar{x}$ for $\bar{y} = F(\bar{x})$ if and only if $DF(\bar{x})$ is surjective.

Proposition 3 *Let $F : \mathbb{R} \to \mathbb{R}$ be a polynomial function in one variable; then F has the metric regularity property only at the points $(\bar{x}, \bar{y}) \in gphF$ where $DF(\bar{x})$ is different from null mapping.*

Proof. We have

$$\lim_{x \to \bar{x}} \frac{F(x) - F(\bar{x}) - F'(\bar{x})(x - \bar{x})}{|x - \bar{x}|} =$$

$$= \lim_{x \to \bar{x}} \frac{\sum\limits_{i=0}^{n} a_i x^i - \sum\limits_{i=0}^{n} a_i \bar{x}^i - F'(\bar{x})(x - \bar{x})}{|x - \bar{x}|} =$$

$$= \lim_{x \to \bar{x}} \frac{(x - \bar{x}[F'(x) - F'(\bar{x})])}{|x - \bar{x}|} = 0$$

so $DF(\bar{x})(t) = F'(\bar{x})t \in L(\mathbb{R}, \mathbb{R})$. The surjection of $DF(\bar{x})$ requires to be different from null mapping that is the points are not critical and even more extremal.

Proposition 4 *If $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = ax^2 + by^2 + cxy + dx + ey + f$ then F has the metric regularity property at the graph points $(\bar{x}, \bar{y}) \in gph\, F$ for which $DF(\bar{x}, \bar{y})$ is different from null mapping, where $\bar{x} = (x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and the coefficients are real.*

Proof. We have $(\bar{x}, \bar{y}) \in gphF$, so $F(x_0, y_0) = \bar{y} \in \mathbb{R}$. Consider $DF(x)h = \frac{\partial f}{\partial x}(x_0, y_0)h_1 + \frac{\partial f}{\partial y}(x_0, y_0)h_2$, $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}$. Consequently, $DF(\bar{x})h = (2ax_0 + cy_0 + d)h_1 + (2by_0 + cx_0 + e)h_2$ and we have

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x, y) - F(x_0, y_0) - DF(\bar{x})(x - x_0, y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} =$$

$$= \lim_{(x,y)\to(x_0,y_0)} \frac{a(x - x_0)^2 + b(y - y_0)^2 + c(x - x_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} =$$

$$= \lim_{(u,v)\to(0,0)} \frac{au^2 + bv^2 + cuv}{\sqrt{u^2 + v^2}},$$

where $u = x - x_0$, $v = y - y_0$. From

$$|au^2 + bv^2 + cuv| \leq \frac{max(|a|, |b|, |c|)(u^2 + uv + v^2)}{\sqrt{u^2 + v^2}} \leq$$

$$\leq \frac{max(|a|, |b|, |c|)\frac{3}{2}(u^2 + v^2)}{\sqrt{u^2 + v^2}} = max(|a|, |b|, |c|)\frac{3}{2}\sqrt{u^2 + v^2},$$

we have

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x, y) - F(x_0, y_0) - DF(\bar{x})(x - x_0, y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

and $DF(\bar{x}) : \mathbb{R}^2 \to \mathbb{R} \in L(\mathbb{R}^2, \mathbb{R})$. For $\bar{x} \in \mathbb{R}^2$ for which $DF(\bar{x})$ is different from null mapping, $DF(\bar{x})$ is surjective, so due to Theorem 1 F has the metric regularity property at these graph points.

Proposition 5 *If $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x) = \sum\limits_{i+j\leq n} a_{ij}x^i y^j$, then F has the metric regularity property at the graph points $(\bar{x}, \bar{y}) \in gph\, F$ for which $DF(\bar{x}, \bar{y})$ is different from null mapping, where $\bar{x} = (x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ and $a_{ij} \in \mathbb{R}$.*

Proof. Let $\bar{y} = F(x_0, y_0) \in \mathbb{R}$ such that

$$DF(\bar{x})h = \frac{\partial f}{\partial x}(x_0, y_0)h_1 + \frac{\partial f}{\partial y}(x_0, y_0)h_2 = \tag{12}$$
$$= \sum_{k=0}^{n-1} \sum_{i=0}^{n-k-1} (n-k-i)a_{n-k-i,i}x_0^{n-k-i-1}y_0^i h_1 +$$
$$+ \sum_{k=0}^{n-1} \sum_{i=0}^{n-k-1} (n-k-i)a_{i,n-k-i}x_0^i y_0^{n-k-i-1} h_2$$

and we have

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x,y) - F(x_0,y_0) - DF(\bar{x})(x-x_0, y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0,$$

because the partial derivatives are continuous at (x_0, y_0) as polynomial functions, the differential of F at point is the one given by (12).

Obviously, we do not have the metric regularity property at saddle points because $DF(\bar{x})$ is not surjective. In the case of several variables polynomial functions, the differential at the point assures the metric regularity property in the surjectivity case.

Remark 1 In the case of functions which are not polynomial, at extremal points or at saddle points when the function is not differentiable, the metric regularity property may or may not the place. Let for example the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 > 0 \\ 0, & x^2+y^2 = 0 \end{cases}$ and $\bar{x} = (1,1)$ which is maximum point, since $f(x,t) \leq f(1,1)$. Because $\frac{\partial f}{\partial x}(1,1) = \frac{\partial f}{\partial y}(1,1) = 0$, we have $DF(1,1)h = 0$, for all $h \in \mathbb{R}^2$. This can be deduced also from

$$\lim_{(x,y)\to(1,1)} \frac{f(x,y) - 1/2 - DF(1,1)(x-1, y-1)}{\sqrt{(x-1)^2 + (y-1)^2}} = \lim_{(u,v)\to(0,0)} \frac{(u-v)^2}{\sqrt{u^2+v^2}},$$

where $u = x-1$, $v = y-1$. From $|u^2 + v^2 - 2uv| \leq 2(u^2 + v^2)$, the limit above is 0. Since $DF(1,1)$ is not surjective, f has not the metric regularity property at $(1,1)$.

Remark 2 At extremal points where the functions not differentiable we can have metric regularity property. For example, let the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = \begin{cases} 1, & x = (1,y) \in \mathbb{R}^2 \\ 0, & otherwise \end{cases}$ for which we have $d(x, f^{-1}(y)) \leq Kd(y, f(x))$, for $(x,y) \in V \times W$, $V = [0,2]$, $W = [1,3]$ and from $f(1,y) \geq f(x,y) \in \{0,1\}$ the points $(1,y)$ are all the points of maximum. At $\alpha = (1,2)$, for which $V \times W$ is a neighborhood, we have $d(x, f^{-1}(y)) = 0$, because $\{(1,y)|y \in [1.3]\} \subset [0,2] \times [1,3]$ and we choose $K = 0$,because $f^{-1}(y) = \{(1,y)|y \in \mathbb{R}\} \cup \{(1+\epsilon, y)|\epsilon, y \in \mathbb{R}\}$, $\epsilon \neq 0$. Due the continuity, f is not differentiable at α and at the points $(1,y)$, $y \in \mathbb{R}$.

Proposition 6 *For the function* $F : \mathbb{R}^p \to \mathbb{R}$ *defined by*

$$F(x) = \sum_{i_1+i_2+\ldots+i_p \leq n} a_{i_1 i_2 \ldots i_p} x_1^{i_1} x_2^{i_2} \ldots x_p^{i_p},$$

where $x = (x_1, x_2, \ldots, x_p)$ *and* $\bar{x}_0 = \{x_0^1, x_0^2, \ldots, x_0^p\}$, *we note* $\alpha_i = \frac{\partial F}{\partial x_i}(\bar{x}_0)$, $i \in \{1, 2, \ldots, p\}$ *and we suppose that* $\sum\limits_{i=0}^{p} \alpha_i^2 \neq 0$. *Then we have*

$$reg\ F(\bar{x}|\bar{y}) = \frac{1}{\sqrt{\sum\limits_{i=1}^{p} \alpha_i^2}}. \tag{13}$$

Proof. Because $\sum\limits_{i=0}^{p} \alpha_i^2 \neq 0$, from Theorem 1.2 from [4], we have that the application $DF(\bar{x}_0)$ is a surjection for $\bar{x}_0 \in \mathbb{R}^p$ and that $reg\ F(\bar{x}|\bar{y}) = reg\ DF(\bar{x}_0)$, where $(\bar{x}_0, \bar{y}_0) \in gph\ F$. Due to Example 1.1 from [4] and $DF(\bar{x}_0) \in L(\mathbb{R}^p, \mathbb{R})$, then the regularity modulus $reg\ DF(\bar{x}_0)(\bar{x}_0|\bar{y}_0)$ is the same for all $(\bar{x}_0, \bar{y}_0) \in gph\, F$ and that common value, denoted by $reg\, DF(\bar{x}_0)$ is given by

$$\begin{aligned} reg\ DF(\bar{x}_0) &= inf\{k \in (0,\infty)|kDF(\bar{x}_0)(\mathbb{B}_X) \supset int\, \mathbb{B}_Y\} = \\ &= sup\{d(0, DF^{-1}(\bar{x}_0)(y))|y \in \mathbb{B}_Y\}. \end{aligned} \tag{14}$$

Let $x \in DF^{-1}(\bar{x}_0)y$, so that $DF^{-1}(\bar{x}_0)x = y$, $\sum\limits_{i=1}^{p} \frac{\partial F}{\partial x_i}(\bar{x}_0)x_i = y$ and more, $\sum\limits_{i=1}^{p} \alpha_i x_i = y$. It results that $\left|\sum\limits_{i=1}^{p} \alpha_i x_i\right| \leq 1$ and we will calculate min $\sqrt{\sum\limits_{i=1}^{p} x_i^2}$

because $d(0, A) = \min\{\|x\| : x \in A\}$, $A = DF^{-1}(\bar{x}_0)y$. Consider

$$h(x, \lambda) = \sum_{i=1}^{p} x_i^2 - \lambda \left(\sum_{i=1}^{p} \alpha_i x_i - s \right), \tag{15}$$

where $s \in [0, 1]$. From the Lagrange multipliers method, we have

$$\begin{cases} 2x_i - \lambda \alpha_i = 0, \ i \in \{1, 2, \ldots, p\} \\ \sum\limits_{i=1}^{p} \alpha_i x_i = s \end{cases} \tag{16}$$

from where we obtain $x_i = \frac{\lambda \alpha_i}{2}$ and $\lambda = \frac{2s}{\sum\limits_{i=1}^{p} \alpha_i^2}$, so that $x_i = \alpha_i s \cdot \frac{1}{\sum\limits_{i=1}^{p} \alpha_i^2}$, $\sum\limits_{i=1}^{p} x_i^2 = s^2 \cdot \frac{1}{\sum\limits_{i=1}^{p} \alpha_i^2}$ and $\|x\| \leq \frac{|s|}{\sqrt{\sum\limits_{i=1}^{p} \alpha_i^2}} \leq \frac{1}{\sqrt{\sum\limits_{i=1}^{p} \alpha_i^2}}$; the last inequality is in fact (13), since the supremum is reached in the case of equality.

References

[1] A.D. Yoffe, *Metric regularity and subdifferential calculus*, Uspekhi Mat. Nauk, **55**(2000), 103-162

[2] B. Sh. Mordukhovich, *Complete characterization of openness, metric regularity and lipschitzian properties of multifunctions*, Trans. Amer. Math. Soc., **340**(1993), 1-36

[3] S.M. Robinson, *Regularity and stability for convex multifunctions*, Math. Oper. Res., **1**(1976), 130-143

[4] A.I. Dontchev, A.S. Lewis, R.T. Rochafellar, *The radius of metric regularity*, Trans. Amer. Math. Soc., vol. 355, nr. 2(2002), 493-517

[5] A.V. Dmitriuk, A.A. Milyutin, N.P. Osmolowskii, *The Lyusternik theorem and the theory of extremum*, Uspekhi Math. Nauk, **35**(1980), nr.6, 11-46

[6] B.S. Mordukhovich, *Variational Analysis and Generalized Differentiation*, Vol. I, II, Springer, 2005

[7] M. Mureşan, *Nonsmooth Analysis and Appllcations*, Ed. Risoprint, Cluj-Napoca, 2001(Romanian)

[8] I. Muntean, *Functional Analysis*, Cluj-Napoca, 1993(Romanian)

[9] A.D. Yoffe, *Nonsmooth analysis: differential calculus of subdifferentiable mappings*, Trans. Amer. Math. Soc., **266**(1981), 1-56

[10] W. Rudin, *Functional Analysis, second edition*, Ed. McGraw-Hill, New-York, 1991

Author:

Alexandru V. Blaga
National College "Mihai Eminescu"
5 Mihai Eminescu Street
Satu Mare 440014,Romania
e-mail:*alblaga2005@yahoo.com*