ON SOME INTERESTING TRIGONOMETRIC SUMS

Dorin Andrica and Mihai Piticari

Abstract. We give three different proofs to the identity (2). Some applications are also presented.

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1. Introduction

In [3] we proposed the following question: Evaluate the sum

$$\sum_{k=0}^{n-1} \frac{1}{1+8\sin^2\left(\frac{k\pi}{n}\right)}. \tag{1}$$

In the next issue of the journal "Mathematical Reflexions" no solutions were given to this problem. The main purpose of this note is to present three different proofs to the following general result.

Theorem. *For any real number $a \in \mathbb{R} \setminus \{-1, 1\}$ the following relation holds:*

$$\sum_{k=0}^{n-1} \frac{1}{a^2 - 2a\cos\frac{2k\pi}{n} + 1} = \frac{n(a^n+1)}{(a^2-1)(a^n-1)}. \tag{2}$$

Let us remark that the particular case $n = 7$ and $a \in (-1, 1)$ represents Problem 49 in the Longlisted Problems of IMO1988 (see [4] page 217).

In the last section we present four applications of our main result.

2. Proofs of the theorem

Proof 1. (Dorin Andrica and Mihai Piticari) Let $P \in \mathbb{R}[X]$ be a polynomial with real coefficients of degree $n-1$, $P = a_0 + a_1 X + \cdots + a_{n-1}X^{n-1}$. If $\alpha \in U_n$ is an n^{th} root of unity, then we have

$$\begin{aligned}|P(\alpha)|^2 &= P(\alpha)\cdot\overline{P(\alpha)} = P(\alpha)\cdot P(\overline{\alpha}) = P(\alpha)\cdot P\left(\frac{1}{\alpha}\right)\\ &= (a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1})\left(a_0 + \frac{a_1}{\alpha} + \cdots + a_{n-1}\frac{1}{\alpha^{n-1}}\right)\\ &= a_0^2 + a_1^2 + \cdots + a_{n-1}^2 + \sum_{k=1}^{n-1} A_k\alpha^k + \sum_{j=1}^{n-1} B_j\frac{1}{\alpha^j},\end{aligned}$$

where coefficients A_k, B_j are different from zero. Using the relation (see [2, Proposition 3, page 46])

$$\sum_{\alpha\in U_n} \alpha^k = \begin{cases} n, & \text{if } n|k \\ 0, & \text{otherwise}\end{cases}$$

we get the following nice formula

$$\sum_{\alpha\in U_n} |P(\alpha)|^2 = n(a_0^2 + a_1^2 + \cdots + a_{n-1}^2) \tag{3}$$

In order to prove (2) we consider the polynomial

$$P = 1 + aX + \cdots + a^{n-1}X^{n-1} = \frac{a^nX^n - 1}{aX - 1}.$$

Applying formula (3) it follows

$$\sum_{\alpha\in U_n} |P(\alpha)|^2 = n(1 + a^2 + \cdots + a^{2n-2}) = n\frac{a^{2n}-1}{a^2-1} \tag{4}$$

On the other hand we have

$$|P(\alpha)|^2 = \left|\frac{a^n\alpha^n - 1}{a\alpha - 1}\right|^2 = \frac{(a^n-1)^2}{(a\alpha-1)(a\overline{\alpha}-1)} = \frac{(a^n-1)^2}{a^2 - 2\text{Re }\alpha\cdot a + 1}$$

and formula (2) follows.

Proof 2. (Gabriel Dospinescu, Ecole Normale Superieure, Paris, France) We will use so called Poisson kernel formula

$$\frac{1-r^2}{1-2r\cos t+r^2}=\sum_{m=-\infty}^{+\infty} r^{|m|}z^m, \quad \text{where} \quad |r|<1 \tag{5}$$

and $z=\cos t+i\sin t$. Applying this formula in our case we have

$$\sum_{k=0}^{n-1}\frac{1-a^2}{1-2a\cos\dfrac{2k\pi}{n}+a^2}=\sum_{k=0}^{n-1}\left(\sum_{m=-\infty}^{+\infty}a^{|m|}e^{im\frac{2k\pi}{n}}\right)$$

$$=\sum_{m=-\infty}^{+\infty}a^{|m|}\left(\sum_{k=0}^{n-1}\left(e^{\frac{2\pi im}{n}}\right)^k\right)=\sum_{m\in\mathbb{Z}}na^{|m|}$$

$$=n\sum_{j\in\mathbb{Z}}a^{n|j|}=n\frac{1-a^{2n}}{(1-a^n)^2}=n\frac{1+a^n}{1-a^n},$$

and formula (2) follows.

Proof 3. (Dorin Andrica) Consider the polynomial $P\in\mathbb{C}[X]$ having the factorization $P=\prod_{k=1}^{n}(X^2+a_kX+b)$. Then

$$\frac{P'}{P}=\sum_{k=1}^{n}\frac{2X+a_k}{X^2+a_kX+b}=\frac{1}{X}\sum_{k=1}^{n}\frac{2X^2+a_kX+b-b}{X^2+a_kX+b}$$

$$=X\sum_{k=1}^{n}\frac{1}{X^2+a_kX+b}+\frac{n}{X}-\frac{b}{X}\sum_{k=1}^{n}\frac{1}{X^2+a_kX+b}$$

$$=\frac{X^2-b}{X}\sum_{k=1}^{n}\frac{1}{X^2+a_kX+b}+\frac{n}{X},$$

and we derive the formula

$$\frac{XP'-nP}{(X^2-b)P}=\sum_{k=1}^{n}\frac{1}{X^2+a_kX+b}. \tag{6}$$

For the polynomial

$$P=(X^n-1)^2=\prod_{k=0}^{n-1}\left(X^2-2X\cos\frac{2k\pi}{n}+1\right)$$

we have $P'=2nX^{n-1}(X^n-1)$, hence

$$\frac{XP'-nP}{(X^2-1)P}=\frac{2nX^n(X^n-1)-n(X^n-1)^2}{(X^2-1)(X^n-1)^2}=n\frac{X^n+1}{(X^2-1)(X^n-1)},$$

and (2) follows from (6).

3. Applications

Application 1. In order to evaluate the sum (1) we take $a=2$ in identity (2) and get

$$\sum_{k=0}^{n-1}\frac{1}{4-4\cos\dfrac{2k\pi}{n}+1}=\frac{n(2^n+1)}{3(2^n-1)},$$

that is

$$\sum_{k=0}^{n-1}\frac{1}{1+8\sin^2\left(\dfrac{k\pi}{n}\right)}=\frac{n(2^n+1)}{3(2^n-1)}. \tag{7}$$

Application 2. Let us use the identity (2) to prove the equality

$$\sum_{k=1}^{n-1}\frac{1}{\sin^2\left(\dfrac{k\pi}{n}\right)}=\frac{(n-1)(n+1)}{3}. \tag{8}$$

Indeed, the identity (2) is equivalent to

$$\sum_{k=1}^{n-1}\frac{1}{a^2-2a\cos\dfrac{2k\pi}{n}+1}=\frac{n(a^n+1)}{(a^2-1)(a^n-1)}-\frac{1}{(a-1)^2}.$$

Taking in the right hand side the limit for $a\to 1$ we get

$$\lim_{a\to 1}\left[\frac{n(a^n+1)}{(a^2-1)(a^n-1)}-\frac{1}{(a-1)^2}\right]$$

$$= \frac{1}{2n} \lim_{a\to 1} \frac{(n-1)a^{n+1} - (n+1)a^n + (n+1)a - n + 1}{(a-1)^3}$$

$$= \frac{1}{2n} \lim_{a\to 1} \frac{(n+1)(n-1)a^n - n(n+1)a^{n-1} + (n+1)}{3(a-1)^2}$$

$$= \frac{1}{2n} \lim_{a\to 1} \frac{(n+1)(n-1)na^{n-2}(a-1)}{6(a-1)} = \frac{(n-1)(n+1)}{12}.$$

It follows

$$\lim_{a\to 1} \sum_{k=1}^{n-1} \frac{1}{a^2 - 2a\cos\dfrac{2k\pi}{n} + 1} = \frac{(n-1)(n+1)}{12},$$

that is

$$\sum_{k=1}^{n-1} \frac{1}{4\sin^2\left(\dfrac{k\pi}{n}\right)} = \frac{(n-1)(n+1)}{12},$$

so we obtain identity (8).

Remarks. 1) Using the symmetry

$$\sin^2\left(\frac{k\pi}{n}\right) = \sin^2\frac{(n-k)\pi}{n}, \quad k = 1, 2, \ldots, n-1,$$

from identity (8) we get:

If n is odd, $n = 2m+1$, then

$$\sum_{k=1}^{m} \frac{1}{\sin^2\left(\dfrac{k\pi}{2m+1}\right)} = \frac{2m(2m+2)}{6} = \frac{m(2m+2)}{3}$$

That is equivalent to

$$\sum_{k=1}^{m} \operatorname{ctg}^2\frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}. \tag{9}$$

If n is even, then we obtain

$$\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\sin^2\left(\dfrac{k\pi}{n}\right)} = \frac{1}{2}\left(\frac{n^2-1}{3} - 1\right) = \frac{n^2-4}{6}. \tag{10}$$

2) A different method to prove (9) is given in [1, page 147]. Consider the trigonometric equation $\sin(2m+1)x = 0$, with roots

$$\frac{\pi}{2m+1}, \frac{2\pi}{2m+1}, \ldots, \frac{m\pi}{2m+1}$$

Expressing $\sin(2m+1)x$ in terms of $\sin x$ and $\cos x$, we obtain

$$\sin(2m+1)x = \binom{2m+1}{1}\cos^{2m} x \sin x - \binom{2m+1}{3}\cos^{2m-2} x \sin^3 x + \ldots$$

$$= \sin^{2m+1} x \left(\binom{2m+1}{1} \operatorname{ctg}^{2m} x - \binom{2m+1}{3} \operatorname{ctg}^{2m-2} x + \ldots \right)$$

Set $x = \dfrac{k\pi}{2m+1}$, $k = 1, 2, \ldots, m$. Since $\sin^{2m+1} x \neq 0$, we have

$$\binom{2m+1}{1} \operatorname{ctg}^{2m} x - \binom{2m+1}{3} \operatorname{ctg}^{2m-2} x + \cdots = 0$$

Substituting $y = \operatorname{ctg}^2 x$ yields

$$\binom{2m+1}{1} y^m - \binom{2m+1}{3} y^{m-1} + \cdots = 0,$$

an algebraic equation with roots

$$\operatorname{ctg}^2 \frac{\pi}{2m+1}, \operatorname{ctg}^2 \frac{2\pi}{2m+1}, \ldots, \operatorname{ctg}^2 \frac{m\pi}{2m+1}$$

Using the Vietà's relation between coefficients and roots, we obtain

$$\sum_{k=1}^{m} \operatorname{ctg}^2 \frac{k\pi}{2m+1} = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{m(2m-1)}{3}$$

Application 3. We use now the result in our Theorem to prove the following identities:

If n is odd and $n \geq 3$, then

$$\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)} = \frac{n^2-1}{2} \tag{11}$$

If n is even and $n \geq 4$, then

$$\sum_{k=1}^{\frac{n}{2}-1} \frac{1}{\cos^2\left(\frac{k\pi}{n}\right)} = \frac{n^2-4}{6} \tag{12}$$

In order to prove the identity (11) we note that from (2) we have

$$\sum_{k=0}^{n-1} \frac{1}{2+2\cos\frac{2k\pi}{n}} = \lim_{a\to -1} \frac{n(a^n+1)}{(a^2-1)(a^n-1)}$$
$$= \frac{n}{4}\lim_{a\to -1}\frac{a^n+1}{a+1} = \frac{n^2}{4}.$$

That is
$$\sum_{k=0}^{n-1} \frac{1}{\cos^2\left(\frac{k\pi}{n}\right)} = n^2,$$
therefore
$$\sum_{k=1}^{n-1} \frac{1}{\cos^2\left(\frac{k\pi}{n}\right)} = n^2-1.$$

Using the symmetry

$$\cos^2\left(\frac{k\pi}{n}\right) = \cos^2\left(\frac{(n-k)\pi}{n}\right), \quad k=1,\ldots,n-1,$$

the identity (11) follows.

To prove the identity (12) by the same method is much more complicated. From (2) we have

$$\sum_{\substack{k=0\\ k\neq \frac{n}{2}}}^{n-1} \frac{1}{2+2\cos\frac{2k\pi}{n}} = \lim_{a\to -1}\left[\frac{n(a^n+1)}{(a^2-1)(a^n-1)} - \frac{1}{(a+1)^2}\right]$$

$$= \lim_{t\to 0}\left\{\frac{n[(t-1)^n+1]}{t(t-2)[(t-1)^n-1]} - \frac{1}{t^2}\right\}$$

$$= -\frac{1}{2}\lim_{t\to 0}\frac{n[(t-1)^n+1]t-(t-2)[(t-1)^n-1]}{t^2[(t-1)^n-1]}$$

$$= -\frac{1}{2}\lim_{t\to 0}\frac{nt\left(2-nt+\dfrac{n(n-1)}{2}t^2+t^3f(t)\right)}{t^2\left(-nt+\dfrac{n(n-1)}{2}t^2+t^3h(t)\right)}$$

$$-\frac{(t-2)\left(-nt+\dfrac{n(n-1)}{2}t^2-\dfrac{n(n-1)(n-2)}{6}+t^3+t^4g(t)\right)}{t^2\left(-nt+\dfrac{n(n-1)}{2}t^2+t^3h(t)\right)}$$

$$= -\frac{1}{2}\cdot\frac{\dfrac{n^2(n-1)}{2}-2\dfrac{n(n-1)(n-2)}{6}-\dfrac{n(n-1)}{2}}{-n}=\frac{n^2-1}{12},$$

since f,g,h are polynomials in t, and so $f(0),g(0),h(0)$ are finite. It follows

$$\sum_{\substack{k=1\\k\neq\frac{n}{2}}}^{n-1}\frac{1}{\cos^2\left(\dfrac{k\pi}{n}\right)}=\frac{n^2-4}{3}$$

and then using again the symmetry

$$\cos^2\left(\frac{k\pi}{n}\right)=\cos^2\left(\frac{(n-k)\pi}{n}\right),\quad k=1,\ldots,n-1,$$

we get the identity (10).

Remark. It is possible to derive the relation (12) directly from (8) by using the symmetry

$$\cos^2\frac{k\pi}{n}=\sin^2\left(\frac{\pi}{2}-\frac{k\pi}{n}\right)=\sin^2\frac{(n-2k)\pi}{2n}=\sin^2\frac{\left(\dfrac{n}{2}-k\right)\pi}{n},\ k=1,\ldots,\frac{n}{2}.$$

Application 4. If $x\in\mathbb{R}$ and $|x|>1$, then

$$\sum_{k=0}^{n-1}\frac{1}{x-\cos\dfrac{2k\pi}{n}}=\frac{2n\left(x+\sqrt{x^2-1}\right)\left[\left(x+\sqrt{x^2-1}\right)^n+1\right]}{\left[\left(x+\sqrt{x^2-1}\right)^2-1\right]\left[\left(x+\sqrt{x^2-1}\right)^n-1\right]}\tag{13}$$

Indeed, we have

$$\sum_{k=0}^{n-1} \frac{1}{a^2 - 2a\cos\dfrac{2k\pi}{n} + 1} = \frac{1}{2a}\sum_{k=0}^{n-1} \frac{1}{\dfrac{1}{2}\left(a + \dfrac{1}{a}\right) - \cos^2\dfrac{2k\pi}{n}}.$$

Let $x = \dfrac{1}{2}\left(a + \dfrac{1}{a}\right)$. Then $a = x + \sqrt{x^2 - 1}$, and from identity (2) we get

$$\frac{1}{2a}\sum_{k=0}^{n-1} \frac{1}{x - \cos\dfrac{2k\pi}{n}} = \frac{n(a^n + 1)}{(a^2 - 1)(a^n - 1)},$$

i.e. the relation (13).

Taking in (13), $x = 2$, we get

$$\sum_{k=0}^{n-1} \frac{1}{1 + 2\sin^2\left(\dfrac{k\pi}{n}\right)} = \frac{2n\left(2+\sqrt{3}\right)\left[\left(2+\sqrt{3}\right)^n + 1\right]}{\left(3+3\sqrt{3}\right)\left[\left(2+\sqrt{3}\right)^n - 1\right]}. \qquad (14)$$

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Authors:

Dorin Andrica
"Babeş-Bolyai" University
Faculty of Mathematics and Computer Science

Cluj-Napoca, Romania
e-mail: *dandrica@math.ubbcluj.ro*

Mihai Piticari
"Dragoş-Vodă" National College
Câmpulung Moldovenesc, Romania