NOTES ON INTEGRAL MEANS INEQUALITIES

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Abstract.Applying subordination theorem of J. E. Littlewood in [1], and Lemma of S. S. Miller and P. T. Monanu in [2] to certain analytic functions and the Koebe function, we show an integral means inequality. Further, we obtain an integral means inequality for the first derivative.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open unit disk $\mathrm{U} = \{z \in \mathrm{C} : |z| < 1\}$.

Denote by $k(z)$ the Koebe function

$$k(z) = \frac{z}{(1-z)^2} \quad (z \in \mathrm{U}). \tag{2}$$

Further we denote by $h(z)$ the analytic function in U defined by

$$h(z) = \frac{1}{1+z}. \tag{3}$$

In this paper, we discuss the integral means inequalities of $f(z)$ in $\mathcal{A}$ and the Koebe function $k(z)$ given by (2), and $f'(z)(f(x) \in \mathcal{A})$ and $h(z)$ of the form (3). Moreover we show an estimate of $f'(z)$.

We recall the concept of subordination between analytic functions. Given two functions $f(z)$ and $g(z)$, which are analytic in U, the function $f(z)$ is said to be subordinate to $g(z)$ in U if there exists a function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in U, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathrm{U}) \subset g(\mathrm{U})$.

We need the following subordination theorem of J. E. Littltewood.

Lemma A(Littlewood [1]) *If $f(z)$ and $g(z)$ are analytic in U with $f(z) \prec g(z)$, then, for $\mu > 0$ and $z = re^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \leq \int_0^{2\pi} |g(z)|^\mu \, d\theta.$$

Applying the lemma of Littlewood above, H. Silverman [5] showed the integral means inequalities for univalent functions with negative coefficients. S. Owa and T. Sekine [3] proved integral means inequalities with coefficients inequalities for normalized analytic functions and polynomials(see also Sekine et al. [4]).

In addition we need the following Lemma of S. S. Miller and P. T. Mocanu.

Lemma A(S. S. Miller and P. T. Mocanu [2]) *Let $g(z) = g_n z^n + g_{n+1} z^{n+1} + \cdots$ be analytic in U with $g(z) \neq 0$ and $n \geq 1$. If $z_0 = r_0 e^{i\theta_0}$ $(r_0 < 1)$ and*

$$|g(z_0)| = \max_{|z| \leq |z_0|} |g(z)|$$

then

(i) $\dfrac{z_0 g'(z_0)}{g(z_0)} = k$

and

(ii) $\mathrm{Re}\left(\dfrac{z_0 g''(z_0)}{g'(z_0)}\right) + 1 \geq k,$

where $k \geq n \geq 1$.

2. Integral means inequalities for $f(z)$ and $k(z)$

Theorem 1. *Let $f(z)$ be in $\mathcal{A}$ and $k(z)$ be the Koebe function given by (2). If the function $f(z)$ satisfies*

$$\mathrm{Re}\left\{\alpha \frac{zf'(z)}{f(z)} - \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \tag{4}$$

for $\alpha \in \mathrm{R}$ *and* $\beta \geq 0$, *then, for* $\mu > 0$ *and* $z = re^{i\theta}$ $(0 < r < 1)$

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|k\left(re^{i\theta}\right)\right|^{\mu} d\theta. \tag{5}$$

Proof. By applying Lemma A, it suffices to show that

$$f(z) \prec \frac{z}{(1-z)^2}.$$

Let us define the function $w(z)$ by

$$f(z) = \frac{w(z)}{(1-w(z))^2} \quad (w(z) \neq 1). \tag{6}$$

Thus we have an analytic function $w(z)$ in U such that $w(0) = 0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1 (z \in \mathrm{U})$ for

$$\begin{aligned} &\mathrm{Re}\left\{\alpha\frac{zf'(z)}{f(z)} - \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \\ = \;& \mathrm{Re}\left\{\alpha\left(\frac{zw'(z)}{w(z)} + \frac{2zw'(z)}{1-w(z)}\right) - \beta\left(1 + \frac{zw''(z)}{w'(z)} + \frac{zw'(z)}{1+w(z)} + \frac{3zw'(z)}{1-w(z)}\right)\right\} \\ > \;& 0 \quad (z \in \mathrm{U}). \end{aligned}$$

If there exists $z_0 \in \mathrm{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then we have by Lemma B,

$$w(z_0) = e^{i\theta}, \; \frac{z_0 w'(z_0)}{w(z_0)} = k \geq 1, \; \mathrm{Re}\left(\frac{z_0 w''(z_0)}{w'(z_0)}\right) + 1 \geq k.$$

For such a point $z_0 \in \mathrm{U}$, we obtain that

$$
\begin{aligned}
&\operatorname{Re}\left\{\alpha\frac{z_0f'(z_0)}{f(z_0)}-\beta\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)\right\}\\
&= \operatorname{Re}\left\{\alpha\left(\frac{z_0w'(z_0)}{w(z_0)}+\frac{2z_0w'(z_0)}{1-w(z_0)}\right)-\beta\left(1+\frac{z_0w''(z_0)}{w'(z_0)}+\frac{z_0w'(z_0)}{1+w(z_0)}+\frac{3z_0w'(z_0)}{1-w(z_0)}\right)\right\}\\
&= \operatorname{Re}\left\{\alpha k+\alpha\frac{2kw(z_0)}{1-w(z_0)}-\beta-\frac{\beta z_0w''(z_0)}{w'(z_0)}-\frac{\beta kw(z_0)}{1+w(z_0)}-\frac{3\beta kw(z_0)}{1-w(z_0)}\right\}\\
&= \operatorname{Re}\left\{\alpha k+(2\alpha k-3\beta k)\frac{w(z_0)}{1-w(z_0)}-\beta-\frac{\beta z_0w''(z_0)}{w'(z_0)}-\frac{\beta kw(z_0)}{1+w(z_0)}\right\}\\
&\leq \alpha k+(2\alpha k-3\beta k)\left(-\frac{1}{2}\right)-\beta+\beta(1-k)-\beta k\left(\frac{1}{2}\right)\\
&= 0 \quad (\alpha\in \mathrm{R},\ \beta\geq 0),
\end{aligned}
$$

which contradicts the hypothesis (4) of the theorem. Therefore there is no $z_0 \in \mathrm{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathrm{U}$. Thus we have that

$$f(z) \prec \frac{z}{(1-z)^2},$$

which shows that

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|k\left(re^{i\theta}\right)\right|^{\mu} d\theta.$$

This comletes the proof.

Corollary 1. *Let the function $f(z)$ in $\mathcal{A}$ and the Koebe function given by* (2) *satisfy the conditions in* Theorem 1. *Then, for $\mu > 0$ and $z = r^{i\theta}$* $(0 < r < 1)$

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \leq 2\pi\left(\frac{r}{1+r^2}\right)^{\mu}\left\{1+\sum_{n=1}^{\infty}\frac{\prod_{j=0}^{2n-1}(j+\mu)}{(n!)^2}\left(\frac{r}{1+r^2}\right)^{2n}\right\}. \tag{7}$$

Proof.

$$\begin{aligned}
\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta &\leq \int_0^{2\pi} \left|\frac{re^{i\theta}}{\left(1-re^{i\theta}\right)^2}\right|^{\mu} d\theta \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \int_0^{2\pi} \left(1-\frac{2r\cos\theta}{1+r^2}\right)^{-\mu} d\theta \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \int_0^{2\pi} \left\{\sum_{n=0}^{\infty} \binom{-\mu}{n} \left(-\frac{2r\cos\theta}{1+r^2}\right)^{n}\right\} d\theta \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \int_0^{2\pi} \left\{1+\sum_{n=1}^{\infty} \binom{-\mu}{n} (-1)^n \left(\frac{2r\cos\theta}{1+r^2}\right)^{n}\right\} d\theta \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \left\{2\pi+\int_0^{2\pi} \sum_{n=1}^{\infty} \binom{-\mu}{n} (-1)^n \left(\frac{2r\cos\theta}{1+r^2}\right)^{n} d\theta\right\} \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \left\{2\pi+\sum_{n=1}^{\infty} \binom{-\mu}{n} (-1)^n \left(\frac{2r}{1+r^2}\right)^{n} \int_0^{2\pi} \cos^n\theta\, d\theta\right\} \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \left\{2\pi+\sum_{n=1}^{\infty} \binom{-\mu}{2n} (-1)^{2n} \left(\frac{2r}{1+r^2}\right)^{2n} \int_0^{2\pi} \cos^{2n}\theta\, d\theta\right\} \\
&= \left(\frac{r}{1+r^2}\right)^{\mu} \left\{2\pi+\sum_{n=1}^{\infty} \binom{-\mu}{2n} \left(\frac{r}{1+r^2}\right)^{2n} \cdot 2^{2n} \cdot \frac{4(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}\right\} \\
&= 2\pi \left(\frac{r}{1+r^2}\right)^{\mu} \left\{1+\sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+j)}{(2n)!} \left(\frac{r}{1+r^2}\right)^{2n} \cdot \frac{(2n)!}{(n!)^2}\right\} \\
&= 2\pi \left(\frac{r}{1+r^2}\right)^{\mu} \left\{1+\sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+j)}{(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n}\right\}.
\end{aligned}$$

3. Integral means inequalities for the first derivative

The proof for the first derivative is similar.

Theorem 2. *Let $f(z)$ be in $\mathcal{A}$ and $h(z)$ be the analytic function given by* (3). *If the function $f(z)$ satisfies*

$$\operatorname{Re}\left\{\alpha f'(z)+\beta\frac{zf''(z)}{f'(z)}-\gamma\frac{zf'''(z)}{f''(z)}\right\}>\frac{\alpha-\beta-2\gamma}{2} \tag{8}$$

for $\alpha\in\mathrm{R}$, $\beta+4\gamma\geq 0$ *and* $\gamma\geq 0$, then, for $\mu>0$ *and* $z=re^{i\theta}$ $(0<r<1)$

$$\int_0^{2\pi}\left|f'\left(re^{i\theta}\right)\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|h\left(re^{i\theta}\right)\right|^{\mu}d\theta. \tag{9}$$

Proof. By Lemma A, it suffices to show that

$$f'(z)\prec\frac{1}{1+z}.$$

Let us define the function $w(z)$ by

$$f'(z)=\frac{1}{1+w(z)}\qquad(w(z)\neq 1)\,. \tag{10}$$

Then we have an analytic function $w(z)$ in U such that $w(0)=0$. Further, we prove that the analytic function $w(z)$ satisfies $|w(z)|<1(z\in\mathrm{U})$ for

$$\begin{aligned}&\operatorname{Re}\left\{\alpha f'(z)+\beta\frac{zf''(z)}{f'(z)}-\gamma\frac{zf'''(z)}{f''(z)}\right\}\\ =&\ \operatorname{Re}\left\{\alpha\frac{1}{1+w(z)}-\beta\frac{zw'(z)}{1+w(z)}-\gamma\left(\frac{zw''(z)}{w'(z)}+\frac{2zw'(z)}{1+w(z)}\right)\right\}\\ >&\ \frac{\alpha-\beta-2\gamma}{2}\qquad(z\in\mathrm{U}).\end{aligned}$$

If there exists $z_0\in\mathrm{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1,$$

then we have by Lemma B,

$$w(z_0)=e^{i\theta},\ \frac{z_0w'(z_0)}{w(z_0)}=k\geq 1,\ \operatorname{Re}\left(\frac{z_0w''(z_0)}{w'(z_0)}\right)+1\geq k.$$

For such a point $z_0 \in \mathrm{U}$, we obtain that

$$\mathrm{Re}\left\{\alpha f'(z_0)+\beta\frac{z_0 f''(z_0)}{f'(z_0)}-\gamma\frac{z_0 f'''(z_0)}{f''(z_0)}\right\}$$

$$= \mathrm{Re}\left\{\alpha\frac{1}{1+w(z_0)}-\beta\frac{z_0 w'(z_0)}{1+w(z_0)}-\gamma\left(\frac{z_0 w''(z_0)}{w'(z_0)}+\frac{2z_0 w'(z_0)}{1+w(z_0)}\right)\right\}$$

$$= \mathrm{Re}\left(\frac{\alpha}{1+w(z_0)}\right)-\mathrm{Re}\left(\frac{\beta k w(z_0)}{1+w(z_0)}\right)-\mathrm{Re}\left(\frac{\gamma z_0 w''(z_0)}{w'(z_0)}\right)-2\mathrm{Re}\left(\frac{\gamma k w(z_0)}{1+w(z_0)}\right)$$

$$\begin{aligned}
&= \frac{\alpha}{2}-\frac{\beta k}{2}-\mathrm{Re}\left(\frac{\gamma z_0 w''(z_0)}{w'(z_0)}\right)-\gamma k\\
&\leq \frac{\alpha}{2}-\frac{\beta k}{2}+\gamma(1-k)-\gamma k\\
&= \frac{\alpha}{2}-k\left(\frac{\beta}{2}+2\gamma\right)+\gamma\\
&\leq \frac{\alpha-\beta-2\gamma}{2} \quad (\alpha\in\mathrm{R},\ \beta+4\gamma\geq 0,\ \gamma\geq 0),
\end{aligned}$$

which contradicts the hypothesis (8) of the Theorem 2. Therefore there is no $z_0 \in \mathrm{U}$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in \mathrm{U}$. Thus we have that

$$h(z) \prec \frac{1}{1+z},$$

which shows that

$$\int_0^{2\pi}\left|f'\left(re^{i\theta}\right)\right|^{\mu}d\theta \leq \int_0^{2\pi}\left|h\left(re^{i\theta}\right)\right|^{\mu}d\theta.$$

This completes the proof.

Corollary 2. *Let the function $f(z)$ in $\mathcal{A}$ and the function $h(z)$ given by (3) satisfy the conditions in* Theorem 2. *Then, for $\mu > 0$ and $z = r^{i\theta}(0 < r < 1)$*

$$\int_0^{2\pi} \left|f'\left(re^{i\theta}\right)\right|^{\mu} d\theta \leq \frac{2\pi}{(1+r^2)^{\frac{\mu}{2}}} \left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+2j)}{2^{2n}(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n}\right\}. \tag{11}$$

Proof.

$$\begin{aligned}
\int_0^{2\pi} \left|f'\left(re^{i\theta}\right)\right|^{\mu} d\theta &\leq \int_0^{2\pi} \left|\frac{1}{1+re^{i\theta}}\right|^{\mu} d\theta \\
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \int_0^{2\pi} \left(1 + \frac{2r\cos\theta}{1+r^2}\right)^{\frac{-\mu}{2}} d\theta \\
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \int_0^{2\pi} \left\{\sum_{n=0}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(\frac{2r\cos\theta}{1+r^2}\right)^{n}\right\} d\theta
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \int_0^{2\pi} \left\{1 + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(\frac{2r\cos\theta}{1+r^2}\right)^{n}\right\} d\theta \\
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \int_0^{2\pi} \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(\frac{2r\cos\theta}{1+r^2}\right)^{n} d\theta\right\} \\
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{n} \left(\frac{2r}{1+r^2}\right)^{n} \int_0^{2\pi} \cos^n\theta\, d\theta\right\} \\
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{2n} \left(\frac{2r}{1+r^2}\right)^{2n} \int_0^{2\pi} \cos^{2n}\theta\, d\theta\right\} \\
&= \frac{1}{(1+r^2)^{\frac{\mu}{2}}} \left\{2\pi + \sum_{n=1}^{\infty} \binom{-\frac{\mu}{2}}{2n} \left(\frac{r}{1+r^2}\right)^{2n} \cdot 2^{2n} \cdot \frac{4(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}\right\} \\
&= \frac{2\pi}{(1+r^2)^{\frac{\mu}{2}}} \left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+2j)}{2^{2n}(2n)!} \left(\frac{r}{1+r^2}\right)^{2n} \cdot \frac{(2n)!}{(n!)^2}\right\} \\
&= \frac{2\pi}{(1+r^2)^{\frac{\mu}{2}}} \left\{1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(\mu+2j)}{2^{2n}(n!)^2} \left(\frac{r}{1+r^2}\right)^{2n}\right\}.
\end{aligned}$$

Putting $\mu = 1$ in Corollary 2, we have the following.

Corollary 3. *Let the function* $f(z)$ *in* $\mathcal{A}$ *and the function h(z) given by* (3) *satisfy the conditions in* Theorem 2. *Then, for* $0 < r < 1$

$$|f(z)| \leq \frac{2\pi}{(1+r^2)^{\frac{1}{2}}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{2n-1}(1+2j)}{2^{2n}(n!)^2} \left(\frac{r}{1+r^2} \right)^{2n} \right\}.$$

References

[1] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc., (2) **23** (1925), 481-519.

[2] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 289-305.

[3] S. Owa and T. Sekine, *Integral means for analytic functions*, J. Math. Anal. Appl., **304**(2005), 772-782.

[4] T. Sekine, S. Owa and R. Yamakawa, *Integral means of certain analytic functions*, General Math., **13**(2005),99-108.

[5] H. Silverman, *Integral means for univalent functions with negative coeff icients*, Houston J. Math., **23**(1997),169-174.

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