PRODUCTS OF MULTIALGEBRAS AND THEIR FUNDAMENTAL ALGEBRAS

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Abstract. An important tool in the hyperstructure theory is the fundamental relation. The factorization of a multialgebra modulo its fundamental relation provides a functor into the category of universal algebras. The question that lead us to the results we will present is whether this functor commutes with the products.

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1. Introduction

Multialgebras (also called hyperstructures) have been studied for more than sixty years and they are used in different areas of mathematics as well as in some applied sciences (see [2]). The results we present here refer to some categorical problems in hyperstructure theory and their complete proofs are given in [8]. The basic categorical notions we use can be found in [10].

The multialgebras of a given type determine a category for which the morphisms are the multialgebra homomorphisms. The direct product is the product in this category (see [7]). An important tool in hyperstructure theory is the fundamental relation of a multialgebra. From [6] it follows that the factorization of a multialgebra by the fundamental relation provides a covariant functor. The problem we investigate here is whether, or when this functor commutes with the products. In general, the fundamental algebra of a product of multialgebras is not necessarily (isomorphic to) the product of their fundamental algebras, but we found classes of multialgebras for which this property holds. Since the main tools of our investigation are the term functions of the universal algebra of the nonvoid subsets of a multialgebra, we considered two particular classes of multialgebras for which we know the form of these term functions: the class of hypergroups, and the class of the complete multialgebras (see [6]). This lead us to good results on complete hypergroups.

2. Preliminaries

Let $\tau = (n_\gamma)_{\gamma < o(\tau)}$ be a sequence of nonnegative integers, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation and let us consider the algebra of the n-ary terms (of type τ)

$$\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma < o(\tau)}).$$

Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$,

$$f_\gamma : A^{n_\gamma} \to P^*(A)$$

is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$.

If, for any $\gamma < o(\tau)$ and for any $A_0, \dots, A_{n_\gamma - 1} \in P^*(A)$, we define

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) = \bigcup \{ f_\gamma(a_0, \dots, a_{n_\gamma-1}) \mid \ a_i \in A_i, \ i \in \{0, \dots, n_\gamma - 1\}\},$$

we obtain a universal algebra on $P^*(A)$ (see [9]). We denote this algebra by $\mathfrak{P}^*(A)$. As in [4], we can construct, for any $n \in \mathbb{N}$, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(A))$ of the n-ary term functions on $\mathfrak{P}^*(A)$.

Let $\mathfrak{A}$ be a multialgebra and let ρ be an equivalence relation on its support set A. We obtain, as in [3], a multialgebra on A/ρ by defining the multioperations in the factor multialgebra $\mathfrak{A}/\rho$ as follows: for any $\gamma < o(\tau)$,

$$f_\gamma(\rho\langle a_0\rangle, \dots, \rho\langle a_{n_\gamma-1}\rangle) = \{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \dots, b_{n_\gamma-1}),\, a_i \rho b_i,\, i = 0, \dots, n_\gamma - 1\}$$

($\rho\langle x\rangle$ denotes the class of x modulo ρ).

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma-1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})). \tag{1}$$

As in [11] we can see a multialgebra $\mathfrak{A}$ as a relational system $(A, (r_\gamma)_{\gamma < o(\tau)})$ if we consider that, for any $\gamma < o(\tau)$, r_γ is the $n_\gamma + 1$-ary relation defined by

$$(a_0, \dots, a_{n_\gamma-1}, a_{n_\gamma}) \in r_\gamma \ \Leftrightarrow \ a_{n_\gamma} \in f_\gamma(a_0, \dots, a_{n_\gamma-1}). \tag{2}$$

Thus, the definition of the multialgebra homomorphism follows from the definition of the homomorphism among relational systems.

A bijective mapping h is a multialgebra isomorphism if both h and h^{-1} are multialgebra homomorphisms. As it results from [9], the multialgebra isomorphisms can be characterized as being those bijective homomorphisms h for which we have equality in (1).

Remark 1 *From the steps of construction of a term (function) it follows that for a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \ldots, a_{n-1} \in A$ then*

$$h(p(a_0, \ldots, a_{n-1})) \subseteq p(h(a_0), \ldots, h(a_{n-1})).$$

The definition of the multioperations of $\mathfrak{A}/\rho$ allows us to see the canonical mapping from A to A/ρ as a multialgebra homomorphism.

The fundamental relation of the multialgebra $\mathfrak{A}$ is the transitive closure $\alpha^* = \alpha^*_{\mathfrak{A}}$ of the relation $\alpha = \alpha_{\mathfrak{A}}$ given on A as follows: for $x, y \in A$, $x\alpha y$ if and only if

$$\exists n \in \mathbb{N},\ \exists \mathbf{p} \in \mathbf{P}^{(n)}(\tau),\ \exists a_0, \ldots, a_{n-1} \in A:\ x, y \in p(a_0, \ldots, a_{n-1}) \tag{3}$$

($p \in P^{(n)}(\mathfrak{P}^*(A))$ denotes the term function induced by $\mathbf{p}$ on $\mathfrak{P}^*(A)$). The relation α^* is the smallest equivalence relation on A with the property that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra (see [5] and [6]). The universal algebra $\overline{\mathfrak{A}} = \mathfrak{A}/\alpha^*$ is called the fundamental algebra of $\mathfrak{A}$. We denote by φ_A the canonical projection of $\mathfrak{A}$ onto $\overline{\mathfrak{A}}$ and by $\overline{a}$ the class $\alpha^*\langle a\rangle = \varphi_A(a)$ of an element $a \in A$.

In [6] we proved the following theorem:

Theorem 1 *If $\mathfrak{A}$, $\mathfrak{B}$ are multialgebras and $\overline{\mathfrak{A}}$, $\overline{\mathfrak{B}}$ respectively, are their fundamental algebras and if $f\colon A \to B$ is a homomorphism then there exists only one homomorphism of universal algebras $\overline{f}\colon \overline{A} \to \overline{B}$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"] & B \arrow[d, "\varphi_B"] \\ \overline{A} \arrow[r, "\overline{f}"] & \overline{B} \end{tikzcd} \tag{4}$$

(φ_A and φ_B denote the canonical projections).

Corollary 1 *a) If $\mathfrak{A}$ is a multialgebra then $\overline{1_A} = 1_{\overline{A}}$.*
b) If $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are multialgebras of type τ and if $f\colon A \to B$, $g\colon B \to C$ are homomorphisms, then $\overline{g \circ f} = \overline{g} \circ \overline{f}$.

The multialgebras of the same type τ, the multialgebra homomorphisms and the usual mapping composition form a category denoted here by $\mathbf{Malg}(\tau)$. Clearly, the universal algebras of the same type τ together with the homomorphisms between them form a full subcategory in the above category (denoted $\mathbf{Alg}(\tau)$).

Remark 2 *From Corollary 1 it results a functor*

$$F\colon \mathbf{Malg}(\tau) \longrightarrow \mathbf{Alg}(\tau)$$

defined as follows: for any multialgebra $\mathfrak{A}$ *of type* τ,

$$F(\mathfrak{A}) = \overline{\mathfrak{A}},$$

and, using the notations from diagram (4),

$$F(f) = \overline{f}.$$

Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. Using the model offered by [4] and the definitions of the hyperstructures from [1] and of the generalizations presented in [13], named H_v-structures, we say that the n-ary (strong) identity

$$\mathbf{q} = \mathbf{r}$$

is satisfied on the multialgebra $\mathfrak{A}$ of type τ if

$$q(a_0, \ldots, a_{n-1}) = r(a_0, \ldots, a_{n-1}),\ \forall a_0, \ldots, a_{n-1} \in A,$$

(q and r are the term functions induced by $\mathbf{q}$ and $\mathbf{r}$ respectively on $\mathfrak{P}^*(A)$). We also say that a weak identity

$$\mathbf{q} \cap \mathbf{r} \neq \emptyset$$

is satisfied on a multialgebra $\mathfrak{A}$ of type τ if

$$q(a_0, \ldots, a_{n-1}) \cap r(a_0, \ldots, a_{n-1}) \neq \emptyset,\ \forall a_0, \ldots, a_{n-1} \in A.$$

Many important particular multialgebras can be defined by using identities.

Example 1 *A hypergroupoid* $(H, \circ)$ *is a semihypergroup if the identity*

$$(\mathbf{x}_0 \circ \mathbf{x}_1) \circ \mathbf{x}_2 = \mathbf{x}_0 \circ (\mathbf{x}_1 \circ \mathbf{x}_2) \tag{5}$$

is satisfied on $(H, \circ)$.

Remark 3 *Let H be a nonempty set. A hypergroup $(H,\circ)$ is a semihypergroup which satisfies the reproductive law:*

$$a\circ H = H\circ a = H,\ \forall a\in H.$$

It results that the mappings $/,\backslash\colon H\times H\to P^*(H)$ defined by

$$a/b=\{x\in H\mid a\in x\circ b\},\ b\backslash a=\{x\in H\mid a\in b\circ x\}$$

are two binary multioperations on H. Thus, as we have seen in [6], the hypergroups can be identified with those multialgebras $(H,\circ,/,\backslash)$ for which $H\neq\emptyset$, $\circ$ is associative and the multioperations $/,\backslash$ are obtained from $\circ$ using the above equalities. It results that a semihypergroup $(H,\circ)$ (with $H\neq\emptyset$) is a hypergroup if and only if there exist two binary multioperations $/,\backslash$ on H such that the following weak identities:

$$\mathbf{x}_1\cap\mathbf{x}_0\circ(\mathbf{x}_0\backslash\mathbf{x}_1)\neq\emptyset,\ \mathbf{x}_1\cap(\mathbf{x}_1/\mathbf{x}_0)\circ\mathbf{x}_0\neq\emptyset,$$

$$\mathbf{x}_1\cap\mathbf{x}_0\backslash(\mathbf{x}_0\circ\mathbf{x}_1)\neq\emptyset,\ \mathbf{x}_1\cap(\mathbf{x}_1\circ\mathbf{x}_0)/\mathbf{x}_0\neq\emptyset$$

are satisfied on $(H,\circ,/,\backslash)$ (see again [6]).

If we replace above (5) *by*

$$(\mathbf{x}_0\circ\mathbf{x}_1)\circ\mathbf{x}_2\cap\mathbf{x}_0\circ(\mathbf{x}_1\circ\mathbf{x}_2)\neq\emptyset, \tag{5'}$$

we obtain the class of the H_v-groups (see [12]).

A mapping $h:H\to H'$ between two hypergroups is called hypergroup homomorphism if

$$h(a\circ b)\subseteq h(a)\circ h(b),\ \forall a,b\in H.$$

Clearly, h is a homomorphism between $(H,\circ,/,\backslash)$ and $(H',\circ,/,\backslash)$ since

$$h(a/b)\subseteq h(a)/h(b),\ h(a\backslash b)\subseteq h(a)\backslash h(b),\ \forall a,b\in H.$$

Remark 4 *Any (weak or strong) identity satisfied on a multialgebra $\mathfrak{A}$ is satisfied (in a strong manner) in $\overline{\mathfrak{A}}$ (see [6]). So, the fundamental algebra of a hypergroup or of a H_v-group is a group.*

Remark 5 *Since the hypergroups and the hypergroup homomorphisms form a category it follows immediately that the mappings from Remark 2 define a functor F from the category* **HG** *of hypergroups into the category of groups* **Grp**.

In [6] we introduced a new class of multialgebras which generalize the notion of complete hypergroup that appears in [1] and that is why we suggested we should name them complete multialgebras.

Proposition 1 *[6] Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra of type τ. The following conditions are equivalent:*

(i) for all $\gamma < o(\tau)$, for all $a_0, \dots, a_{n_\gamma-1} \in A$,

$$a \in f_\gamma(a_0, \dots, a_{n_\gamma-1}) \Rightarrow \overline{a} = f_\gamma(a_0, \dots, a_{n_\gamma-1}).$$

(ii) for all $m \in \mathbb{N}$, for all $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, m-1\}\}$, for all $a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1} \in A$,

$$q(a_0, \dots, a_{m-1}) \cap r(b_0, \dots, b_{m-1}) \neq \emptyset \Rightarrow q(a_0, \dots, a_{m-1}) = r(b_0, \dots, b_{m-1}).$$

Remark 6 *Since for any $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $m \in \mathbb{N}$, $m \geq n$ there exists $q \in P^{(m)}(\mathfrak{P}^*(\mathfrak{A}))$ such that*

$$p(A_0, \dots, A_{n-1}) = q(A_0, \dots, A_{m-1}),\ \forall A_0, \dots, A_{m-1} \in P^*(A),$$

it follows that the arities of $\mathbf{q}$ *and* $\mathbf{r}$ *from condition (ii) need not be equal.*

Definition 1 *A multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ of type τ is complete if it satisfies one of the two equivalent conditions from* **Proposition 1**.

Remark 7 *[6] We notice that if $\mathfrak{A}$ is a complete multialgebra, then the relation $\alpha_{\mathfrak{A}}$ given by (3) is transitive (so $\alpha_{\mathfrak{A}}^* = \alpha_{\mathfrak{A}}$).*

Remark 8 *The complete multialgebras of type τ form a subcategory* $\mathbf{CMalg}(\tau)$ *of* $\mathbf{Malg}(\tau)$. *So, if we compose F from Remark 2 with the inclusion functor we get a functor (which we will denote by F, too) from* $\mathbf{CMalg}(\tau)$ *into* $\mathbf{Alg}(\tau)$.

Let $(\mathfrak{A}_i = (A_i, (r_\gamma)_{\gamma<o(\tau)}) \mid i \in I)$ be a family of relational systems of type $\tau = (n_\gamma + 1)_{\gamma<o(\tau)}$. In [4] is defined the direct product of this family as being the relational system obtained on the Cartesian product $\prod_{i\in I} A_i$ considering that for $(a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma})_{i\in I} \in \prod_{i\in I} A_i$,

$$((a_i^0)_{i\in I}, \dots, (a_i^{n_\gamma})_{i\in I}) \in r_\gamma \Leftrightarrow (a_i^0, \dots, a_i^{n_\gamma}) \in r_\gamma,\ \forall i \in I.$$

Let $(\mathfrak{A}_i \mid i \in I)$ be a family of multialgebras of type τ and consider the relational systems defined by (2). The relational system obtained on the Cartesian product $\prod_{i\in I} A_i$ as in the above considerations is a multialgebra of type τ with the multioperations:

$$f_\gamma((a_i^0)_{i\in I},\dots,(a_i^{n_\gamma-1})_{i\in I}) = \prod_{i\in I} f_\gamma(a_i^0,\dots,a_i^{n_\gamma-1}),$$

for any $\gamma < o(\tau)$. This multialgebra is called the direct product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$. We observe that the canonical projections of the product, e_i^I, $i \in I$, are multialgebra homomorphisms.

Proposition 2 *[7] The multialgebra $\prod_{i\in I} \mathfrak{A}_i$ constructed this way, together with the canonical projections, is the product of the multialgebras $(\mathfrak{A}_i \mid i \in I)$ in the category* $\mathbf{Malg}(\tau)$.

Lemma 1 *[7] If $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $(a_i^0)_{i\in I},\dots,(a_i^{n-1})_{i\in I} \in \prod_{i\in I} A_i$, then*

$$p((a_i^0)_{i\in I},\dots,(a_i^{n-1})_{i\in I}) = \prod_{i\in I} p(a_i^0,\dots,a_i^{n-1}). \tag{6}$$

Proposition 3 *[7] If $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras such that the weak identity $\mathbf{q}\cap\mathbf{r} \neq \emptyset$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q}\cap\mathbf{r} \neq \emptyset$ is also satisfied on the multialgebra $\prod_{i\in I} \mathfrak{A}_i$.*

Proposition 4 *[7] If $(\mathfrak{A}_i \mid i \in I)$ is a family of multialgebras such that $\mathbf{q} = \mathbf{r}$ is satisfied on each multialgebra $\mathfrak{A}_i$ then $\mathbf{q} = \mathbf{r}$ is also satisfied on the multialgebra $\prod_{i\in I} \mathfrak{A}_i$.*

From Example 3, Proposition 3 and Proposition 4 we have the following:

Corollary 2 *The subcategory* $\mathbf{HG}$ *of* $\mathbf{Malg}((2,2,2))$ *is closed under products.*

Corollary 3 *[8] The direct product of complete multialgebras is a complete multialgebra.*

Corollary 4 *The subcategory* $\mathbf{CMalg}(\tau)$ *of* $\mathbf{Malg}(\tau)$ *is closed under products.*

3. On the fundamental algebras of a direct product of multialgebras

Let us consider the universal algebra $\prod_{i\in I}\overline{\mathfrak{A}_i}$ and its canonical projections

$$p_i\colon \prod_{i\in I}\overline{A_i}\to\overline{A_i}\ (i\in I).$$

There exists a unique homomorphism φ of universal algebras such that the following diagram is commutative:

$$\begin{tikzcd} \prod_{i\in I}\overline{\mathfrak{A}_i} \arrow[r, "p_j"] & \overline{\mathfrak{A}_j} \\ \overline{\prod_{i\in I}\mathfrak{A}_i} \arrow[u, "\varphi"] \arrow[ur, "\overline{e^I_j}"'] & \end{tikzcd}.$$

This homomorphism is given by

$$\varphi(\overline{(a_i)_{i\in I}})=(\overline{a_i})_{i\in I},\ \forall(a_i)_{i\in I}\in\prod_{i\in I}A_i.$$

It is clear that φ is surjective, so the universal algebra $\overline{\prod_{i\in I}\mathfrak{A}_i}$, with the homomorphisms $(\overline{e^I_i}\mid i\in I)$ is the product of the family $(\overline{\mathfrak{A}_i}\mid i\in I)$ if and only if φ is also injective.

But this does not always happen, as it results from the following example.

Example 2 *[8, Example 2] Let us consider the hypergroupoids $(H_1,\ \circ)$ and $(H_2,\ \circ)$ on the three elements sets H_1 and H_2 given by the following tables:*

H_1	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a

H_2	x	y	z
x	x	$y,\ z$	$y,\ z$
y	$y,\ z$	$y,\ z$	$y,\ z$
z	$y,\ z$	$y,\ z$	$y,\ z$

then in $\overline{H_1}\times\overline{H_2}$, $(\overline{b},\overline{y})=(\overline{b},\overline{z})$ but in $\overline{H_1\times H_2}$ the supposition $\overline{(b,y)}=\overline{(b,z)}$ leads us to the fact that $y=z$, which is false.

We will use the above notations and we will search for necessary and sufficient conditions expressed with the aid of term functions for φ to be injective. We will deal only with the cases when I is finite or $\alpha_{\mathfrak{A}_i}=\alpha^*_{\mathfrak{A}_i}$ for all $i\in I$ (even if I is not finite).

Lemma 2 *[8, Lemma 2] If I is finite or $\alpha_{\mathfrak{A}_i}$ is transitive for any $i \in I$, then the homomorphism φ is injective if and only if for any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \ldots, a_i^{n_i-1} \in A_i$ $(i \in I)$ and for any*

$$(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} q_i(a_i^0, \ldots, a_i^{n_i-1})$$

there exist $m, k_j \in \mathbb{N}$, $\mathbf{q}^j \in \mathbf{P}^{(k_j)}(\tau)$ and

$$(b_i^0)_{i\in I}^j, \ldots, (b_i^{k_j-1})_{i\in I}^j \in \prod_{i\in I} A_i \ (j \in \{0, \ldots, m-1\})$$

such that

$$(x_i)_{i\in I} \in q^0((b_i^0)_{i\in I}^0, \ldots, (b_i^{k_0-1})_{i\in I}^0),\ (y_i)_{i\in I} \in q^{m-1}((b_i^0)_{i\in I}^{m-1}, \ldots, (b_i^{k_{m-1}-1})_{i\in I}^{m-1})$$

and

$$q^{j-1}((b_i^0)_{i\in I}^{j-1}, \ldots, (b_i^{k_{j-1}-1})_{i\in I}^{j-1}) \cap q^j((b_i^0)_{i\in I}^j, \ldots, (b_i^{k_j-1})_{i\in I}^j) \neq \emptyset, \qquad (7)$$

for all $j \in \{1, \ldots, m-1\}$.

It seems uncomfortable to work with the condition from the above statement, but we can deduce a sufficient condition which will prove to be useful in the next part of our paper. Of course, we use the same notations and the same hypothesis as before.

Corollary 5 *[8, Corollary 5] The (complicated) condition from the previous lemma is verified if there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \ldots, b_i^{n-1} \in A_i$ $(i \in I)$ such that*

$$\prod_{i\in I} q_i(a_i^0, \ldots, a_i^{n_i-1}) \subseteq q((b_i^0)_{i\in I}, \ldots, (b_i^{n-1})_{i\in I}). \qquad (8)$$

Let us take a subcategory $\mathcal{C}$ of $\mathbf{Malg}(\tau)$ and the functor $F \circ U$ obtained as the composition of the functor F introduced in Remark 2 with the inclusion functor $U : \mathcal{C} \longrightarrow \mathbf{MAlg}(\tau)$. Since we know how U is defined, we will refer to $F \circ U$ as F. It follows immediately the following statements:

Proposition 5 *[8, Proposition 5] Let $\mathcal{C}$ be a subcategory of* $\mathbf{Malg}(\tau)$ *closed under finite products and let us consider that for any finite set I, for any family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras from $\mathcal{C}$ and any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that (8) holds. Then the functor $F\colon \mathcal{C} \longrightarrow \mathbf{Alg}(\tau)$ commutes with the finite products.*

Proposition 6 *[8, Proposition 6] Let $\mathcal{C}$ be a subcategory of* $\mathbf{Malg}(\tau)$ *closed under products and let us consider that $\alpha_{\mathfrak{A}}$ is transitive for each $\mathfrak{A} \in \mathcal{C}$. If for any set I, any family $(\mathfrak{A}_i \mid i \in I)$ of multialgebras from $\mathcal{C}$ and any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \dots, a_i^{n_i-1} \in A_i$ $(i \in I)$ there exist $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ and $b_i^0, \dots, b_i^{n-1} \in A_i$ $(i \in I)$ such that (8) holds, then the functor $F : \mathcal{C} \longrightarrow \mathbf{Alg}(\tau)$ commutes with the products.*

In the following we will present the results of the investigation we started for two particular classes of multialgebras for which the relation α defined by (3) is transitive: the class of hypergroups and the class of complete multialgebras.

The case of hypergroups.

What happens with finite products of hypergroups? We remind that the fundamental relation on a hypergroup $(H, \circ, /, \backslash)$ is the transitive closure of the relation $\beta = \bigcup_{n \in \mathbb{N}^*} \beta_n$ where for any $x, y \in H$,

$$x\beta_n y \text{ if and only if there exist } a_0, \dots, a_{n-1} \in H, \text{ with } x, y \in a_0 \circ \cdots \circ a_{n-1}.$$

The relation β is transitive, so $\beta^* = \beta$ (see [1]). As we can easily see, the term functions q_i which interest us are only those which are involved in the definition of the fundamental relations of the multialgebras $\mathfrak{A}_i$. As it results immediately, in the case of hypergroups, these term functions can be obtained from the canonical projections using only the hyperproduct $\circ$, and from the property of reproducibility of a hypergroup follows that any hyperproduct with n factors is a subset of a hyperproduct with $n+1$ factors. So, $\beta_n \subseteq \beta_{n+1}$ for any $n \in \mathbb{N}^*$. It means that for any two hypergroups $(H_0, \circ)$, $(H_1, \circ)$ we can apply Corollary 5 and it follows that $\overline{H_0 \times H_1}$, together with the homomorphisms $\overline{e_0^2}$, $\overline{e_1^2}$, is the product of the groups $\overline{H_0}$ and $\overline{H_1}$. Thus we have proved the following:

Proposition 7 *[8, Proposition 7] The functor* $F\colon \mathbf{HG} \longrightarrow \mathbf{Grp}$ *commutes with the finite products of hypergroups.*

Yet, F does not commute with the arbitrary products of hypergroups, as it follows from the next eample:

Example 3 *[8, Example 3] Let us consider the hypergroupoid* $(\mathbb{Z}, \circ)$, *where* $\mathbb{Z}$ *is the set of the integers and for any* $x, y \in \mathbb{Z}$, $x \circ y = \{x+y, x+y+1\}$. *It results immediately that* $(\mathbb{Z}, \circ)$ *is a hypergroup with the fundamental relation* $\beta = \mathbb{Z} \times \mathbb{Z}$. *It means that the fundamental group of* $(\mathbb{Z}, \circ)$ *is a one-element group. Now let us consider the product* $(\mathbb{Z}^{\mathbb{N}}, \circ)$. *The fundamental group of this hypergroup has more than one element. Indeed,*

$$f, g\colon \mathbb{N} \to \mathbb{Z},\ f(n) = 0,\ g(n) = n+1\ (n \in \mathbb{N})$$

are not in the same equivalence class of the fundamental relation of the hypergroup $(\mathbb{Z}^{\mathbb{N}}, \circ)$.

As for arbitrary products (not necessarily finite) of hypergroups we have:

Theorem 2 *[8, Theorem 2] Let* I *be a set and consider the hypergroups* H_i $(i \in I)$ *with the fundamental relations* β^{H_i}. *The group* $\overline{\prod_{i\in I} H_i}$, *with the homomorphisms* $(\overline{e_i^I} \mid i \in I)$, *is the product of the family of groups* $(\overline{H_i} \mid i \in I)$ *if and only if there exists* $n \in \mathbb{N}^*$ *such that* $\beta^{H_i} \subseteq \beta_n^{H_i}$, *for all the elements* i *from* I, *except for a finite number of* i*'s.*

The case of complete multialgebras.

It is known that for a complete multialgebra $\mathfrak{A}$ the classes from $\overline{A}$ have the form $\{a\}$ or $f_\gamma(a_0, \ldots, a_{n_\gamma - 1})$, with $\gamma < o(\tau)$, $a, a_0, \ldots, a_{n_\gamma-1} \in A$ (situations which not exclude each other). Using this the following can be proved:

Theorem 3 *[8, Theorem 3] For a family* $(\mathfrak{A}_i \mid i \in I)$ *of complete multialgebras of the same type* τ, *the following statements are equivalent:*

i) $\overline{\prod_{i\in I} \mathfrak{A}_i}$ *(together with the homomorphisms* $\overline{e_i^I}$ $(i \in I)$*) is the product of the family of the universal algebras* $(\overline{\mathfrak{A}_i} \mid i \in I)$;

ii) For any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \ldots, a_i^{n_i-1} \in A_i$, $(i \in I)$ *there exist* $n \in \mathbb{N}$, $\mathbf{q} \in \mathbf{P}^{(n)}(\tau)$ *and* $b_i^0, \ldots, b_i^{n-1} \in A_i$ $(i \in I)$ *such that (8) holds;*

iii) For any $n_i \in \mathbb{N}$, $\mathbf{q}_i \in \mathbf{P}^{(n_i)}(\tau)$, $a_i^0, \ldots, a_i^{n_i-1} \in A_i$ $(i \in I)$ *either*

$$\left| \prod_{i \in I} q_i(a_i^0, \ldots, a_i^{n_i-1}) \right| = 1$$

or there exist $\gamma < o(\tau)$, $b_i^0, \ldots, b_i^{n_\gamma-1} \in A_i$ $(i \in I)$ *such that*

$$\prod_{i \in I} q_i(a_i^0, \ldots, a_i^{n_i-1}) \subseteq f_\gamma((b_i^0)_{i \in I}, \ldots, (b_i^{n_\gamma-1})_{i \in I}). \tag{9}$$

Remark 9 *If all* $\mathfrak{A}_i$ *are universal algebras then* $iii)$ *is trivially satisfied.*

Remark 10 *For a family of complete multialgebras* $(\mathfrak{A}_i \mid i \in I)$ *the following conditions are equivalent:*

a) there exist $n \in \mathbb{N}$ *and* $\mathbf{p} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, .., n-1\}\}$ *such that for each* $i \in I$ *and for any* $a_i \in A_i$ *we have* $a_i \in p(a_i^0, \ldots, a_i^{n-1})$ *for some* $a_i^0, \ldots, a_i^{n-1} \in A_i$;

b) there exists a $\gamma < o(\tau)$ *such that for each* $i \in I$ *and for any* $a_i \in A_i$ *we have* $a_i \in f_\gamma(a_i^0, \ldots, a_i^{n_\gamma-1})$ *for some* $a_i^0, \ldots, a_i^{n_\gamma-1} \in A_i$.

Corollary 6 *[8, Corollary 8] If for a family of complete multialgebras one of the equivalent conditions* $a)$ *or* $b)$ *is satisfied, then the condition* $i)$ *from the previous theorem holds.*

Remark 11 *The condition a), respectively b) from above are not necessary for* $i)$ *to be satisfied, and the exception is not covered by the case when all* $\mathfrak{A}_i$ *are universal algebras.*

Example 4 *[8, Example 4] Let us consider the multialgebras* $\mathfrak{A}_0$ *and* $\mathfrak{A}_1$, *of the same type (2,3,4) obtained on the sets* $A_0 = \{1, 2, 3\}$ *and* $A_1 = \{1, 2, 3, 4\}$ *as follows:*

$$\mathfrak{A}_0 = (A, f_0^0, f_1^0, f_2^0),\ \mathfrak{A}_1 = (A, f_0^1, f_1^1, f_2^1),$$

where $f_j^i\colon A_i^{j+2} \to P^*(A_i),\ i = 0, 1,\ j = 0, 1, 2,$

$$f_0^0(x, y) = \{1\},\ f_1^0(x, y, z) = \{2, 3\},\ f_2^0(x, y, z, t) = \{2, 3\},$$

$$f_0^1(x, y) = \{1, 2, 3\},\ f_1^1(x, y, z) = \{4\},\ f_2^1(x, y, z, t) = \{1, 2, 3\}.$$

These complete multialgebras satisfy condition $iii)$ *and, consequently, the condition* $i)$, *but they do not verify condition b).*

Either using the fact that for the complete hypergroups we have $\beta = \beta_2$ and Theorem 2 or using Corollary 6 we obtain the following result:

Corollary 7 *The functor F commutes with the products of complete hypergroups.*

Remark 12 *The complete semihypergroups from [1, Proposition 346] satisfy the condition b) from Remark 10.*

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