SOME REMARKS ON THE ENDS OF GROUPS

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Abstract.This note in an expanded version of the talk I gave at the ICTAMI 2007. After a brief introduction on the topics of the geometric group theory and on the theory of ends, we will study a refinement of the one-endness condition for groups, introducing a function whose growth distinguish spaces which are one-ended in a trivial way (e.g. Gromov-hyperbolic or CAT(0) spaces) from spaces which are one-ended but "looks" like infinite-ended spaces.

2000 Mathematics Subject Classification: 20 F 32.

Keywords and Phrases: Ends, rate of vanishing of the simple connectivity at infinity, dead-end elements, quasi-isometries.

1.Definitions and basic properties

This section is devoted to the introduction of the basic tools of the Geometric Group Theory. We would like to explain how and why it can be useful to do geometry with groups. All groups referred to in this paper will be infinite, unless the contrary will be stated. The reason is that we will consider universal coverings spaces of compact polyhedra with a given fundamental group.

The main references for this section are [5], [3]and [8].

1.1.Cayley graphs and quasi-isometries

The fundamental notions that must be understood if one is to be comfortable with geometric group theory are group presentations, Cayley's graphs and quasi-isometries.

A group presentation has the form $G =< C|R >$. In this presentation the set C is a finite set of letters called the *generating set* for the presentation. The set C has the property that each letter $c \in C$ is paired with another letter c^{-1} called the inverse letter. We will always assume that $C = C^{-1}$. A word in the alphabet C is a finite sequence of elements of C. The set R is a collection of words in the alphabet C and is called the *defining relator set* for the presentation. The symbol G refers to the group we will define below. Let W denote the set of all words in the generating set C. Let T denote the set of all trivial words, that is words of the form cc^{-1}, where $c \in C$. Declare two words w and w' to be equivalents if w' can be obtained from w by either inserting or deleting at some point a copy of a word from $R \cup T$. Extend this notion of equivalence to an equivalence relation $\sim$. Let $G = W/\sim$ denote the set of equivalence classes, and define the product of two such classes $\overline{a}$ and $\overline{b}$ to be the class $\overline{ab}$ represented by the concatenation ab of a and b. It is easy to check that G with this well-defined multiplication is a group, that we will write $G =< C|R >$. The group G is *finitely presented* if C and R are finite. (We will always assume that $C = C^{-1}$ is finite, i.e. G will be always *finitely generated*).

An easy example of a presentation is represented by the fundamental group of a connected simplicial complex. Indeed, assume that X be such a space, triangulated, with base point x_0 a vertex of the triangulation. Collapse a maximal subtree of the 1-skeleton to the base point. Then the image of the 1-skeleton of X is a bouquet of loops, each loop supplying a generator for the fundamental group. Each 2-cell is attached along its boundary to the bouquet by an attaching map which can be realized as a word in the generators and their inverses. These words supply the defining relators of a presentation. If the complex is finite, then the presentation is finite.

Finite presentation of groups arise naturally in a wide range of mathematical contexts (e.g. surface groups, 3-manifold groups, Coxeter groups, Artin groups, co-compact lattices in Lie groups, etc.).

As already said, we would like to do geometry. Actually, we would like to attach to any group a "good" space, that we will call a *geometry*. For our aim, a geometry is a topological space endowed with a proper *path metric* (this is one of the underlying ideas on Gromov approach [7]), namely a metric such that the distance between each pair of points is realized as the length of some path in the space joining these points; the metric is *proper* if closed metric balls of finite radius are compact. The typical example is a complete

Riemannian manifold (by Hopf-Rinow theorem).
Let X be a geometry and G a group acting on X. The action is said to be:

- *isometric* if for each $g \in G$ and for any $x, y \in X$, $d(gx, gy) = d(x, y)$,
- *cocompact* if the orbit space X/G is compact,
- *properly discontinuous* if the set $\{g \in G : K \cap gK \neq \emptyset\}$ is finite for any K compact,
- *geometric* whenever all these three properties hold.

Now, let G be a finitely generated group, with neutral element denoted by e. Let S be a finite generating set (for simplicity, we will always assume that $e \notin S$ and that $S = S^{-1}$). Define the *length* $l_S(g)$ of any element of G to be the smallest integer n such that there exists a sequence $(s_1, s_2, \cdots, s_n)$ of generators in S for which $g = s_1 s_2 \dots s_n$. Then we can define the distance d_S by $d_S(a, b) = l_S(a^{-1}b)$. This distance makes G a metric space, and d_S is called the *word metric* with respect to the generating set S. Since d_S takes integral values, the space (G, d_S) is discrete, and this may impede geometric understanding. Actually, we want find a "natural" geometry on which a finitely generated group acts. This space is the *Cayley graph* (see [5] for an extensive discussion).

Definition 1.1. *Let G be a finitely generated group. The Cayley graph $\mathcal{C}(G, S)$ of G with respect to a finite generating set S is the graph whose vertices are the elements of G and two vertices g_1, g_2 are the two ends of an edge if and only if $d_S(g_1, g_2) = 1$ (or equivalently $g_1^{-1}g_2 \in S$).*

This graph is infinite whenever G is, and G acts naturally on it by left multiplication. Each edge of $\mathcal{C}(G, S)$ can be made a metric space isometric to the segment $[0, 1]$, in such a way the action becomes isometric. One define naturally the length of a path between two points of the graph and the distance between two points is defined as the infimum of the appropriate path-length. In this way we have associated to any infinite group a metric space on which the group acts geometrically.

Remark 1.1. Actually the Cayley graph is a geometry if and only if the generating set S is finite.

Examples.

- If $G = \mathbb{Z}$ generated by $\{+1, -1\}$, then the Cayley graph is isometric to the real line.
- The group $\mathbb{Z}^2$ generated by $\{(1,0),(0,1),(-1,0),(0,-1)\}$ has a Cayley graph isometric to the standard square grid in $\mathbb{R}^2$.
- The Cayley graph of the free group generated by the set $S = (\{a, b, a^{-1}, b^{-1}\}$ is a 4-valence tree (i.e. with 4 branches coming out from any vertex).

The alert reader will note that whenever one changes the generating system, the Cayley graph and the distance d_S can change a lot (but, actually, not so deeply...). Indeed, these definitions depend on S. However, if one stands far back, then two Cayley graphs of the same group looks alike, that is: in the *large-scale* they are the same. This motivates the following definition [7].

Definition 1.2. *The metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there are constants $\lambda > 0$,$C \geq 0$ and maps $f : X \to Y$ and $g : Y \to X$ (called (λ, C)-quasi-isometries) so that, for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, the following holds:*

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C$$

$$d_X(g(y_1), g(y_2)) \leq \lambda d_Y(y_1, y_2) + C$$

$$d_X(gf(x), x) \leq C$$

$$d_Y(fg(y), y) \leq C.$$

This relation is an equivalence relation between metric spaces. Observe also that the maps f and g are not necessarily continuous.
For example, the real line $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric: it suffices to check with the map f which takes a real number r to its integral part.
More generally, any (finitely generated) group G is quasi-isometric to its Cayley graph.

Proposition 1.1. *Let S and S' be two finite generating sets of the same group G, and let d_S and $d'_{S'}$ be the distances defined on G by S and S' respectively. Then (G, d_s) and $(G, d'_{S'})$ are quasi-isometrics.*

Proof. Let f be the identity map of G. Set $\lambda_1 = \max\{d'_{S'}(s,e) : s \in S\}$ and $\lambda_2 = \max\{d_S(s',e) : s' \in S'\}$. It is easy to check that $d'_{S'}(f(x), f(y)) \leq \lambda_1 d_S(x,y)$ and similarly, $d_S(f^{-1}(x), f^{-1}(y)) \leq \lambda_2 d'_{S'}(x,y)$. This ends the proof.

This proposition means that, on a finitely generated group, the *word metric* is unique up to quasi-isometry. It also follows that the Cayley graph associated to a (finitely-generated) group is a well-defined geometry up to quasi-isometry (hence, any quasi-isometry property of the Cayley graph can be viewed as a property of the group itself).

More generally, it is an exercise to prove that if a group G acts geometrically on two geometries X and Y, then X and Y are quasi-isometric (see [3]).

Remark 1.2.

- There exist different groups with isomorphic Cayley graphs. For examples, if A_n, B_n are groups of order n, then the Cayley graphs of the free product $A_p * B_k$ and $B_p * A_k$ are the same, though these groups are, in general, not isomorphic.

- A metric space if quasi-isometric to a point if and only if its diameter is finite. In particular, finite groups are all quasi-isometric to the trivial group.

The next theorem is the fundamental observation in geometric group theory. We sketch a proof, for details see [5].

Proposition 1.2. *Let X be a geometry and G be a group acting geometrically (i.e. cocompactly, isometrically and proper discontinuously). Then G is finitely generated and quasi-isometric to X.*

Proof. Let $\pi : X \to X/G$ be the canonical projection. The space X/G has a canonical metric defined by $d(p,q) = \inf\{d(x,y) : x \in \pi^{-1}(p) \text{ and } y \in \pi^{-1}(q)\}$. As X/G is compact, its diameter R is finite. Choose a base point x_0 in X and set $B = \{x \in X : d(x_0, x) \leq R\}$. Set $S = \{g \in G : g \neq e, \text{ and } gB \cap B \neq \emptyset\}$. Observe that $S = S^{-1}$ and that it is finite, since the action is proper discontinuous. Finally, set $r = \inf\{d(B, gB) : g \in G - (S \cup e)\}$. One can prove that S generates G and that $d_S(e,g) = r^{-1} d(x_0, gx_0) + 1$.

Consider now the map $f : G \to X$ sending an element g to the point gx_0. We have that $d_S(g_1, g_2) \leq r^{-1} d(f(g_1), f(g_2)) + 1$. Furthermore, since

$d(x_0, g_{x_0}) \leq \lambda d_S(e, g)$ for all $g \in G$, where $\lambda = \sup\{d(x_0, sx_0) : s \in S\}$, one has that $d_S(g_1, g_2) \leq \lambda d_S(g_1, g_2)$. To finish, we have that $d(f(G), x) \leq R$ for all $x \in X$ because $(gB)_{g \in G}$ covers X.

Corollary 1.1. *Let H be a finite index subgroup of a finitely generated group G. Then H is finitely generated and quasi-isometric to G.*

1.2. Geometric properties

A property (P) of finitely generated groups is said to be a *geometric* property if, whenever G_1 and G_2 are two quasi-isometric (finitely generated) groups, G_1 has property (P) if and only if G_2 has property (P). It is remarkable that there is an abundance of geometric properties, so that quasi-isometry is a very interesting relation between groups: though it ignores finite details, it preserves a lot of distinct properties; for example one can prove the following proposition [5].

Proposition 1.3. *Being of finite presentation is a geometric property.*

Other examples of geometric properties of groups are: being virtually free, virtually cyclic and virtually nilpotent (this is one of the deepest theorem by Gromov in Geometric Group Theory), and the number of ends (see below for a definition). Among properties that are *not* geometric, one has: being virtually solvable and virtually torsion-free.

1.3. Ends

Let X denote a locally compact, connected metric space. If K is a compact subset of X, then we say that a connected component C of the complementary of K in X is *unbounded* if its closure in X is noncompact. Denote by $e(X, K)$ the number of unbounded components of $X - K$. Then one can define the *number of ends* $e(X)$ of X to be the supremum over all K of the numbers $e(X, K)$ ([3]).

Examples.

- a compact metric space has 0 ends;

- the real line has 2 ends;
- all Euclidean spaces of dimension ≥ 2 are one-ended;
- a tree of valence $n \geq 3$ has infinitely many ends.

Let G be a finitely generated group. The number of ends of G is by definition the number of ends of its Cayley graph (it is easy to check that this number does not depend on the generating set). It takes some more time to show Hopf's result on the number of ends of a group, namely:

Theorem 1.1.[Hopf]. *A group has either* $0, 1, 2$ *or infinitely many ends.*

The groups with 0 ends are finite. On the other hand, there is a deep result of J. Stallings ([10]) on the structure of group with more then one end, which states:

Theorem 1.2.[Stallings]. *A finitely generated group G has exactly two ends if and only if G has an infinite cyclic finite index subgroup. A finitely generated group G has infinitely many ends if and only if G can be factored in one of the two following ways:*

- *G is a free product with finite amalgamating subgroup where this amalgamating subgroup is properly contained in both factors and of index > 2 in at least one factor;*
- *G is a HNN extension amalgamated over a finite subgroup which is properly embedded in base group.*

This theorem reduced the study of infinitely ended groups to that of one-ended groups. Hence, in order to "classify" groups (up to quasi-isometry), one has to look for invariants of one-ended groups in order to better understand the collection of one-ended groups.

2. Quasi-isometries and the end-depth

Definition 2.1. *A connected, locally compact, topological space X with $\pi_1 X = 0$ is simply connected at infinity (abbreviated s.c.i. and one writes also $\pi_1^\infty X = 0$) if for each compact $k \subseteq X$ there exists a larger compact $k \subseteq K \subseteq X$ such that any closed loop in $X - K$ is null homotopic in $X - k$.*

This notion was extended by Brick in [2] to a group-theoretical framework as follows:

Definition 2.2. *A group G is simply connected at infinity if for some (equivalently any) finite complex X such that $\pi_1 X = G$ one has $\pi_1^\infty \widetilde{X} = 0$, where $\widetilde{X}$ denotes the universal covering of X.*

Definition 2.3. *Let X be a simply connected non-compact metric space with $\pi_1^\infty X = 0$. The rate of vanishing of π_1^∞, denoted $V_X(r)$, is the infimal $N(r)$ with the property that any loop which sits outside the ball $B(N(r))$ of radius $N(r)$ bounds a 2-disk outside $B(r)$.*

Remark 2.1. It is easy to see that V_X can be an arbitrary large function.

It is customary to introduce the following equivalence relation on functions: $f \sim g$ if there exists constants c_i, C_j (with $c_1, c_2 > 0$) such that

$$c_1 f(c_2 R) + c_3 \leq g(R) \leq C_1 f(C_2 R) + C_3.$$

It is proved in [6] that the equivalence class of $V_X(r)$ is a quasi-isometry invariant. In particular $V_G = V_{\widetilde{X_G}}$ is a quasi-isometry invariant of the group G, where $\widetilde{X_G}$ is the universal covering space of a compact simplicial complex X_G, with $\pi_1(X_G) = G$ and $\pi_1^\infty(G) = 0$.
If V_G is defined and linear we say that G has linear s.c.i. The simple connectivity at infinity and its refinement (the π_1^∞ growth) are "1-dimensional" invariants at infinity for a group G, in the sense that they take care about loops and disks. The "0-dimensional" analogous of the simple connectivity (at infinity) is the connectivity at infinity, namely the condition to be one-ended. Hence, we can adapt the notion of the sci growth to a sort of growth of the end.

In order to measure the "kind" of connectivity at infinity of a one-ended metric space X, we introduce a functions measuring the "depth" of those connected components of $X - B(r)$ which are bounded, as $r \to \infty$.

Definition 2.4. *Let X be a one-ended metric space. Let $B(r)$ be the r-ball of X, centered at the identity. The end-depth of X (or the growth rate of the connectivity at infinity), denoted $V_0(X)$, is the infimal $N(r)$ with the property that any two points which sit outside the ball $B(N(r))$ can be joined by a path outside $B(r)$.*

Remark 2.2.

- For spaces which are k-connected at infinity, one can also consider the function $V_k(X) = \inf(N(r))$ such that any k-sphere out of $B(N(r))$ bounds a $(k+1)$-sphere outside $B(r)$.

- It is easy to see that these functions can have arbitrary large growth for metric spaces (which are not Cayley's complexes).

Now we study the function V_0 for groups. It is easy to see that the function itself depends on the presentation of a group, but the growth of the function (i.e. whether it is linear, polynomial or exponential) does not. Actually, the growth of the function V_0 (that we shall continue to call end-depth) is a geometric property of groups (following M.Gromov [7]). The aim of this note is to prove the following statement.

Proposition 2.1. *The growth rate of V_0 is a well-defined quasi-isometry invariant of finitely presented groups.*

Observe that the function V_0 depends on the presentation of a group. Let us recall a definition. An element g of a group G is a *dead-end element* with respect to a generating set S if it is not adjacent to an element further from the identity; that is, if a geodesic ray in the Cayley graph of (G, S) from the identity to g cannot be extended beyond g. The *dead-end depth*, with respect to S, of $g \in G$, is the distance in the word metric d_S between g and the complement in G of the closed ball B_g of radius $d_S(1, g)$ centered at 1. If $G - B_g$ is empty one defines the depth of g to be infinite. So $g \in G$ is a dead end when its depth is at least 1. The next lemma indicates a relation between our function V_0 and dead-ends elements.

Lemma 2.1. *Let $\mathcal{P}$ be a presentation for G. Let V_0 be the end-depth with respect to $\mathcal{P}$. If there exists r such that $V_0(r) > r$, then there exists a dead-element g in the sphere of radius $V_0(r)$ of depth $V_0(r) - r + 1$.*

Proof. Since $V_0(r) > r$, there exist $x \in B(V_0(r))$ and $y \in X - B(V_0(r))$ such that any path from x to y goes through $B(r)$. To any such a x one can associate a dead-end element in $B(V_0(r))$ in this way: if x is a dead-end element there is nothing to do, otherwise x is adjacent to some element x_1 further from the identity. One can do the same for x_1 and find x_2 and so on. This process that has to stop after a finite number of steps, since the norm of x_i cannot be greater then $V_0(r)$, otherwise the point x could be joined with y outside $B(r)$.

Thus the set $D_r \subset B(V_0(r))$, consisting in all the dead-ends elements constructed in such a way, is non-empty. Let $\bar{x}$ be the dead-end element in D_r with maximal norm. By construction of $\bar{x}$, any point further from the identity can be joined with y by a path outside $B(r)$. Hence the norm of $\bar{x}$

is exactly $V_0(r)$. Now, since any path joining $\bar{x}$ with a point out of $B(V_0(r))$ goes into $B(r)$, then the depth of $\bar{x}$ is $V_0(r) - r + 1$.

Notice that Cleary and Riley have constructed a group G such that G contains a sequence of dead-end elements g_n at distance $4n$ from the identity, and of depth n with respect to one presentation, while G with respect to another presentation $< S', R' >$ has dead-end depth bounded by 2 (see [4]). This implies that for a group, the property of having unbounded dead-end depth is not an invariant. Actually, they write that the behavior of the depth is hard to understand, for example the dead-end depth of $\mathbb{Z}$ is not uniformly bounded.

According to the previous equivalence relation on functions, we will say that G has a linear (or polynomial, exponential....) end-depth if (the class of) V_0 is. From now on, we will refer to V_0 (or end-depth) to indicate the equivalence class of the function V_0. Our aim is to show that the equivalent class of V_0 only depends on the group. Let G be a group and consider a compact complex X with G as fundamental group. We first observe that the end-depth of $\widetilde{X}$ only depends on its 2-skeleton (actually the 1-skeleton). We can then restrict to 2-dimensional complexes with a given fundamental group.

Lemma 2.2. *Suppose X is a finite two-complex. Let $\widetilde{X}$ be its universal covering. Suppose $\widetilde{X}$ one-ended. Let T be a maximal tree in the 1-skeleton $X^{(1)}$ of X and set $\widetilde{Y} = \widetilde{X}/\widetilde{T}$. Then the end-depth of $\widetilde{X}$ and $\widetilde{Y}$ are equivalent.*

Proof. Let C be the diameter of the (finite) maximal tree T of X. Let p the quotient map $p : \widetilde{X} \to \widetilde{Y}$. Denote V_X and V_Y the end depth of $\widetilde{X}$ and $\widetilde{Y}$. Let $B_Y(r)$ be the r-ball of $\widetilde{Y}$. We claim that two points a, b out of $B_Y(V_X(r+C)+C)$ can be joined by a path out of $B_Y(r)$. In fact, the inverse image of such two points $p^{-1}(a)$ and $p^{-1}(b)$ sit in $\widetilde{X} - B_X(V_X(r+C))$. Hence they can be joined by a path α of $\widetilde{X}$ out of $B_X(r + C)$. The image $p(\alpha)$ is still a path, joining a and b, which sits out of $B_Y(r)$, since p collapse a tree of diameter C. The reverse implication is the same.

Since finite 2-complexes with one vertex and with isomorphic fundamental groups are standard 2-complexes associated to two distinct presentations of the same group, for the end-depth to be a well-defined group property, we need to analyze how it change with respect to the presentation.

Proposition 2.2. *The end-depth of a group G does not change with the presentation.*

Proof. Let $\mathcal{P}$ and $\mathcal{L}$ two distinct (finite) presentations of the group G. One can pass from one to the other by an application of a sequence of Tietze transformations. Let us prove the result when $\mathcal{P}$ is gotten from $\mathcal{L}$ by applying a single transformation. Since the desired relation is clearly transitive, this suffices.

Consider the different transformations:

- (T_1): add a new relator r, which is a consequence of the existing relators;
- (T_2): the inverse of (T_1);
- (T_3): add a new generator y and a new relator of the form yu^{-1}, where u is an arbitrary word in the old generators;
- (T_4): the inverse of (T_3).

We want to prove that these transformations does not change the growth of the function V_0. Let $\widetilde{X}_1$ be the Cayley 2-complex associated to $\mathcal{L}$ and $\widetilde{X}_2$ that of $\mathcal{P}$.

For transformations of type (T_1) and (T_2) it is obvious. Any edge path joining two points needs not to use the 2-cells.

Consider now the transformations of type (T_3) or (T_4). It this case one needs to compare the metrics of $\widetilde{X_1}$ and $\widetilde{X_2}$. Let d_1 be the metric on $\widetilde{X_1}$ and d_2 be the metric of $\widetilde{X_2}$. Since the difference in the generating set is the presence of a new generator whose length in the other generating set is the norm of the word w, then we have the following inequalities: $d_1 \leq d_2 \leq \|w\| d_1$. Since $\|w\|$ is constant, the end-depth of $\widetilde{X_1}$ and $\widetilde{X_2}$ are equivalent.

An application of the previous two results yields the following corollaries:

Corollary 2.1. *Let K_1 and K_2 be finite connected complexes with isomorphic fundamental groups. Then $V_0(K_1)$ and $V_0(K_2)$ are equivalent.*

Corollary 2.2. *Let G be a finitely presented group and H a subgroup of finite index. Then $V_0(G) \approx V_0(H)$.*

Proof. The results follows immediately from the fact that G and H have the same universal covering.

Remark 2.3. This implies that the end-depth is a well-defined property of groups. Furthermore, the group of Cleary and Riley ([4]) belong to the class of groups with linear end-depth.

Another interesting property of the end-depth is its quasi-isometry invariance.

Proposition 2.3. *The end depth is a quasi-isometry invariant for groups.*

Proof. The strategy of the proof is as follows. Denote by X and Y the Cayley 2-complexes associated to two quasi-isometric groups G, H respectively. Since G and H are quasi-isometric, there is a (λ, c)-quasi-isometry $f : X \to Y$ with quasi-inverse f'. Let V_H be the end depth of H; in order to show that V_G is equivalent to V_H we use f to maps two points in X to two points in Y, we choose a path joining them in Y and map it back to X using the quasi-inverse f'; a suitable approximation to the resulting (non-continuous) map of an interval to X yields an arc in X joining our original points. Let $m = \max\{\lambda, c\}$, and $B_X(r)$ the ball of radius r in X. Let a, b be two points out of the ball $B_X(mV_H(mr + m) + 4m)$ such that $d(a, b) > m$ (otherwise they are obviously joined by a path outside $B_X(r)$). Thus the image points $f(a)$ and $f(b)$ are distinct and they sit outside the ball of radius $V_H(mr + m)$ (thanks to the property of f). Then there exists a edge-loop l of length L in Y joining $f(a)$ and $f(b)$ out of $B_Y(r)$. We can consider l as a map $l : [0, L] \to Y$. Now, we associate to any point $t \in [0, L]$ an element $h_t \in H$ such that either $l(t) = h_t$ or else h_t is a vertex of the edge in which $l(t)$ lies. Now define $\phi : \{0, 1, 2, \ldots L\} \to X$ such that $\phi(0) = a$, $\phi(1) = b$ and $\phi(n) = f'(h_n)$ for $n = 1, 2, \ldots, L - 1$. We claim that ϕ sends each pair of consecutive naturals to elements of G at distance at most $3m$. Hence, one can send each edge $[j, j + 1]$ to a geodesic in X joining $\phi(j)$ with $\phi(j + 1)$ outside $B(r)$, provided the claim is satisfied. This will allow us to define a continuous map ϕ from an interval to X joining a and b and lying out of $B(r)$, as wanted.

We are left with $d(\phi(j), \phi(j + 1)) \leq 3m$. If j and $j + 1$ are different from 0 and L, then $d((\phi(j), \phi(j + 1)) = d(f'(h_j), f'_{j+1}) \leq md(h_j, h_{j+1}) + m = 2m$. In the case when $j = 0$ (or $j + 1 = L$), we have $d(\phi(0), \phi(1)) = d(a, f'(h_1)) \leq d(a, f'(f(a))) + d(f'(f(a)), f'(h_1)) \leq m + (md(f(a), h_1) + m) = m + 2m = 3m$.

2.1. Geometric examples

Now we want to study the end-depth for groups having some nice geometric properties, in particular Gromov-hyperbolic groups and groups acting properly discontinuously and co-compactly on CAT(0) spaces.

Proposition 2.4. *If G is $CAT(0)$ or hyperbolic (and 1-ended) then the end depth function V_0 is linear.*

Proof. Ontaneda in [9] proved that a proper cocompact $CAT(0)$ space X is almost geodetically complete (for hyperbolic groups the same was proved by Mihalik, Bestvina-Mess [1]). This means that there exists $c > 0$ such that for any $x \in X$ there exists an infinite geodesics ray staring at 1 and which passes within c of x.
Take two points $x, y \in X - B(r + c)$; we have the rays r_x, r_y as above. Thus there are two points on the rays $x' \in r_x, y' \in r_y$ closed to x, y at distance $< c$. Moreover the points of the rays will eventually belong to the same connected component (because 1-ended) and could be joined by a faraway segment not intersecting $B(r)$. Further we connect x to x' by a segment of length $< c$ and thus x' is outside $B(r)$; and x' can be connected to y' by going along rays sufficiently far and then connecting the rays as above. This proves that we can take $V_0(r) = r + c$.

Acknowledgement: Partially supported by GNSAGA.

References

[1] M.Bestvina and G.Mess, *The boundary of negatively curved groups*, J.A.M.S. **4**(1991), 469-481.

[2] S.G.Brick, *Quasi-isometries and ends of groups*, J.Pure Appl. Algebra **86** (1993), 23–33.

[3] J.W.Cannon, *Geometric group theory*, Handbook of geometric topology, 261–305, North-Holland, Amsterdam, 2002.

[4] S.Cleary and T.Riley, *A finitely presented group with unbounded dead-end depth*, math.GR/0406443.

[5] P.de la Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press, 2000.

[6] L. Funar and D.E. Otera, *A refinement of the simple connectivity at infinity of groups*, Archiv Math. (Basel) 81(2003), 360-368.

[7] M. Gromov, *Asymptotic invariants of infinite groups* Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, (G.A. Niblo and M. A. Roller, Ed.) London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, 1993.

[8] R.C.Lyndon and P.E.Shupp, *Combinatorial group theory*, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[9] P.Ontaneda, *Cocompact CAT(0) spaces are almost geodesically complete*, Topology **44** (2005), no. 1, 47–62.

[10] J.R.Stallings, *On torsion-free groups with infinitely many ends*, Ann. of Math., **88** (1968), 312–334.

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