MULTIPLE HAMILTONIAN STRUCTURES FOR THE ISHII'S EQUATION

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Abstract.The Ishii's equation is considered and some aspects of its Poisson geometry are pointed out.

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1. Introduction

The dynamics of Ishii's equation using an Hamilton-Poisson formulation was studied in [1]. The authors show that the system

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ \dot{x_3} &= x_1 x_2, \end{cases} \tag{1}$$

has the Hamilton-Poisson realization $(R^3, \{\cdot,\cdot\}_1, H_1)$, where the Poisson structure $\{\cdot,\cdot\}_1$ is generated by the matrix

$$\Pi_1(x_1,x_2,x_3) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & x_1 \\ 0 & -x_1 & 0 \end{bmatrix}, \tag{2}$$

and the Hamiltonian H_1 is given by

$$H_1(x_1,x_2,x_3) = x_1x_3 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3. \tag{3}$$

Also, the function $H_2 \in C^\infty(R^3, R)$ given by

$$H_2(x_1,x_2,x_3) = x_3 - \frac{1}{2}x_1^2 \tag{4}$$

is a Casimir of the configuration $(R^3, \{\cdot,\cdot\}_1)$.

Next, we find new Hamilton-Poisson formulations for the system (1), write the system (1) as a multi-gradient system and construct geometric integrators that preseve some "qualitative" features (constants of motion, Poisson structure) of the system (1).

2. Multi-Hamiltonian realization of the system (1)

Let $C^\infty(R^3, R)$ be the space of smooth real valued functions defined on R^3 and the bracket $\{\cdot,\cdot\}_2$ on $C^\infty(R^3, R)$ defined by

$$\{f, g\}_2 = (\nabla f)^t \Pi_2 (\nabla g), \tag{5}$$

where the matrix Π_2 is given by

$$\Pi_2(x_1, x_2, x_3) = \begin{bmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 - x_1^2 \\ -x_2 & x_1^2 - x_3 & 0 \end{bmatrix}. \tag{6}$$

Proposition 1. *The bracket* (5) *defines a Poisson structure on* R^3.
Proof. It is easy to see that the bracket (5) is bilinear, skew-symmetric and satisfies Leibniz' rule. The Jacobi identity reduces in the three dimensional case to the following single relation

$$\pi_{12}\left(\frac{\partial \pi_{31}}{\partial x_1} - \frac{\partial \pi_{23}}{\partial x_2}\right) + \pi_{13}\left(\frac{\partial \pi_{12}}{\partial x_1} - \frac{\partial \pi_{23}}{\partial x_3}\right) + \pi_{23}\left(\frac{\partial \pi_{12}}{\partial x_2} - \frac{\partial \pi_{31}}{\partial x_3}\right) = 0,$$

which is, also, easily verified.
Proposition 2. *The Poisson structures* $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ *are compatible.*
Proof. It is well known that $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ are compatible if and only if $[\Pi_1, \Pi_2]_S = 0$, where $[\cdot,\cdot]_S$ is the Schouten bracket. Computing the components in local coordinates of $[\Pi_1, \Pi_2]_S$ given by (see [2])

$$[\Pi_1, \Pi_2]_S^{ijk} = -\sum_{m=1}^{3}\left(\Pi_2^{mk}\frac{\partial \Pi_1^{ij}}{\partial x_m} + \Pi_1^{mk}\frac{\partial \Pi_2^{ij}}{\partial x_m} + cycle(i, j, k)\right)$$

we obtain the desired result.

Proposition 3. *The system* (1) *is a bi-Hamiltonian system.*
Proof. Indeed, the Poisson structures $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ are not constant multiples of each other, compatible and

$$\dot{x} = \Pi_1(x) \cdot \nabla H_1(x) = \Pi_2(x) \cdot \nabla H_2(x), \quad x \in R^3.$$

Remark 1. Let us observe that $\Pi_1 \cdot \nabla H_2 = 0$ and $\Pi_2 \cdot \nabla H_1 = 0$, so the function H_2 is a Casimir of the configuration $(R^3, \{\cdot,\cdot\}_1)$ and H_1 is a Casimir of the configuration $(R^3, \{\cdot,\cdot\}_2)$. ∎

The fact that the Poisson structures $\{\cdot,\cdot\}_1$, $\{\cdot,\cdot\}_2$ are compatible i.e. $a\{\cdot,\cdot\}_1 + b\{\cdot,\cdot\}_2$ is a Poisson structure for all $a, b \in R$, helps us show that the system (1) may be realized as a Hamilton-Poisson system in an infinite number of different ways. More exactly, we can prove

Proposition 4. *The system* (1) *has the following Hamilton-Poisson realizations*:

$$(R^3, \Pi_{ab}, H_{cd}),$$

where $\Pi_{ab} = a\Pi_1 + b\Pi_2$, $H_{cd} = cH_1 - dH_2$ *and* $a, b, c, d \in R$, $ac - bd = 1$.

Remark 2. The function $C_{ab} \in C^\infty(R^3, R)$ given by

$$C_{ab}(x_1, x_2, x_3) = a(\frac{1}{2}x_1^2 - x_3) + b(x_1x_3 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3)$$

is a Casimir of the configuration (R^3, Π_{ab}).

3.The system (1) like a multi-gradient system

Let H_1, $H_2 \in C^\infty(R^3, R)$ be the first integrals of the system (1) given by (3) and (4). Then we have

Proposition 5. *The system* (1) *can be written as a multi-gradient system*

$$\dot{x} = S(x) \cdot \nabla H_1(x) \cdot \nabla H_2(x), \quad x = (x_1, x_2, x_3) \in R^3, \tag{7}$$

where S *is a completely skew symmetric* $3-$*tensor.*

Proof. If we take $S = \epsilon_{ijk}$ (the Levi-Civita $3-$tensor), then, a direct computation shows us that

$$\dot{x}_i = \sum_{j,k=1}^{3} S_{ijk} \frac{\partial H_1(x)}{\partial x_j} \frac{\partial H_2(x)}{\partial x_k}, \quad i = 1, 2, 3,$$

as required. ■

Let us now consider the discretization of the system (7) given by (see [3], [4]):

$$\frac{x^{n+1} - x^n}{h} = \widetilde{S}(x^n, x^{n+1}, h) \cdot \overline{\nabla} H_1(x^n, x^{n+1}) \cdot \overline{\nabla} H_2(x^n, x^{n+1}), \tag{8}$$

where the discrete gradients $\overline{\nabla} H_1$, $\overline{\nabla} H_2$ are any solution of

$$\left\{ \begin{array}{ccc} H(x^{n+1}) - H(x^n) & = & (\overline{\nabla} H) \cdot (x^{n+1} - x^n) \\ \overline{\nabla} H(x^n, x^{n+1}) & = & \overline{\nabla} H(x^n) + \mathcal{O}(h) \end{array} \right.$$

and $\widetilde{S}$ is a completely skew symmetric 3−tensor that verifies

$$\widetilde{S}(x^n, x^{n+1}, h) = S(x^n) + \mathcal{O}(h).$$

Choosing discrete gradients $\overline{\nabla} H_1$, $\overline{\nabla} H_2$ as follows:

$$\overline{\nabla} H_1(x^n, x^{n+1}) = \left(-\frac{1}{3} \left[(x_1^{n+1})^2 + x_1^{n+1} x_1^n + (x_1^n)^2 \right] + x_3^n, -\frac{1}{2} \left(x_2^{n+1} + x_2^n \right), x_1^{n+1} \right),$$

$$\overline{\nabla} H_2(x^n, x^{n+1}) = \left(-\frac{1}{2} \left(x_1^{n+1} + x_1^n \right), 0, 1 \right)$$

and $\widetilde{S}(x^n, x^{n+1}, h) = S(x^n)$ we obtain, via (8), an explicit first order numerical integrator for the system (1), given by

$$\begin{cases} \dfrac{x_1^{n+1} - x_1^n}{h} &= \dfrac{1}{2} \left(x_2^{n+1} + x_2^n \right) \\ \dfrac{x_2^{n+1} - x_2^n}{h} &= \dfrac{1}{6} (x_1^{n+1})^2 + \dfrac{1}{6} x_1^{n+1} x_1^n - \dfrac{1}{3} (x_1^n)^2 + x_3^n \\ \dfrac{x_3^{n+1} - x_3^n}{h} &= \dfrac{1}{4} \left(x_1^{n+1} + x_1^n \right) \left(x_2^{n+1} + x_2^n \right) \end{cases} \tag{9}$$

where h is the size step of the integrator.

Proposition 6. *The integrator* (9) *preserves both the first integrals* H_1, H_2 *of the system* (1).

Proof. Indeed, for $i = 1, 2$ we have:

$$\begin{aligned} & H_i(x_1^{n+1}, x_2^{n+1}, x_3^{n+1}) - H_i(x_1^n, x_2^n, x_3^n) = \\ &= \overline{\nabla} H_i(x^n, x^{n+1})^t \cdot (x^{n+1} - x^n) \\ &= h \overline{\nabla} H_i(x^n, x^{n+1})^t \cdot \widetilde{S}(x^n, x^{n+1}, h) \cdot \overline{\nabla} H_1(x^n, x^{n+1}) \cdot \overline{\nabla} H_2(x^n, x^{n+1}) \\ &= 0, \end{aligned}$$

as required.

4. The system (1) and the midpoint rule

The midpoint integrator for the system (1) is given by

$$\begin{cases} \dfrac{x_1^{n+1} - x_1^n}{h} &= \dfrac{1}{2} \left(x_2^{n+1} + x_2^n \right) \\ \dfrac{x_2^{n+1} - x_2^n}{h} &= \dfrac{1}{2} \left(x_3^{n+1} + x_3^n \right) \\ \dfrac{x_3^{n+1} - x_3^n}{h} &= \dfrac{1}{4} \left(x_1^{n+1} + x_1^n \right) \left(x_2^{n+1} + x_2^n \right). \end{cases} \tag{10}$$

Then, we can prove

Proposition 7. *The midpoint integrator* (10) *preserves the first integral* H_2 *given by* (4).

Remark 3. Let us observe that if we modify the midpoint rule as follows:

$$\begin{cases} \dfrac{x_1^{n+1}-x_1^n}{h} & = & \dfrac{1}{2}\left(x_2^{n+1}+x_2^n\right) \\ \dfrac{x_2^{n+1}-x_2^n}{h} & = & \dfrac{1}{2}\left(x_3^{n+1}+x_3^n\right)+\Delta \\ \dfrac{x_3^{n+1}-x_3^n}{h} & = & \dfrac{1}{4}\left(x_1^{n+1}+x_1^n\right)\left(x_2^{n+1}+x_2^n\right), \end{cases}$$

where $\Delta = \dfrac{1}{6}(x_1^{n+1})^2 + \dfrac{1}{6}x_1^{n+1}x_1^n - \dfrac{1}{3}(x_1^n)^2 + \dfrac{1}{2}x_3^n - \dfrac{1}{2}x_3^{n+1}$, we obtain the integrator (9), that preserves both the first integrals H_1, H_2 of the system (1). The midpoint integrator lies on $H_2 = const.$ and with Δ we "drag" it on the phase curves of the system (1), the intersection of

$$x_1x_3 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3 = const.$$

with

$$x_3 - \frac{1}{2}x_1^2 = const.$$

Remark 4. Unfortunately, none of these integrators preserves the Poisson structures Π_1, Π_2 defined by (2), (6).■

Next, using the midpoint rule, combined with splitting and composition methods, we construct an Poisson integrator for the system (1). Let us observe that the system (1) can be written as (see [5])

$$\dot{x} = (\Pi_{21}(x) + \Pi_{22}(x)) \cdot \nabla H_2(x), \quad x \in R^3,$$

where

$$\Pi_{21} = \begin{bmatrix} 0 & x_1 & 0 \\ -x_1 & 0 & x_3 - x_1^2 \\ 0 & x_1^2 - x_3 & 0 \end{bmatrix}, \quad \Pi_{22} = \begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & 0 \\ -x_2 & 0 & 0 \end{bmatrix}.$$

If φ_1, φ_2 are the midpoint integrators of the systems

$$\dot{x} = \Pi_{2i}(x) \cdot \nabla H_2(x), \quad i = 1, 2,$$

then, the splitting midpoint integrator $\varphi = \varphi_1 \circ \varphi_2$ is given by

$$\begin{cases} x_1^{n+1} &= x_1^n + hx_2^n + h^2 x_3^n \\ x_2^{n+1} &= x_2^n + hx_2^n \\ x_3^{n+1} &= x_3^n + hx_1^n x_2^n + h^2 x_1^n x_3^n + h^3 x_2^n x_3^n + \dfrac{h^2 (x_2^n)^2}{2} + \dfrac{h^4 (x_3^n)^2}{2}, \end{cases} \tag{11}$$

where h is the size step of the integrator. Now, we can prove

Proposition 9. *The numerical integrator* (11) *has the following properties*:
(i) *It preserves the Poisson structure* Π_1 *given by* (2).
(ii) *It preserves the Casimir* H_2 *of the configuration* (R^3, Π_1) *given by* (4).

Proof. (i) A direct computation gives us

$$D\varphi(x) \cdot \Pi_1(x) \cdot [D\varphi(x)]^t = \Pi_1(\varphi(x)),$$

as required.

(ii) Indeed, via (11), we have $H_2(\varphi(x_1^n, x_2^n, x_3^n)) = H_2(x_1^n, x_2^n, x_3^n)$.

Remark 5. In a similar manner we obtain a Poisson integrator for the Poisson structure Π_2, given by (6).

References

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