MONADIC INVOLUTIVE PSEUDO-BCK ALGEBRAS

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Abstract.In this paper we introduce and study the monadic involutive pseudo-BCK algebras (porims). As particular cases we obtain the monadic involutive pseudo-BCK(pP) lattices (pseudo-residuated lattices), monadic pseudo-IMTL algebras, monadic pseudo-NM algebras, monadic pseudo-MV algebras, etc. and all the commutative corresponding algebras.

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1. Introduction

In this paper we generalize some results from [29] and [10] concerning monadic MV algebras to the most general, non-commutative, involutive case. Since in [29], all the non-commutative algebras, connected to logics, were redefined as pseudo-BCK algberas, we shall consider involutive pseudo-BCK algebras as the most general non-commutative involutive algebras. Therefore, we define in this paper the monadic involutive pseudo-BCK algebra and we obtain the principal properties. As particular cases, we get monadic involutive pseudo-BCK(pP) lattices (pseudo-residuated lattices), monadic pseudo-IMTL algebras, monadic pseudo-NM algebras, monadic pseudo-MV algebras, etc. [29], [30] and all the commutative corresponding algebras. The study will be continued in a future paper.

We assume the reader is familiar with [29], [30], [31], but the paper is self-contained as much as possible.

The already known results are presented without proof.

2. Classes of pseudo-BCK algebras

In this section we recall definitions and results needed in the sequel, that the reader can find in [29], [31].

2.1. Bounded pseudo-BCK algebras, pseudo-BCK(pP) algebras

BCK algebras were introduced in 1966 by Iséki as "right" algebras with the only bound 0, starting from the systems of positive implicational calculus, weak positive implicational calculus by A. Church and BCI, BCK-systems by C.A. Meredith (cf. [35]).

Pseudo-BCK algebras were introduced in 2001 [19], as non-commutative generalizations of BCK algebras, i.e. as "right" algebras. We need the "left" definition and the reversed definition (see [29] for details).

Definition 1. A *reversed left-pseudo-BCK algebra* [28] is a structure

$$\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1),$$

where "$\leq$" is a binary relation on A, "$\to$" and "$\rightsquigarrow$" are binary operations on A and "1" is an element of A verifying, for all $x, y, z \in A$, the axioms:

(I) $(z \to x) \rightsquigarrow (y \to x) \geq y \to z, \quad (z \rightsquigarrow x) \to (y \rightsquigarrow x) \geq y \rightsquigarrow z,$
(II) $(y \to x) \rightsquigarrow x \geq y, \quad (y \rightsquigarrow x) \to x \geq y,$
(III) $x \geq x,$
(IV) $1 \geq x,$
(V) $x \geq y,\ y \geq x \Longrightarrow x = y,$
(VI) $x \geq y \Longleftrightarrow y \to x = 1 \Longleftrightarrow y \rightsquigarrow x = 1.$

We shall freely write $x \geq y$ or $y \leq x$ in the sequel. From now on we shall simply say "pseudo-BCK algebra" instead of "reversed left-pseudo-BCK algebra".

Let $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. We shall say that $\mathcal{A}$ is *commutative* if $x \to y = x \rightsquigarrow y$, for all $x, y \in A$ [27]. Any commutative pseudo-BCK algebra is a left-BCK algebra [29].

Proposition 1. (see [28], [27]) *The following properties hold in a pseudo-BCK algebra:*

$$x \leq y \Longrightarrow y \to z \leq x \to z \text{ and } y \rightsquigarrow z \leq x \rightsquigarrow z, \tag{1}$$

$$x \leq y,\ y \leq z \Longrightarrow x \leq z. \tag{2}$$

$$z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x). \tag{3}$$

$$z \leq y \to x \Longleftrightarrow y \leq z \rightsquigarrow x, \tag{4}$$

$$z \to x \leq (y \to z) \to (y \to x), \quad z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x) \tag{5}$$

$$x \leq y \to x, \quad x \leq y \rightsquigarrow x, \tag{6}$$

$$1 \to x = x = 1 \rightsquigarrow x, \tag{7}$$

$$x \leq y \Longrightarrow z \to x \leq z \to y \text{ and } z \rightsquigarrow x \leq z \rightsquigarrow y, \tag{8}$$

$$[(y \to x) \rightsquigarrow x] \to x = y \to x, \quad [(y \rightsquigarrow x) \to x] \rightsquigarrow x = y \rightsquigarrow x. \tag{9}$$

Recall that "$\leq$" is a partial order relation and that $(A, \leq, 1)$ is a poset with greatest element 1.

Definition 2. [27] A *pseudo-BCK algebra with (pP) condition* (i.e. *with pseudo-product*) or a *pseudo-BCK(pP) algebra* for short, is a pseudo-BCK algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$ satisfying (pP) condition:

(pP) for all $x, y \in A$, there exists $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\}$.

Note that any linearly ordered bounded pseudo-BCK algebra is with (pP) condition.

Proposition 2. (see [27], Theorem 2.15) *Let $\mathcal{A}$ be a pseudo-BCK(pP) algebra. Then, (pRP) condition holds, where:*

(pRP) for all x, y, z, $x \odot y \leq z \Longleftrightarrow x \leq y \to z \Longleftrightarrow y \leq x \rightsquigarrow z$.

Lemma 1. [29] *A pseudo-BCK(pP) algebra $\mathcal{A}$ is commutative iff $x \odot y = y \odot x$, for any $x, y \in A$.*

Proposition 3. [28] *Let us consider the pseudo-BCK(pP) algebra $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 1)$. Then, for all $x, y, z \in A$:*

$$x \odot y \leq x,\ y \tag{10}$$

$$(x \to y) \odot x \leq x, y, \quad x \odot (x \rightsquigarrow y) \leq x,\ y \tag{11}$$

$$y \leq x \to (y \odot x), \quad y \leq x \rightsquigarrow (x \odot y) \tag{12}$$

$$x \to y \leq (x \odot z) \to (y \odot z), \quad x \rightsquigarrow y \leq (z \odot x) \rightsquigarrow (z \odot y) \tag{13}$$

$$x \odot (y \to z) \leq y \to (x \odot z), \quad (y \rightsquigarrow z) \odot x \leq y \rightsquigarrow (z \odot x) \tag{14}$$

$$(y \to z) \odot (x \to y) \leq x \to z, \quad (x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z \tag{15}$$

$$x \to (y \to z) = (x \odot y) \to z, \quad x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z \tag{16}$$

$$(x \odot z) \to (y \odot z) \leq x \to (z \to y), \quad (z \odot x) \rightsquigarrow (z \odot y) \leq x \rightsquigarrow (z \rightsquigarrow y) \tag{17}$$

$$x \to y \le (x \odot z) \to (y \odot z) \le x \to (z \to y), \tag{18}$$

$$x \rightsquigarrow y \le (z \odot x) \rightsquigarrow (z \odot y) \le x \rightsquigarrow (z \rightsquigarrow y)$$

$$x \le y \quad \Rightarrow \quad x \odot z \le y \odot z, \quad z \odot x \le z \odot y. \tag{19}$$

Proposition 4. [27] *Let $\mathcal{A} = (A, \le, \to, \rightsquigarrow, 1)$ be a pseudo-BCK(pP) algebra. Then the algebra $(A, \le, \odot, 1)$ is a partially ordered, integral (left-) monoid, or, equivalently, the operation $\odot$ is a pseudo-t-norm on the poset $(A, \le, 1)$ with greatest element* 1.

Pseudo-BCK(pP) algebras are termwise equivalent with left-porims (partially ordered, residuated, integral left-monoids) [27].

* * *

Definition 3. [28] If there is an element, 0, of a pseudo-BCK algebra $\mathcal{A} = (A, \le, \to, \rightsquigarrow, 1)$, satisfying $0 \le x$ (i.e. $0 \to x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then 0 is called the *zero* of $\mathcal{A}$. A pseudo-BCK algebra with zero is called to be *bounded* and it is denoted by: $(A, \le, \to, \rightsquigarrow, 0, 1)$.

Let $\mathcal{A} = (A, \le, \to, \rightsquigarrow, 0, 1)$ be a bounded pseudo-BCK algebra. Define, for all $x \in A$, two negations, $^-$ and $^\sim$, by [28]: for all $x \in A$,

$$x^- \stackrel{def}{=} x \to 0, \quad x^\sim \stackrel{def}{=} x \rightsquigarrow 0. \tag{20}$$

Proposition 5. [28] *In a bounded pseudo-BCK algebra $\mathcal{A}$ the following properties hold, for all $x, y \in A$:*

$$1^- = 0 = 1^\sim, \ 0^- = 1 = 0^\sim, \tag{21}$$

$$x \le (x^-)^\sim, \quad x \le (x^\sim)^- \tag{22}$$

$$x \to y \le y^- \rightsquigarrow x^-, \quad x \rightsquigarrow y \le y^\sim \to x^\sim, \tag{23}$$

$$x \le y \Rightarrow y^- \le x^-, \ y^\sim \le x^\sim \tag{24}$$

$$y \rightsquigarrow x^- = x \to y^\sim, \tag{25}$$

$$((x^-)^\sim)^- = x^-, \quad ((x^\sim)^-)^\sim = x^\sim. \tag{26}$$

Proposition 6. [28] *Let us consider the bounded pseudo-BCK(pP) algebra* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$. *Then, for all* $x, y, z \in A$:

$$0 \odot x = x \odot 0 = 0. \tag{27}$$

Proposition 7. [29] *Let* $\mathcal{A}$ *be a bounded pseudo-BCK(pP) algebra. Then,*

$$x \odot x^{\sim} = 0 = x^{-} \odot x.$$

Definition 4. [29] If a bounded pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ verifies, for every $x \in A$:

$$(DN^1)\ (x^{\sim})^{-} = x, \qquad (DN^2)\ (x^{-})^{\sim} = x,$$

then we shall say that $\mathcal{A}$ is *with (pDN) (pseudo-Double Negation) condition* or is an *involutive pseudo-BCK algebra* (note that the more correct name would be "pseudo-involutive").

We write: (pDN)=(DN^1) + (DN^2).

Lemma 2. *Let* $\mathcal{A}$ *be an involutive pseudo-BCK algebra. Then, for all* $x, y \in A$ *(see [28]):*

$$x \leq y \Leftrightarrow y^{-} \leq x^{-} \Leftrightarrow y^{\sim} \leq x^{\sim} \tag{28}$$

$$x \rightsquigarrow y = y^{\sim} \rightarrow x^{\sim}, \quad x \rightarrow y = y^{-} \rightsquigarrow x^{-}, \tag{29}$$

$$y^{-} \rightsquigarrow x = x^{\sim} \rightarrow y. \tag{30}$$

Proposition 8. [19] *Let* $\mathcal{A}$ *be an involutive pseudo-BCK algebra. Then, for all* $x, y \in A$:

$$(x \rightarrow y^{-})^{\sim} = (y \rightsquigarrow x^{\sim})^{-}. \tag{31}$$

We recall now the following important result:

Theorem 1. [29] *Let* $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ *be an involutive pseudo-BCK algebra. Then* $\mathcal{A}$ *is with (pP) condition and we have, for all* $x, y \in A$:

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\} = \min\{z \mid y \leq x \rightsquigarrow z\} = \tag{32}$$

$$= (x \rightarrow y^{-})^{\sim} = (y \rightsquigarrow x^{\sim})^{-},$$

$$x \rightarrow y = (x \odot y^{\sim})^{-}, \quad x \rightsquigarrow y = (y^{-} \odot x)^{\sim}. \tag{33}$$

* * *

In a pseudo-BCK algebra $\mathcal{A}$ we define, for all $x, y \in A$ (see [19], [28]):

$$x \vee y \stackrel{def}{=} (x \to y) \rightsquigarrow y. \tag{34}$$

$$x \cup y \stackrel{def}{=} (x \rightsquigarrow y) \to y. \tag{35}$$

Definition 5. [19], [28]

(i) If $x \vee y = y \vee x$, for all $x, y \in A$, then the pseudo-BCK algebra $\mathcal{A}$ is called to be $\vee$-*commutative*.

(i') If $x \cup y = y \cup x$, for all $x, y \in A$, then the pseudo-BCK algebra $\mathcal{A}$ is called to be $\cup$-*commutative*.

Lemma 3. (see [19], [28])

(i) A pseudo-BCK algebra is $\vee$-commutative iff it is a semilattice with respect to $\vee$ (under $\leq$).

(i') A pseudo-BCK algebra is $\cup$-commutative iff it is a semilattice with respect to $\cup$ (under $\leq$).

Definition 6. [19], [28] We say that a pseudo-BCK algebra is *sup-commutative* if it is both $\vee$-commutative and $\cup$-commutative.

Theorem 2. [19], [28] *A pseudo-BCK algebra is sup-commutative iff it is a semilattice with respect to both $\vee$ and $\cup$.*

Corollary 1. (see [19], [28] Corollary 3.27) *Let $\mathcal{A}$ be a bounded, sup-commutative pseudo-BCK algebra. Then, $\mathcal{A}$ is with (pDN) condition (and hence it is with (pP) condition, by Theorem 1).*

In a bounded, sup-commutative pseudo-BCK algebra $\mathcal{A}$, define, for all $x, y \in A$ (see [19], [28]):

$$x \wedge y \stackrel{def}{=} (x^- \vee y^-)^\sim, \tag{36}$$

$$x \cap y \stackrel{def}{=} (x^\sim \cup y^\sim)^-. \tag{37}$$

Theorem 3. (see [19], [28] Theorem 3.33) *If a pseudo-BCK algebra is bounded and sup-commutative, then it is a lattice with respect to both $\vee$, $\wedge$ and $\cup$, $\cap$ (under $\leq$) and we have, for all x, y:*

$$x \vee y = x \cup y, \quad x \wedge y = x \cap y.$$

Note that a sup-commutative pseudo-BCK algebra can be a lattice without being bounded.

You can find in [19], [28] some properties of bounded, sup-commutative pseudo-BCK algebras and that bounded, sup-commutative pseudo-BCK algebras coincide (are termwise equivalent) with pseudo-MV algebras.

2.2. The self-duality of involutive pseudo-BCK algebras

We have discussed in details the self-duality of involutive pseudo-BCK algebras in [31]; we recall the following dual operations and their connections, needed in the sequel.

Let $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ be an involutive (left) pseudo-BCK algebra, where:
- the "left" operations are: the two "left" implications $\rightarrow$, $\rightsquigarrow$, the two "left" corresponding negations $x^{-_L} = x^- = x \rightarrow 0$, $x^{\sim_L} = x^\sim = x \rightsquigarrow 0$ and the pseudo-t-norm $\odot$;
- the "right" operations are: the two "right" implications $\Rightarrow$, $\approx>$, the two "right" corresponding negations $x^{-_R} = x \Rightarrow 1$, $x^{\sim_R} = x \approx> 1$ and the pseudo-t-conorm $\oplus$;
- the two "left" negations and the corresponding "right" negations coincide, therefore we shall use only the two symbols $^-$ and $^\sim$, where for all $x \in A$, $x^- = x \rightarrow 0 = x \Rightarrow 1$, $\quad x^\sim = x \rightsquigarrow 0 = x \approx> 1$;
- the "left" partial order relation is $\geq$, while the "right" partial order relation is $\leq$, where $x \geq y$ iff $y \leq x$.

Thus, $(A, \leq, \Rightarrow, \approx>, 1, 0)$ is the dual involutive (right) pseudo-BCK algebra, and conversely.

Then [31], for all $x, y \in A$:

• the connections between the "left" operations are:

$$x \odot y = (x \rightarrow y^-)^\sim = (y \rightsquigarrow x^\sim)^-, \tag{38}$$

$$x \rightarrow y = (x \odot y^\sim)^-, \quad x \rightsquigarrow y = (y^- \odot x)^\sim; \tag{39}$$

• the connections between the "right" operations are:

$$x \oplus y = (x \Rightarrow y^-)^\sim = (y \approx> x^\sim)^-, \tag{40}$$

$$x \Rightarrow y = (x \oplus y^{\sim})^{-}, \quad x \approx> y = (y^{-} \oplus x)^{\sim}; \tag{41}$$

• the "left" operations expressed in terms of "right" operations are:

$$x \odot y = (y^{-} \oplus x^{-})^{\sim} = (y^{\sim} \oplus x^{\sim})^{-}, \tag{42}$$

$$x \to y = (x^{\sim} \Rightarrow y^{\sim})^{-}, \quad x \sim> y = (x^{-} \approx> y^{-})^{\sim}; \tag{43}$$

• the "right" operations expressed in terms of "left" operations are:

$$x \oplus y = (y^{-} \odot x^{-})^{\sim} = (y^{\sim} \odot x^{\sim})^{-}, \tag{44}$$

$$x \Rightarrow y = (x^{\sim} \to y^{\sim})^{-}, \quad x \approx> y = (x^{-} \sim> y^{-})^{\sim}. \tag{45}$$

By Lemma 2, we have: for all $x, y \in A$,

$$x \sim> y = y^{\sim} \to x^{\sim}, \quad x \to y = y^{-} \sim> x^{-}, \tag{46}$$

$$y^{-} \sim> x = x^{\sim} \to y \tag{47}$$

and dually

$$x \approx> y = y^{\sim} \Rightarrow x^{\sim}, \quad x \Rightarrow y = y^{-} \approx> x^{-}, \tag{48}$$

$$y^{-} \approx> x = x^{\sim} \Rightarrow y. \tag{49}$$

2.3. Bounded pseudo-BCK(pP) lattices

Definition 7. Let $\mathcal{A} = (A, \leq, \sim>, \to, 0, 1)$ be a bounded pseudo-BCK (pseudo-BCK(pP)) algebra. If the poset $(A, \leq)$ is a lattice, then we shall say that $\mathcal{A}$ is a bounded *pseudo-BCK (pseudo-BCK(pP)) lattice*.

A bounded pseudo-BCK (pseudo-BCK(pP)) lattice $\mathcal{A} = (A, \leq, \to, \sim>, 0, 1)$ will be denoted:

$$\mathcal{A} = (A, \wedge, \vee, \to, \sim>, 0, 1).$$

Denote by **pBCK(pP)-L**^{b} the class of bounded pseudo-BCK(pP) lattices.

(Bounded) pseudo-BCK(pP) lattices are termwise equivalent with (bounded) pseudo-residuated lattices[27].

Lemma 4. [29] *Let $(A, \wedge, \vee, \rightarrow, \rightsquigarrow, 1)$ be a pseudo-BCK lattice. Then, for any $x, y, z \in A$, we have:*
(i) $z \rightarrow (x \wedge y) \leq (z \rightarrow x) \wedge (z \rightarrow y)$ *and* $z \rightsquigarrow (x \wedge y) \leq (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)$;
(ii) $z \rightarrow (x \vee y) \geq (z \rightarrow x) \vee (z \rightarrow y)$ *and* $z \rightsquigarrow (x \vee y) \geq (z \rightsquigarrow x) \vee (z \rightsquigarrow y)$;
(iii) $(x \wedge y) \rightarrow z \geq (x \rightarrow z) \vee (y \rightarrow z)$ *and* $(x \wedge y) \rightsquigarrow z \geq (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$.

Let I be an arbitrary set.

Proposition 9. [29] *Let $\mathcal{A}$ be a pseudo-BCK(pP) lattice. Then the following properties hold, for all $x, y, z \in A$:*

$$a \odot (\vee_{i \in I} b_i) = \vee_{i \in I} (a \odot b_i), \qquad (\vee_{i \in I} b_i) \odot a = \vee_{i \in I} (b_i \odot a), \tag{50}$$

$$g \vee (h \odot k) \geq (g \vee h) \odot (g \vee k), \tag{51}$$

$$(\vee_{i \in I} x_i) \rightsquigarrow y = \wedge_{i \in I} (x_i \rightsquigarrow y), \qquad (\vee_{i \in I} x_i) \rightarrow y = \wedge_{i \in I} (x_i \rightarrow y), \tag{52}$$

$$y \rightsquigarrow (\wedge_{i \in I} x_i) = \wedge_{i \in I} (y \rightsquigarrow x_i), \qquad y \rightarrow (\wedge_{i \in I} x_i) = \wedge_{i \in I} (y \rightarrow x_i), \tag{53}$$

whenever the arbitrary unions and meets exist.

Proposition 10. [29] *Let $\mathcal{A}$ be a pseudo-BCK(pP) lattice. Then we have:*

$$(x \rightarrow y) \odot x \leq x \wedge y, \quad x \odot (x \rightsquigarrow y) \leq x \wedge y, \tag{54}$$

$$x \rightarrow (x \wedge y) = x \rightarrow y, \quad x \rightsquigarrow (x \wedge y) = x \rightsquigarrow y. \tag{55}$$

Proposition 11. [29] *In a bounded pseudo-BCK(pP) lattice we have the properties:*

$$(x \vee y)^{\sim} = x^{\sim} \wedge y^{\sim}, \quad (x \vee y)^{-} = x^{-} \wedge y^{-}, \tag{56}$$

$$x^{-} \vee y^{-} \leq (x \wedge y)^{-}, \quad x^{\sim} \vee y^{\sim} \leq (x \wedge y)^{\sim} \tag{57}$$

$$x \rightarrow y^{-} = (x \odot y)^{-}, \quad x \rightsquigarrow y^{\sim} = (y \odot x)^{\sim}. \tag{58}$$

* * *

We say that a pseudo-BCK lattice $\mathcal{A} = (A, \wedge, \vee, \rightarrow, \rightsquigarrow, 1)$ is *with (pC) condition* if, for all $x, y \in A$,

$$(C^1) \quad x \vee y = (y \rightsquigarrow x) \rightarrow x, \qquad (C^2) \quad x \vee y = (y \rightarrow x) \rightsquigarrow x.$$

We write: (pC)=(C^1) + (C^2).

Remark 1. [29] The pseudo-BCK(pP) lattice with (pC) condition is a duplicate name for the sup-commutative pseudo-BCK(pP) algebra.

We say that a bounded pseudo-BCK lattice $\mathcal{A}$ is *with (pDN) condition* or a pseudo-BCK$_{(pDN)}$ lattice or an *involutive pseudo-BCK algbera*, for short, if the associated bounded pseudo-BCK algebra is with (pDN) condition. An involutive pseudo-BCK lattice is an involutive pseudo-BCK(pP) lattice.

Note that **involutive pseudo-BCK lattices are termwise equivalent with involutive pseudo-residuated lattices**.

Corollary 2. [29] *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, \sim>, 0, 1)$ be a bounded pseudo-BCK(pP) lattice with (pC) condition. Then $\mathcal{A}$ is with (pDN) condition.*

Proposition 12. [29] *Let $\mathcal{A} = (A, \leq, \rightarrow, \sim>, 0, 1)$ be an involutive pseudo-BCK(pP) lattice. Then we have:*

$$(x \wedge y)^- = x^- \vee y^-, \quad (x \wedge y)^\sim = x^\sim \vee y^\sim, \tag{59}$$

$$x \wedge y = (x^- \vee y^-)^\sim, \quad x \wedge y = (x^\sim \vee y^\sim)^-. \tag{60}$$

3. Monadic involutive pseudo-BCK algebras

Let $\mathcal{A} = (A, \leq, \rightarrow, \sim>, 0, 1)$ be an involutive pseudo-BCK algebra, hence an involutive pseudo-BCK(pP) algebra, through this section.

Definition 8. We say that an operator $f : A \longrightarrow A$ is a *monadic operator* of $\mathcal{A}$ if the following condition (m) is satisfied:
(m) $(f(x^-))^\sim = (f(x^\sim))^-$.

Proposition 13. *Let $\exists : A \longrightarrow A$ be a monadic operator of $\mathcal{A}$. Define $\forall$ by:*

$$\forall x \stackrel{def}{=} (\exists x^-)^\sim = (\exists x^\sim)^-. \tag{61}$$

Then $\forall$ is a monadic operator of $\mathcal{A}$ and

$$\exists x = (\forall x^-)^\sim = (\forall x^\sim)^-. \tag{62}$$

Proof. (61) $\Rightarrow$ (62):
$\exists x \stackrel{(pDN)}{=} \exists((x^\sim)^-) \stackrel{(pDN)}{=} ([\exists((x^\sim)^-))]^\sim)^- \stackrel{(61)}{=} [\forall x^\sim]^-$ and
$\exists x \stackrel{(pDN)}{=} \exists((x^-)^\sim) \stackrel{(pDN)}{=} ([\exists((x^-)^\sim))]^-)^\sim \stackrel{(61)}{=} [\forall x^-]^\sim$.

Proposition 14. *Let $\forall : A \longrightarrow A$ be a monadic operator of $\mathcal{A}$. Define $\exists$ by:*

$$\exists x \stackrel{def}{=} (\forall x^-)^\sim = (\forall x^\sim)^-. \tag{63}$$

Then $\exists$ *is a monadic operator of* $\mathcal{A}$ *and*

$$\forall x = (\exists x^-)^\sim = (\exists x^\sim)^-. \tag{64}$$

Proof. (63) $\Rightarrow$ (64):
$\forall x \overset{(pDN)}{=} \forall((x^-)^\sim) \overset{(pDN)}{=} ([\forall((x^-)^\sim))]^-)^\sim \overset{(63)}{=} [\exists x^-]^\sim$ and
$\forall x \overset{(pDN)}{=} \forall((x^\sim)^-) \overset{(pDN)}{=} ([\forall((x^\sim)^-))]^\sim)^- \overset{(63)}{=} [\exists x^\sim]^-$.

Corollary 3. *The following hold: for all* $x \in A$,

$$(\forall x)^- = \exists x^-, \quad (\forall x)^\sim = \exists x^\sim, \tag{65}$$

$$(\exists x^\sim)^- = \forall (x^\sim)^- = \forall x, \quad (\exists x^-)^\sim = \forall (x^-)^\sim = \forall x. \tag{66}$$

and

$$(\exists x)^- = \forall x^-, \quad (\exists x)^\sim = \forall x^\sim, \tag{67}$$

$$(\forall x^\sim)^- = \exists (x^\sim)^- = \exists x, \quad (\forall x^-)^\sim = \exists (x^-)^\sim = \exists x \tag{68}$$

Proof. (65): $(\forall x)^- \overset{(61)}{=} [(\exists x^-)^\sim]^- \overset{(pDN)}{=} \exists x^-$, $(\forall x)^\sim \overset{(61)}{=} [(\exists x^\sim)^-]^\sim \overset{(pDN)}{=} \exists x^\sim$.

(66): $(\exists x^\sim)^- \overset{(61)}{=} \forall x \overset{(pDN)}{=} \forall (x^\sim)^-$, $(\exists x^-)^\sim \overset{(61)}{=} \forall x \overset{(pDN)}{=} \forall (x^-)^\sim$.

(67): $(\exists x)^- \overset{(63)}{=} [(\forall x^-)^\sim]^- \overset{(pDN)}{=} \forall x^-$, $(\exists x)^\sim \overset{(63)}{=} [(\forall x^\sim)^-]^\sim \overset{(pDN)}{=} \forall x^\sim$.

(68): $(\forall x^\sim)^- \overset{(63)}{=} \exists x \overset{(pDN)}{=} \exists (x^\sim)^-$, $(\forall x^-)^\sim \overset{(63)}{=} \exists x \overset{(pDN)}{=} \exists (x^-)^\sim$.

Proposition 15. *The following are equivalent:*
(E0) $\exists 0 = 0$,
(U0) $\forall 1 = 1$.

Proof. (E0) $\Longrightarrow$ (U0): $\forall 1 = \forall (0^-) \overset{(67)}{=} (\exists 0)^- \overset{(E0)}{=} 0^- = 1$.
(U0) $\Longrightarrow$ (E0): $\exists 0 = \exists (1^-) \overset{(65)}{=} (\forall 1)^- \overset{(U0)}{=} 1^- = 0$.

Proposition 16. *The following are equivalent:*
(E1) $\exists x \geq x$,
(U1) $\forall x \leq x$.

Proof. (E1) $\Longrightarrow$ (U1): By (E1), $x^- \leq \exists x^- \overset{(65)}{\Leftrightarrow} x^- \leq (\forall x)^- \Leftrightarrow \forall x \leq x$.

(U1) $\Longrightarrow$ (E1): By (U1), $\forall x^- \leq x^- \overset{(67)}{\Leftrightarrow} (\exists x)^- \leq x^- \Leftrightarrow x \leq \exists x$.

Theorem 4. *The following four double conditions* (E2), (E2'), (U2), (U2') *are equivalent:*

(E2)$_1$ $\exists((\exists x)^\sim \to y) = (\exists x)^\sim \to \exists y$, (E2)$_2$ $\exists((\exists x)^- \rightsquigarrow y) = (\exists x)^- \rightsquigarrow \exists y$;

(E2')$_1$ $\exists(x \oplus \exists y) = \exists x \oplus \exists y$, (E2')$_2$ $\exists(\exists x \oplus y) = \exists x \oplus \exists y$;

(U2)$_1$ $\forall((\forall x)^\sim \Rightarrow y) = (\forall x)^\sim \Rightarrow \forall y$, (U2)$_2$ $\forall((\forall x)^- \approx> y) = (\forall x)^- \approx> \forall y$;

(U2')$_1$ $\forall(x \odot \forall y) = \forall x \odot \forall y$, (U2')$_2$ $\forall(\forall x \odot y) = \forall x \odot \forall y$.

Proof.

(E2) $\Longrightarrow$ (E2'):

$\exists(x \oplus \exists y) \overset{(44)}{=} \exists[((\exists y)^- \odot x^-)^\sim] \overset{(39)}{=} \exists(x^- \rightsquigarrow \exists y) \overset{(46)}{=} \exists((\exists y)^\sim \to x) \overset{(E2)_1}{=} (\exists y)^\sim \to \exists x \overset{(39)}{=} ((\exists y)^\sim \odot (\exists x)^\sim)^- \overset{(44)}{=} \exists x \oplus \exists y$;

$\exists(\exists x \oplus y) \overset{(44)}{=} \exists[(y^- \odot (\exists x)^-)^\sim] \overset{(39)}{=} \exists((\exists x)^- \rightsquigarrow y) \overset{(E2)_2}{=} (\exists x)^- \rightsquigarrow \exists y \overset{(39)}{=} ((\exists y)^- \odot (\exists x)^-)^\sim \overset{(44)}{=} \exists x \oplus \exists y$.

(E2') $\Longrightarrow$ (U2):

$\forall((\forall x)^\sim \Rightarrow y) \overset{(41)}{=} \forall[((\forall x)^\sim \oplus y^\sim)^-] \overset{(67)}{=} [\exists((\forall x)^\sim \oplus y^\sim)]^- \overset{(65)}{=} [\exists(\exists x^\sim \oplus y^\sim)]^- \overset{(E2')_2}{=} (\exists x^\sim \oplus \exists y^\sim)^- \overset{(65)}{=} ((\forall x)^\sim \oplus (\forall y)^\sim)^- \overset{(41)}{=} (\forall x)^\sim \Rightarrow \forall y$;

$\forall((\forall x)^- \approx> y) \overset{(41)}{=} \forall[(y^- \oplus (\forall x)^-)^\sim] \overset{(67)}{=} [\exists(y^- \oplus (\forall x)^-)]^\sim \overset{(65)}{=} [\exists(y^- \oplus \exists x^-)]^\sim \overset{(E2')_1}{=} (\exists y^- \oplus \exists x^-)^\sim \overset{(65)}{=} ((\forall y)^- \oplus (\forall x)^-)^\sim \overset{(41)}{=} (\forall x)^- \approx> \forall y$.

(U2) $\Longrightarrow$ (U2'):

$\forall(x \odot \forall y) \overset{(42)}{=} \forall[((\forall y)^- \oplus x^-)^\sim] \overset{(41)}{=} \forall[x^- \approx> \forall y] \overset{(48)}{=} \forall((\forall y)^\sim \Rightarrow x) \overset{(U2)_1}{=} (\forall y)^\sim \Rightarrow \forall x \overset{(40)}{=} ((\forall y)^\sim \oplus (\forall x)^\sim)^- \overset{(42)}{=} \forall x \odot \forall y$;

$\forall(\forall x \odot y) \overset{(42)}{=} \forall[(y^- \oplus (\forall x)^-)^\sim] \overset{(41)}{=} \forall[(\forall x)^- \approx> y] \overset{(U2)_2}{=} (\forall x)^- \approx> \forall y \overset{(40)}{=} ((\forall y)^- \oplus (\forall x)^-)^\sim \overset{(42)}{=} \forall x \odot \forall y$.

(U2') $\Longrightarrow$ (E2):

$\exists((\exists x)^\sim \to y) \overset{(39)}{=} \exists[((\exists x)^\sim \odot y^\sim)^-] \overset{(65)}{=} [\forall((\exists x)^\sim \odot y^\sim)]^- \overset{(67)}{=} [\forall(\forall x^\sim \odot y^\sim)]^- \overset{(U2')_2}{=} [\forall x^\sim \odot \forall y^\sim]^- \overset{(67)}{=} ((\exists x)^\sim \odot (\exists y)^\sim)^- \overset{(39)}{=} (\exists x)^\sim \to \exists y$;

$\exists((\exists x)^- \rightsquigarrow y) \overset{(39)}{=} \exists[(y^- \odot (\exists x)^-)^\sim] \overset{(65)}{=} [\forall(y^- \odot (\exists x)^-)]^\sim \overset{(67)}{=} [\forall(y^- \odot \forall x^-)]^\sim \overset{(U2')_1}{=} [\forall y^- \odot \forall x^-]^\sim \overset{(67)}{=} ((\exists y)^- \odot (\exists x)^-)^\sim \overset{(39)}{=} (\exists x)^- \rightsquigarrow \exists y$.

Theorem 5. *The following four (double) conditions* (E3), (E3'), (U3), (U3') *are equivalent:*

(E3)$_1$ $\exists(x^\sim \to x) = (\exists x)^\sim \to \exists x$, (E3)$_2$ $\exists(x^- \rightsquigarrow x) = (\exists x)^- \rightsquigarrow \exists x$;
(E3') $\exists(x \oplus x) = \exists x \oplus \exists x$;
(U3)$_1$ $\forall(x^\sim \Rightarrow x) = (\forall x)^\sim \Rightarrow \forall x$, (U3)$_2$ $\forall(x^- \approx> x) = (\forall x)^- \approx> \forall x$;
(U3') $\forall(x \odot x) = \forall x \odot \forall x$.

Proof.

(E3) $\Longrightarrow$ (E3'):
$\exists(x \oplus x) \overset{(44)}{=} \exists[(x^- \odot x^-)^\sim] \overset{(39)}{=} \exists(x^- \rightsquigarrow x) \overset{(E3)_2}{=} (\exists x)^- \rightsquigarrow \exists x \overset{(39)}{=} ((\exists x)^- \odot (\exists x)^-)^\sim \overset{(44)}{=} \exists x \oplus \exists x$.

(E3') $\Longrightarrow$ (U3):
$\forall(x^\sim \Rightarrow x) \overset{(41)}{=} \forall[(x^\sim \oplus x^\sim)^-] \overset{(67)}{=} [\exists(x^\sim \oplus x^\sim)]^- \overset{(E3')}{=} [\exists x^\sim \oplus \exists x^\sim]^- \overset{(65)}{=} [(\forall x)^\sim \oplus (\forall x)^\sim]^- \overset{(41)}{=} (\forall x)^\sim \Rightarrow \forall x$;
$\forall(x^- \approx> x) \overset{(41)}{=} \forall[(x^- \oplus x^-)^\sim] \overset{(67)}{=} [\exists(x^- \oplus x^-)]^\sim \overset{(E3')}{=} [\exists x^- \oplus \exists x^-]^\sim \overset{(65)}{=} [(\forall x)^- \oplus (\forall x)^-]^\sim \overset{(41)}{=} (\forall x)^- \approx> \forall x$.

(U3) $\Longrightarrow$ (U3'):
$\forall(x \odot x) \overset{(42)}{=} \forall[(x^- \oplus x^-)^\sim] \overset{(41)}{=} \forall[x^- \approx> x] \overset{(U3)_2}{=} (\forall x)^- \approx> \forall x \overset{(41)}{=} ((\forall x)^- \oplus (\forall x)^-)^\sim \overset{(42)}{=} \forall x \odot \forall x$.

(U3') $\Longrightarrow$ (E3):
$\exists(x^\sim \to x) \overset{(39)}{=} \exists[(x^\sim \odot x^\sim)^-] \overset{(65)}{=} [\forall(x^\sim \odot x^\sim)]^- \overset{(U3')}{=} [\forall x^\sim \odot \forall x^\sim]^- \overset{(67)}{=} [(\exists x)^\sim \odot (\exists x)^\sim]^- \overset{(39)}{=} (\exists x)^\sim \to \exists x$;
$\exists(x^- \rightsquigarrow x) \overset{(39)}{=} \exists[(x^- \odot x^-)^\sim] \overset{(65)}{=} [\forall(x^- \odot x^-)]^\sim \overset{(U3')}{=} [\forall x^- \odot \forall x^-]^\sim \overset{(67)}{=} [(\exists x)^- \odot (\exists x)^-]^\sim \overset{(39)}{=} (\exists x)^- \rightsquigarrow \exists x$.

Theorem 6. *The following four double conditions* (E4), (E4'), (U4), (U4') *are equivalent*:
(E4)$_1$ $\exists(x \odot \exists y) = \exists x \odot \exists y$, (E4)$_2$ $\exists(\exists x \odot y) = \exists x \odot \exists y$;
(E4')$_1$ $\exists((\exists x)^\sim \Rightarrow y) = (\exists x)^\sim \Rightarrow \exists y$, (E4')$_2$ $\exists((\exists x)^- \approx> y) = (\exists x)^- \approx> \exists y$;
(U4)$_1$ $\forall(x \oplus \forall y) = \forall x \oplus \forall y$, (U4)$_2$ $\forall(\forall x \oplus y) = \forall x \oplus \forall y$;
(U4')$_1$ $\forall((\forall x)^\sim \to y) = (\forall x)^\sim \to \forall y$, (U4')$_2$ $\forall((\forall x)^- \rightsquigarrow y) = (\forall x)^- \rightsquigarrow \forall y$.

Proof.

(E4) $\Longrightarrow$ (E4'):
$\exists((\exists x)^\sim \Rightarrow y) \overset{(45)}{=} \exists[(((\exists x)^\sim)^\sim \to y^\sim)^-] \overset{(46)}{=} \exists[(y \rightsquigarrow (\exists x)^\sim)^-] \overset{(38)}{=} \exists(\exists x \odot y) \overset{(E4)_2}{=} \exists x \odot \exists y \overset{(38)}{=} (\exists y \rightsquigarrow (\exists x)^\sim)^- \overset{(46)}{=} [((\exists x)^\sim)^\sim \to (\exists y)^\sim]^- \overset{(45)}{=} (\exists x)^\sim \Rightarrow$

$\exists y$;

$\exists((\exists x)^- \approx> y) \overset{(45)}{=} \exists[(((\exists x)^-)^- \sim> y^-)^{\sim}] \overset{(46)}{=} \exists[(y \to (\exists x)^-)^{\sim}] \overset{(38)}{=} \exists(y \odot \exists x) \overset{(E4)_1}{=} \exists y \odot \exists x \overset{(38)}{=} (\exists y \to (\exists x)^-)^{\sim} \overset{(46)}{=} [((\exists x)^-)^- \sim> (\exists y)^-]^{\sim} \overset{(45)}{=} (\exists x)^- \approx> \exists y$.

(E4') $\Longrightarrow$ (U4):

$\forall(x \oplus \forall y) \overset{(40)}{=} \forall[(\forall y \approx> x^{\sim})^-] \overset{(67)}{=} [\exists(\forall y \approx> x^{\sim})]^- = [\exists((\exists y^{\sim})^- \approx> x^{\sim})]^- \overset{(E4')_2}{=} [(\exists y^{\sim})^- \approx> \exists x^{\sim}]^- \overset{(65)}{=} (\forall y \approx> (\forall x)^{\sim})^- \overset{(40)}{=} \forall x \oplus \forall y$;

$\forall(\forall x \oplus y) \overset{(40)}{=} \forall[(\forall x \Rightarrow y^-)^{\sim}] \overset{(67)}{=} [\exists(\forall x \Rightarrow y^-)]^{\sim} = [\exists((\exists x^-)^{\sim} \Rightarrow y^-)]^{\sim} \overset{(E4')_1}{=} [(\exists x^-)^{\sim} \Rightarrow \exists y^-]^{\sim} \overset{(65)}{=} (\forall x \Rightarrow (\forall y)^-)^{\sim} \overset{(40)}{=} \forall x \oplus \forall y$.

(U4) $\Longrightarrow$ (U4'):

$\forall((\forall x)^{\sim} \to y) \overset{(39)}{=} \forall[((\forall x)^{\sim} \odot y^{\sim})^-] \overset{(44)}{=} \forall[y \oplus \forall x] \overset{(U4)_1}{=} \forall y \oplus \forall x \overset{(44)}{=} ((\forall x)^{\sim} \odot (\forall y)^{\sim})^- \overset{(39)}{=} (\forall x)^{\sim} \to \forall y$;

$\forall((\forall x)^- \sim> y) \overset{(39)}{=} \forall[(y^- \odot (\forall x)^-)^{\sim}] \overset{(44)}{=} \forall[\forall x \oplus y] \overset{(U4)_2}{=} \forall x \oplus \forall y \overset{(44)}{=} ((\forall y)^- \odot (\forall x)^-)^{\sim} \overset{(39)}{=} (\forall x)^- \sim> \forall y$.

(U4') $\Longrightarrow$ (E4):

$\exists(x \odot \exists y) \overset{(38)}{=} \exists[(\exists y \sim> x^{\sim})^-] \overset{(65)}{=} [\forall(\exists y \sim> x^{\sim})]^- = [\forall((\forall y^{\sim})^- \sim> x^{\sim})]^- \overset{(U4')_2}{=} [(\forall y^{\sim})^- \sim> \forall x^{\sim}]^- \overset{(67)}{=} [\exists y \sim> (\exists x)^{\sim}]^- \overset{(38)}{=} \exists x \odot \exists y$;

$\exists(\exists x \odot y) \overset{(38)}{=} \exists[(\exists x \to y^-)^{\sim}] \overset{(65)}{=} [\forall(\exists x \to y^-)]^{\sim} = [\forall((\forall x^-)^{\sim} \to y^-)]^{\sim} \overset{(U4')_1}{=} [(\forall x^-)^{\sim} \to \forall y^-]^{\sim} \overset{(67)}{=} [\exists x \to (\exists y)^-]^{\sim} \overset{(38)}{=} \exists x \odot \exists y$.

Theorem 7. *The following four (double) conditions* (E5), (E5'), (U5), (U5') *are equivalent:*

(E5) $\exists(x \odot x) = \exists x \odot \exists x$;

(E5')$_1$ $\exists(x^{\sim} \Rightarrow x) = (\exists x)^{\sim} \Rightarrow \exists x$, (E5')$_2$ $\exists(x^- \approx> x) = (\exists x)^- \approx> \exists x$;

(U5) $\forall(x \oplus x) = \forall x \oplus \forall x$;

(U5')$_1$ $\forall(x^{\sim} \to x) = (\forall x)^{\sim} \to \forall x$, (U5')$_2$ $\forall(x^- \sim> x) = (\forall x)^- \sim> \forall x$.

Proof.

(E5) $\Longrightarrow$ (E5'):

$\exists(x^{\sim} \Rightarrow x) \overset{(45)}{=} \exists[((x^{\sim})^{\sim} \to x^{\sim})^-] \overset{(46)}{=} \exists[(x \sim> x^{\sim})^-] \overset{(38)}{=} \exists(x \odot x) \overset{(E5)}{=} \exists x \odot \exists x \overset{(38)}{=} (\exists x \sim> (\exists x)^{\sim})^- \overset{(46)}{=} (((\exists x)^{\sim})^{\sim} \to (\exists x)^{\sim})^- \overset{(45)}{=} (\exists x)^{\sim} \Rightarrow \exists x$;

$\exists(x^- \approx> x) \overset{(45)}{=} \exists[((x^-)^- \sim> x^-)^{\sim}] \overset{(46)}{=} \exists[(x \to x^-)^{\sim}] \overset{(38)}{=} \exists(x \odot x) \overset{(E5)}{=} \exists x \odot \exists x \overset{(38)}{=} (\exists x \to (\exists x)^-)^{\sim} \overset{(46)}{=} (((\exists x)^-)^- \sim> (\exists x)^-)^{\sim} \overset{(45)}{=} (\exists x)^- \approx> \exists x$.

(E5') $\Longrightarrow$ (U5):

$\forall(x \oplus x) \overset{(40)}{=} \forall[(x \Rightarrow x^-)^\sim] \overset{(67)}{=} [\exists(x \Rightarrow x^-)]^\sim \overset{(pDN)}{=} [\exists((x^-)^\sim \Rightarrow x^-)]^\sim \overset{(E5')_1}{=}$ $[(\exists x^-)^\sim \Rightarrow \exists x^-]^\sim \overset{(65)}{=} [\forall x \Rightarrow (\forall x)^-]^\sim \overset{(40)}{=} \forall x \oplus \forall x$.

(U5) $\Longrightarrow$ (U5'):

$\forall(x^\sim \to x) \overset{(39)}{=} \forall[(x^\sim \odot x^\sim)^-] \overset{(44)}{=} \forall(x \oplus x) \overset{(U5)}{=} \forall x \oplus \forall x \overset{(44)}{=} ((\forall x)^\sim \odot (\forall x)^\sim)^- \overset{(39)}{=}$ $(\forall x)^\sim \to \forall x$;

$\forall(x^- \rightsquigarrow x) \overset{(39)}{=} \forall[(x^- \odot x^-)^\sim] \overset{(44)}{=} \forall(x \oplus x) \overset{(U5)}{=} \forall x \oplus \forall x \overset{(44)}{=} ((\forall x)^- \odot$ $(\forall x)^-)^\sim \overset{(39)}{=} (\forall x)^- \rightsquigarrow \forall x$.

(U5') $\Longrightarrow$ (E5):

$\exists(x \odot x) \overset{(38)}{=} \exists[(x \to x^-)^\sim] \overset{(65)}{=} [\forall(x \to x^-)]^\sim \overset{(pDN)}{=} [\forall((x^-)^\sim \to x^-)]^\sim \overset{(U5')_1}{=}$ $[(\forall x^-)^\sim \to \forall x^-]^\sim \overset{(67)}{=} [\exists x \to (\exists x)^-]^\sim \overset{(38)}{=} \exists x \odot \exists x$.

Definition 9. Let $\mathcal{A} = (A, \leq, \to, \rightsquigarrow, 0, 1)$ be an involutive pseudo-BCK algebra, hence an involutive pseudo-BCK(pP) algebra.

(1) An *existential quantifier* on $\mathcal{A}$ is a monadic operator $\exists : A \longrightarrow A$ which verifies the above conditions (E0), (E1), (E2), (E3),(E4) and (E5).

(2) A *universal quantifier* on $\mathcal{A}$ is a monadic operator $\forall : A \longrightarrow A$ which verifies the above conditions (U0), (U1), (U2'), (U3'),(U4') and (U5').

(3) A *monadic involutive pseudo-BCK algebra* is a pair $(\mathcal{A}, \exists)$, where $\exists$ is an existential quantifier on $\mathcal{A}$ or a pair $(\mathcal{A}, \forall)$, where $\forall$ is a universal quantifier on $\mathcal{A}$.

Remark 2. Note that for the particular case of a monadic MV algbera $(\mathcal{A}, \exists)$, we obtain the definition from [20].

Proposition 17.

(1) Let $(\mathcal{A}, \exists)$ be a monadic involutive pseudo-BCK algebra and $\forall$ defined by (61). Then $(\mathcal{A}, \forall)$ is a monadic involutive pseudo-BCK algebra termwise equivalent with $(\mathcal{A}, \exists)$.

(2) Let $(\mathcal{A}, \forall)$ be a monadic involutive pseudo-BCK algebra and $\exists$ defined by (63). Then $(\mathcal{A}, \exists)$ is a monadic involutive pseudo-BCK algebra termwise equivalent with $(\mathcal{A}, \forall)$.

Proof. By Propositions 15 and 16 and Theorems 4, 5, 6, 7.

Proposition 18. *Let $(\mathcal{A}, \exists)$ be a monadic involutive pseudo-BCK algebra and $\forall$ defined by (61). Then the following properties hold:*

$$\exists\exists x = \exists x, \qquad \forall\forall x = \forall x, \tag{69}$$

$$\exists 1 = 1, \qquad \forall 0 = 0, \tag{70}$$

$$\exists(\exists x \odot \exists y) = \exists x \odot \exists y, \qquad \exists(\exists x \oplus \exists y) = \exists x \oplus \exists y, \tag{71}$$

$$\forall\exists x = \exists x, \qquad \exists\forall x = \forall x, \tag{72}$$

$$\exists(\exists x)^- = (\exists x)^-, \qquad \exists(\exists x)^\sim = (\exists x)^\sim, \tag{73}$$

$$\exists(\forall x \rightsquigarrow y) = \forall x \rightsquigarrow \exists y, \qquad \exists(\forall x \to y) = \forall x \to \exists y, \tag{74}$$

$$\exists(x \to \forall y) = \forall x \to \forall y, \qquad \exists(x \rightsquigarrow \forall y) = \forall x \rightsquigarrow \forall y, \tag{75}$$

$$\exists(\exists x \to y) = \exists x \to \exists y, \qquad \exists(\exists x \rightsquigarrow y) = \exists x \rightsquigarrow \exists y, \tag{76}$$

$$\forall(\exists x \rightsquigarrow y) = \exists x \rightsquigarrow \forall y, \qquad \forall(\exists x \to y) = \exists x \to \forall y, \tag{77}$$

$$\forall(x \to \exists y) = \exists x \to \exists y, \qquad \forall(x \rightsquigarrow \exists y) = \exists x \rightsquigarrow \exists y, \tag{78}$$

$$\forall(\forall x \to y) = \forall x \to \forall y, \qquad \forall(\forall x \rightsquigarrow y) = \forall x \rightsquigarrow \forall y, \tag{79}$$

$$x \leq y \Longrightarrow (\exists x \leq \exists y \quad and \quad \forall x \leq \forall y). \tag{80}$$

Proof.

(69): $\exists\exists x = \exists(\exists x \oplus 0) \stackrel{(E2')}{=} \exists x \oplus \exists 0 = \exists x \oplus 0 = \exists x$; the other has a similar proof.

(70): By (E1), $1 \leq \exists 1$, hence $\exists 1 = 1$; by (U1), $\forall 0 \leq 0$, hence $\forall 0 = 0$.

(71): $\exists(\exists x \odot \exists y) \stackrel{(E4)}{=} \exists\exists x \odot \exists y = \exists x \odot \exists y$; $\exists(\exists x \oplus \exists y) \stackrel{(E2')}{=} \exists\exists x \oplus \exists y = \exists x \oplus \exists y$.

(72): By (U1), $\forall\exists x \leq \exists x$; on the other hand, $1 = \forall 1 = \forall(\exists x \to \exists x) = \exists x \to \forall\exists x$, hence $\exists x \leq \forall\exists x$; thus, $\forall\exists x = \exists x$; $\exists\forall x = \exists((\exists x^\sim)^-) = [\forall(\exists x^\sim)]^- = (\exists x^\sim)^- = \forall x$.

(73): $\exists(\exists x)^- = \exists(\forall x^-) = \forall x^- = (\exists x)^-$; $\exists(\exists x)^\sim = \exists(\forall x^\sim) = \forall x^\sim = (\exists x)^\sim$.

(74): $\exists(\forall x \rightsquigarrow y) \overset{(pDN)}{=} \exists(\forall((x^{\sim})^{-}) \rightsquigarrow y) = \exists((\exists x^{\sim})^{-} \rightsquigarrow y) \overset{(E2)}{=} (\exists x^{\sim})^{-} \rightsquigarrow \exists y = \forall x \rightsquigarrow \exists y$; $\exists(\forall x \to y) \overset{(pDN)}{=} \exists(\forall((x^{-})^{\sim}) \to y) = \exists((\exists x^{-})^{\sim} \to y) \overset{(E2)}{=} (\exists x^{-})^{\sim} \to \exists y = \forall x \to \exists y$.

(75): $\exists(x \to \forall y) \overset{(pDN)}{=} \exists((x^{-})^{\sim} \to \forall y) = \exists((\forall y)^{-} \rightsquigarrow x^{-}) = \exists((\exists(\forall y))^{-} \rightsquigarrow x^{-}) \overset{(E2)}{=} (\exists(\forall y))^{-} \rightsquigarrow \exists x^{-} = (\forall y)^{-} \rightsquigarrow (\forall x)^{-} = ((\forall x)^{-})^{\sim} \to ((\forall y)^{-})^{\sim} = \forall x \to \forall y$; the other has a similar proof.

(76): $\exists(\exists x \to y) \overset{(pDN)}{=} \exists((\forall x^{-})^{\sim} \to y) = \exists(y^{-} \rightsquigarrow \forall x^{-}) = \forall y^{-} \rightsquigarrow \forall x^{-} = (\exists y)^{-} \rightsquigarrow (\exists x)^{-} = \exists x \to \exists y$; the other has a similar proof.

(77):$\forall(\exists x \rightsquigarrow y) \overset{(pDN)}{=} \forall[\exists(x^{\sim})^{-} \rightsquigarrow y] = \forall[(\forall x^{\sim})^{-} \rightsquigarrow y] \overset{(U4')}{=} (\forall x^{\sim})^{-} \rightsquigarrow \forall y = \exists x \rightsquigarrow \forall y$; $\forall(\exists x \to y) \overset{(pDN)}{=} \forall[\exists(x^{-})^{\sim} \to y] = \forall[(\forall x^{-})^{\sim} \to y] \overset{(U4')}{=} (\forall x^{-})^{\sim} \to \forall y = \exists x \to \forall y$.

(78): $\forall(x \to \exists y) \overset{(pDN)}{=} \forall((x^{-})^{\sim} \to \exists y) = \forall((\exists y)^{-} \rightsquigarrow x^{-}) = \forall((\forall \exists y)^{-} \rightsquigarrow x^{-}) \overset{(U4')}{=} (\forall \exists y)^{-} \rightsquigarrow \forall x^{-} = (\exists y)^{-} \rightsquigarrow \forall x^{-} = (\exists y)^{-} \rightsquigarrow (\exists x)^{-} = ((\exists x)^{-})^{\sim} \to ((\exists y)^{-})^{\sim} \overset{(pDN)}{=} \exists x \to \exists y$; the other has a similar proof.

(79): $\forall(\forall x \to y) \overset{(pDN)}{=} \forall((\exists x^{-})^{\sim} \to y) = \forall(y^{-} \rightsquigarrow \exists x^{-}) = \exists y^{-} \rightsquigarrow \exists x^{-} = (\forall y)^{-} \rightsquigarrow (\forall x)^{-} = \forall x \to \forall y$; $\forall(\forall x \rightsquigarrow y) \overset{(pDN)}{=} \forall((\exists x^{\sim})^{-} \rightsquigarrow y) = \forall(y^{\sim} \to \exists x^{\sim}) = \exists y^{\sim} \to \exists x^{\sim} = (\forall y)^{\sim} \to (\forall x)^{\sim} = \forall x \rightsquigarrow \forall y$.

(80): Let $x \leq y$; then $x \leq y \leq \exists y$, by (E1); but $x \leq \exists y \Leftrightarrow x \to \exists y = 1$; then, $\exists x \to \exists y = \forall(x \to \exists y) = \forall 1 = 1$, hence $\exists x \leq \exists y$. On the other hand, $\forall x \leq x \leq y$, by (U1); but $\forall x \leq y \Leftrightarrow \forall x \to y = 1$; then, $\forall x \to \forall y = \forall(\forall x \to y) = \forall 1 = 1$, hence $\forall x \leq \forall y$.

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