ON CERTAIN PROPERTIES OF THE CLASS OF GENERALIZED P-VALENT NON-BAZILEVIC FUNCTIONS

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Abstract. In this paper, we introduce a class $N_k(p,\lambda,\mu,\rho)$. We use a method of the mathematical induction and study some of the interesting properties of the operators

$$I_n^\alpha(f(z)) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z t^\alpha (I_{n-1}^\alpha(f(t))dt,$$

where $n \in \mathbb{N}$, $\alpha > -1$, in the class $N_k(p,\lambda,\mu,\rho)$.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{m=p+1}^{\infty} a_m z^m, \quad p \in \mathbb{N} = \{1,2,3,...\} \tag{1.1}$$

which are analytic and p-valent in the unit disk

$$E = \{z : z \in \mathbb{C}, |z| < 1\}.$$

Let $P_k(\rho)$ be the class of functions $h(z)$ analytic in E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re} h(z) - \rho}{1-\rho} \right| d\theta \le k\pi \tag{1.2}$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho \leq 1$. This class has been introduced in [7]. We note, for $\rho = 0$ we obtain the class defined and studied in [8], and for $\rho = 0, k = 2$, we have the well known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ. From (1.2) we can easily deduce that $h \in P_k(\rho)$ if, and only if, there exists $h_1, h_2 \in P(\rho)$ such that for $z \in E$,

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z). \tag{1.3}$$

Assume that $0 < \mu < 1$, a function $f(z) \in A$ is in $N(\mu)$ if and only if

$$\operatorname{Re}\left\{ f'(z) \left(\frac{z}{f(z)}\right)^{1+\mu} \right\} > 0, \ z \in E. \tag{1.4}$$

$N(\mu)$ was introduced by Obradovic [5] recently, he called this class of functions to be of non-Bazilevic type. Until now, the class was studied in a direction of finding necessary conditions over μ that embeds this class into the class of univalent functions or its subclass, which is still an open problem.

Definition 1.1. *Let $f(z) \in A(p)$. Then $f \in N_k(p, \lambda, \alpha, \rho)$, if and only if,*

$$\left\{ (1+\lambda) \left(\frac{z^p}{f(z)}\right)^{\mu} - \lambda \frac{zf'(z)}{pf(z)} \left(\frac{z^p}{f(z)}\right)^{\mu} \right\} \in P_k(\rho), \ z \in E,$$

where $0 < \mu < 1, \lambda \in N^*, k \geq 2$ and $0 \leq \rho < 1$. The powers are understood as principal values. For $k = 2$ and with different choices of p, λ, μ, ρ, these classes have been studied in [5, 9]. In particular $N_2(1, -1, \mu, 0)$ is the class of non-Bazilevic function studied in [5].

We now consider the integral operator defined in [2], as follows:

$$I_n^{\alpha}(f(z)) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z t^{\alpha}(I_{n-1}(f(t)))dt, \ n \in \mathbb{N},$$

with initial gauss $I_0^{\alpha}(f(z)) = I_0(f(z)) = \left(\frac{z^p}{f(z)}\right)^{\alpha}$.

For $\alpha = 0$ the operator I_n^0 was introduced and studied in [3].

2. Preliminaries and main results

In order to prove our main result we shall need the following result.

Lemma 2.1. [1] *Let* $h(z) = 1 + c_1 z + c_2 z^2 + ...$ *be analytic in* E *and let*

$$\left\{h(z) + \frac{zh'(z)}{\gamma}\right\} \in P_k(\rho), \quad for\ \mathrm{Re}\gamma > 0, k \geq 2, 0 \leq \rho < 1.$$

Then

$$h(z) \in P_k(\rho_0), z \in E.$$

where

$$\rho_0 = \frac{\mathrm{Re}\gamma + 2\rho|\gamma|^2}{\mathrm{Re}\gamma + 2|\gamma|^2}. \tag{2.1}$$

Theorem 2.1. *Let* $Re\gamma > 0$ *and* $f \in N_k(p, \lambda, \mu, \rho)$. *Then, for* $z \in E, I_n^\alpha f \in P_k(\rho_n)$, *where*

$$\rho_n = \frac{[(1+a)^n - a^n] + a^n \rho_0}{(1+a)^n}, \tag{2.2}$$

with $a = 2(\alpha + 1)$ *and*

$$\rho_0 = \frac{|\lambda|^2 + 2\rho p\mu\mathrm{Re}\lambda}{|\lambda|^2 + 2p\mu\mathrm{Re}\lambda}.$$

Proof. We use the method of Mathematical induction to prove this result. Let

$$H_0(z) = \left(\frac{z^p}{f(z)}\right)^\mu,$$

where $0 < \mu < 1$.

By choosing a principal branch of $\left(\frac{z^p}{f(z)}\right)^\mu$ we note that $H_0(z)$ is analytic in E with $H_0(0) = 1$. Now by a simple computation, we have

$$\left\{(1+\lambda)\left(\frac{z^p}{f(z)}\right)^\mu - \lambda\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^\mu\right\} = \left\{H_0(z) + \frac{\lambda z H_0'(z)}{p\mu}\right\} \in P_k(\rho),$$

where $z \in E$, and from Lemma 2.1, with $\gamma = \frac{p\mu}{\lambda}$, and $\mathrm{Re}\gamma = \frac{p\mu}{\mathrm{Re}\lambda} > 0$, it implies that $H_0(z) \in P_k(\rho_0)$, where

$$\rho_0 = \frac{|\lambda|^2 + 2p\mu\rho\mathrm{Re}\lambda}{|\lambda|^2 + 2p\mu\mathrm{Re}\lambda}.$$

Thus the result is true for $n = 0$.

For $n = 1$, we proceed as follows:

$$H_1(z) = I_1^\alpha(f(z)) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z t^\alpha (I_0^\alpha(f(t)))dt, \; n \in \mathbb{N}^*, \alpha > 1.$$

And so

$$\left\{ H_1(z) + \frac{zH_1'(z)}{\alpha+1} \right\} \in P_k(\rho_0), \; z \in E.$$

Using again Lemma 2.1 with $\gamma = \alpha + 1$ we obtain that

$$H_1(z) = I_1^\alpha(f(z)) \in P_k(\rho_1),$$

where

$$\rho_1 = \frac{1 + 2(\alpha+1)\rho_0}{1 + 2(\alpha+1)}$$

for $z \in E$, which shows that (2.2) holds true for $n = 1$.

Next we assume that the condition (2.2) holds true for $n = m$. This implies that

$$H_m(z) = I_m^\alpha(f(z)) \in P_k(\rho_m)$$

and

$$\rho_m = \frac{[(1+a)^m - a^m] + a^m\rho_0}{(1+a)^m}, \; a = 2(1+\alpha)$$

Proceeding as before we see that

$$\left\{ H_{m+1}(z) + \frac{zH_{m+1}'(z)}{\alpha+1} \right\} = H_m(z) \in P_k(\rho_m), \; z \in E,$$

where

$$H_{m+1}(z) = I_{m+1}^\alpha(f(z)) = \frac{\alpha+1}{z^{\alpha+1}} \int_0^z t^\alpha (I_m(f(t)))dt, \; \alpha > -1.$$

This implies that

$$H_{m+1}(z) \in P_k(\rho_{m+1})$$

where

$$\begin{aligned}
\rho_{m+1} &= \frac{1 + a\rho_m}{1+a} \\
\rho_{m+1} &= \frac{1 + a\left[\dfrac{\{(1+a)^m - a^m\} + a^m\rho_0}{(1+a)^m}\right]}{1+a} \\
\rho_{m+1} &= \frac{\left[(1+a)^{m+1} - a^{m+1}\right] + a^m}{(1+a)^{m+1}}, \quad a = 2(1+\alpha),
\end{aligned}$$

and

$$\rho_0 = \frac{|\lambda|^2 + 2\rho p\mu \mathrm{Re}\lambda}{|\lambda|^2 + 2p\mu \mathrm{Re}\lambda}$$

Therefore, we conclude that $I_n^\alpha f \in P_k(\rho_n)$ for any integer $n \in \mathbb{N} \cup \{0\}$.

For certain choices of ρ, μ, λ and k we obtain some partial results discussed in [4, 6].

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