INTEGRAL MEANS FOR CERTAIN SUBCLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. We introduce some generalized subclasses $TS_\gamma(f, g; \alpha, \beta)$ of uniformly starlike and convex functions, we settle the Silverman's conjecture for the integral means inequality. In particular, we obtain integral means inequalities for various classes of uniformly $\beta-$starlike and uniformly $\beta-$convex functions in the unit disc.

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1.Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

that are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. Let $f(z) \in S$ be given by (1.1) and $\Phi(z) \in S$ be given by

$$\Phi(z) = z + \sum_{k=2}^{\infty} c_k z^k, \tag{1.2}$$

then for analytic functions f and Φ with $f(0) = \Phi(0)$, f is said to be subordinate to Φ, denoted by $f \prec \Phi$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = \Phi(w(z))$, for all $z \in U$.
The Hadamard product (or convolution) $f * \Phi$ of f and Φ is defined (as usual) by

$$(f * \Phi)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k = (\Phi * f)(z). \tag{1.3}$$

Following Goodman ([7] and [8]), Ronning ([17] and [18]) introduced and studied the following subclasses:

(i) A function $f(z)$ of the form (1.1) is said to be in the class $S_p(\alpha,\beta)$ of uniformly $\beta-$starlike functions if it satisfies the condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right| \qquad (z\in U), \tag{1.4}$$

where $-1\leq\alpha<1$ and $\beta\geq 0$.

(ii) A function $f(z)$ of the form (1.1) is said to be in the class $UCV(\alpha,\beta)$ of uniformly $\beta-$convex functions if it satisfies the condition:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in U), \tag{1.5}$$

where $-1\leq\alpha<1$ and $\beta\geq 0$. We also observe that

$$S_p(\alpha,0)=T^*(\alpha), \quad UCV(\alpha,0)=C(\alpha)$$

are, respectively, well-known subclasses of starlike functions of order α and convex functions of order α. Indeed it follows from (1.4) and (1.5) that

$$f(z)\in UCV(\alpha,\beta) \iff zf'(z)\in S_p(\alpha,\beta). \tag{1.6}$$

For $-1\leq\alpha<1$, $0\leq\gamma\leq 1$ and $\beta\geq 0$, we let $S_\gamma(f,g;\alpha,\beta)$ be the subclass of S consisting of functions $f(z)$ of the form (1.1) and functions $g(z)$ given by

$$g(z)=z+\sum_{k=2}^{\infty}b_kz^k \qquad (b_k\geq 0), \tag{1.7}$$

and satisfying the analytic criterion:

$$\operatorname{Re}\left\{\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-\alpha\right\}$$
$$>\beta\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right|. \tag{1.8}$$

Let T denote the subclass of S consisting of functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_kz^k \qquad (a_k\geq 0)\ . \tag{1.9}$$

Further, we define the class $TS_\gamma(f,g;\alpha,\beta)$ by

$$TS_\gamma(f,g;\alpha,\beta) = S_\gamma(f,g;\alpha,\beta) \cap T. \tag{1.10}$$

We note that:
(i) $TS_0(f,\frac{z}{(1-z)};\alpha,1) = S_pT(\alpha)$ and $TS_0(f,\frac{z}{(1-z)^2};\alpha,1) =$ $TS_1(f,\frac{z}{(1-z)};\alpha,1) = UCT(\alpha)$ $(-1 \le \alpha < 1)$ (see Bharati et al. [4]);
(ii) $TS_1(f,\frac{z}{(1-z)};0,\beta) = UCT(\beta)$ $(\beta \ge 0)$ (see Subramanian et al. [24]);
(iii) $TS_0(f,z+\sum\limits_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k;\alpha,\beta) = TS\,(\alpha,\beta)\,(-1 \le \alpha < 1,\ \beta \ge 0,\ c \ne 0,-1,-2,...)$ (see Murugusundaramoorthy and Magesh [12] and [13]);
(iv) $TS_0(f,z+\sum\limits_{k=2}^{\infty}k^n z^k;\alpha,\beta) = TS\,(n,\alpha,\beta)\,(-1 \le \alpha < 1,\ \beta \ge 0,\ n \in N_0 = N \cup \{0\},\ N = \{1,2,...\})$ (see Rosy and Murugusundaramoorthy [19]);
(v) $TS_0(f,z+\sum\limits_{k=2}^{\infty}\binom{k+\lambda-1}{\lambda}z^k;\alpha,\beta) = D\,(\alpha,\beta,\lambda)\,(-1 \le \alpha < 1, \beta \ge 0, \lambda > -1)$ (see Shams et al. [23]);

(vi) $TS_0(f,z+\sum\limits_{k=2}^{\infty}[1+\lambda(k-1)]^n z^k;\alpha,\beta) = TS_\lambda\,(n,\alpha,\beta)$ $(-1 \le \alpha < 1,\ \beta \ge 0,\ \lambda \ge 0,\ n \in N_0)$ (see Aouf and Mostafa [2]);
(vii) $TS_\gamma(f,z+\sum\limits_{k=2}^{\infty}\frac{(a)_{k-1}}{(c)_{k-1}}z^k;\alpha,\beta) = TS\,(\gamma,\alpha,\beta)\,(-1 \le \alpha < 1,\ \beta \ge 0,\ 0 \le \gamma \le 1,\ c \ne 0,-1,-2,...)$ (see Murugusundaramoorthy et al. [14]);
(viii) $TS_0(f,g;\alpha,\beta) = H_T\,(g,\alpha,\beta)$ $(-1 \le \alpha < 1,\ \beta \ge 0)$ (see Raina and Bansal [16]);
(viii) $TS_\gamma(f,z+\sum\limits_{k=2}^{\infty}\Gamma_k z^k;\alpha,\beta) = TS_q^s(\gamma,\alpha,\beta)$ (see Ahuja et al. [1]), where

$$\Gamma_k = \frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}...(\beta_s)_{k-1}}\frac{1}{(k-1)!} \tag{1.11}$$

$(\alpha_i > 0,\ i = 1,...,q;\ \beta_j > 0,\ j = 1,...,s;\ q \le s+1;\ q,\ s \in N_0)$.
Also we note that
(i) $TS_\gamma(f,z+\sum\limits_{k=2}^{\infty}k^n z^k;\alpha,\beta) = TS_\gamma\,(n,\alpha,\beta)$

$$= \left\{ f \in T : \mathrm{Re}\left\{ \frac{(1-\gamma)z(D^n f(z))' + \gamma z(D^{n+1}f(z))'}{(1-\gamma)D^n f(z) + \gamma D^{n+1}f(z)} - \alpha \right\} \right.$$

$$> \beta \left| \frac{(1-\gamma)z(D^n f(z))' + \gamma z(D^{n+1} f(z))'}{(1-\gamma)D^n f(z) + \gamma D^{n+1} f(z)} - 1 \right|, (-1 \leq \alpha < 1,\ \beta \geq 0,\ n \in N_0,\ z \in U) \Bigg\} \tag{1.12}$$

(ii) $TS_\gamma(f, z + \sum\limits_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) z^k; \alpha, \beta) =$

$$= TS_\gamma(c, \alpha, \beta) = \left\{ f \in T : \mathrm{Re}\left\{ \frac{z(J_c f(z))' + \gamma z^2 (J_c f(z))''}{(1-\gamma)J_c f(z) + \gamma z(J_c f(z))'} - \alpha \right\}\right.$$

$$\left. > \beta \left| \frac{z(J_c f(z))' + \gamma z^2 (J_c f(z))''}{(1-\gamma)J_c f(z) + \gamma z(J_c f(z))'} - 1 \right|, 0 \leq \gamma \leq 1, -1 \leq \alpha < 1, \beta \geq 0, c > -1, z \in U \right\}; \tag{1.13}$$

where J_c is a Bernardi operator [3], defined by

$$J_c f(z) = \frac{c+1}{z^c} \int\limits_0^z t^{c-1} f(t)dt = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) a_k z^k.$$

Note that the operator $J_1 f(z)$ was studied earlier by Libera [9] and Livingston [11];

(iii) $TS_\gamma(f, z + \sum\limits_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}} z^k; \alpha, \beta) = TS_\gamma(\mu, \lambda; \alpha, \beta)$

$$= \left\{ f \in T : \mathrm{Re}\left\{ \frac{z(I_{\lambda,\mu} f(z))' + \gamma z^2 (I_{\lambda,\mu} f(z))''}{(1-\gamma)I_{\lambda,\mu} f(z) + \gamma z(I_{\lambda,\mu} f(z))'} - \alpha \right\}\right.$$

$$> \beta \left| \frac{z(I_{\lambda,\mu} f(z))' + \gamma z^2 (I_{\lambda,\mu} f(z))''}{(1-\gamma)I_{\lambda,\mu} f(z) + \gamma z(I_{\lambda,\mu} f(z))'} - 1 \right|,$$

$$\left. (0 \leq \gamma \leq 1,\ -1 \leq \alpha < 1,\ \beta \geq 0,\ \lambda > -1,\ \mu > 0,\ z \in U) \right\}; \tag{1.14}$$

where $I_{\lambda,\mu}$ is a Choi-Saigo-Srivastava operator [6], defined by

$$I_{\lambda,\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}} a_k z^k \quad (\lambda > -1;\ \mu > 0);$$

(iv) $TS_{\gamma}(f, z + \sum_{k=2}^{\infty} \frac{(c)_{k-1}}{(a)_{k-1}} \frac{(\lambda+1)_{k-1}}{(1)_{k-1}} z^k; \alpha, \beta) = TS_{\gamma}(a, c, \lambda; \alpha, \beta)$

$$= \Bigg\{ f \in T : \; Re\left\{ \frac{z(I^{\lambda}(a,c)f(z))' + \gamma z^2 (I^{\lambda}(a,c)f(z))''}{(1-\gamma)I^{\lambda}(a,c)f(z) + \gamma z(I^{\lambda}(a,c)f(z))'} - \alpha \right\}$$

$$> \beta \left| \frac{z(I^{\lambda}(a,c)f(z))' + \gamma z^2 (I^{\lambda}(a,c)f(z))''}{(1-\gamma)I^{\lambda}(a,c)f(z) + \gamma z(I^{\lambda}(a,c)f(z))'} - 1 \right|,$$

$$\left(0 \le \gamma \le 1, \; -1 \le \alpha < 1, \; \beta \ge 0, \; a, c \in R \backslash Z_0^-, \; \lambda > -1, \; z \in U\right) \Bigg\}; \tag{1.15}$$

where $I^{\lambda}(a,c)$ is a Cho-Kwon-Srivastava operator [5], defined by

$$I^{\lambda}(a,c)f(z) = z + \sum_{k=2}^{\infty} \frac{(c)_{k-1}}{(a)_{k-1}} \frac{(\lambda+1)_{k-1}}{(1)_{k-1}} a_k z^k;$$

(v) $TS_{\gamma}(f, z + \sum_{k=2}^{\infty} \frac{(2)_{k-1}}{(n+1)_{k-1}} z^k; \alpha, \beta) = TS_{\gamma}(n; \alpha, \beta)$

$$= \Bigg\{ f \in T : \; Re\left\{ \frac{z(I_n f(z))' + \gamma z^2 (I_n f(z))''}{(1-\gamma)I_n f(z) + \gamma z(I_n f(z))'} - \alpha \right\}$$

$$> \beta \left| \frac{z(I_n f(z))' + \gamma z^2 (I_n f(z))''}{(1-\gamma)I_n f(z) + \gamma z(I_n f(z))'} - 1 \right|, 0 \le \gamma \le 1, -1 \le \alpha < 1, \beta \ge 0, n > -1, z \in U \Bigg\}; \tag{1.16}$$

where I_n is a Noor integral operator [15], defined by

$$I_n f(z) = z + \sum_{k=2}^{\infty} \frac{(2)_{k-1}}{(n+1)_{k-1}} a_k z^k \quad (n > -1).$$

In [20], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [21] and settled in [22], that

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\eta} d\theta \le \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^{\eta} d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [22], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

In this paper, we prove Silverman's conjecture for the functions in the class $TS_\gamma(f,g;\alpha,\beta)$. By taking appropriate choises of the function g, we obtain the integral means inequalities for several known as well as new subclasses of uniformly convex and uniformly starlike functions in U. In fact, these results also settle the Silverman's conjecture for several other subclasses of T.

2.Lemmas and their proofs

To prove our main results, we need the following lemmas.

Lemma 1. *A function $f(z)$ of the form (1.1) is in the class $TS_\gamma(f,g;\alpha,\beta)$ if*

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]|a_k|\,b_k \le 1-\alpha, \tag{2.1}$$

where $-1 \le \alpha < 1$, $\beta \ge 0$ and $0 \le \gamma \le 1$.

Proof. It suffices to show that

$$\beta\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right|-\operatorname{Re}\left\{\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right\} \le 1-\alpha.$$

We have

$$\beta\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right|-\operatorname{Re}\left\{\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right\}$$

$$\le (1+\beta)\left|\frac{z(f*g)'(z)+\gamma z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\gamma z(f*g)'(z)}-1\right| \le \frac{(1+\beta)\sum\limits_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]|a_k|\,b_k}{1-\sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]|a_k|\,b_k}.$$

This last expression is bounded above by $(1-\alpha)$ if

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]|a_k|\,b_k \le 1-\alpha,$$

and hence the proof is completed.
Lemma 2. *A necessary and sufficient condition for $f(z)$ of the form (1.9) to be in the class $TS_\gamma(f,g;\alpha,\beta)$ is that*

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]a_k b_k \leq 1-\alpha, \tag{2.2}$$

Proof. In view of Lemma 1, we need only to prove the necessity. If $f(z) \in TS_\gamma(f,g;\alpha,\beta)$ and z is real, then

$$\frac{1-\sum\limits_{k=2}^{\infty} k\left[1+\gamma(k-1)\right]a_k b_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]a_k b_k z^{k-1}}-\alpha \geq \beta\left|\frac{\sum\limits_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]a_k b_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left[1+\gamma(k-1)\right]a_k b_k z^{k-1}}\right|.$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]a_k b_k \leq 1-\alpha.$$

Corollary 1.*Let the function $f(z)$ be defined by (1.9) be in the class $TS_\gamma(f,g;\alpha,\beta)$. Then*

$$a_k \leq \frac{1-\alpha}{\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k} \quad (k \geq 2). \tag{2.3}$$

The result is sharp for the function

$$f(z)=z-\frac{1-\alpha}{\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k}z^k \quad (k \geq 2). \tag{2.4}$$

Lemma 3. *The extreme points of $TS_\gamma(f,g;\alpha,\beta)$* are

$$f_1(z)=z \ \text{ and } \ f_k(z)=z-\frac{1-\alpha}{\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+\gamma(k-1)\right]b_k}z^k, \ \text{for } k=2,3,\ldots\ . \tag{2.5}$$

The proof of Lemma 3 is similar to the proof of the theorem on extreme points given in [20] and therefore are omit it.

In 1925, Littlewood [10] proved the following subordination theorem.
Lemma 4. *If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{2.6}$$

3. Main theorem

Applying Lemma 4, Lemma 2 and Lemma 3, we prove the following result.

Theorem 1. *Suppose $f \in TS_\gamma(f, g; \alpha, \beta)$, $\eta > 0$, $-1 \leq \alpha < 1$, $0 \leq \gamma \leq 1$, $\beta \geq 0$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\alpha}{(2+\beta-\alpha)(1+\gamma)b_2} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{3.1}$$

Proof. For $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$, (3.1) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\alpha}{(2+\beta-\alpha)(1+\gamma)b_2} z\right|^\eta d\theta.$$

By Lemma 4, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 - \frac{1-\alpha}{(2+\beta-\alpha)(1+\gamma)b_2} z.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \frac{1-\alpha}{(2+\beta-\alpha)(1+\gamma)b_2} w(z), \tag{3.2}$$

and using (2.2), we obtain

$$|w(z)| = \left|\sum_{k=2}^{\infty} \frac{(2+\beta-\alpha)(1+\gamma)b_2}{1-\alpha} a_k z^{k-1}\right| \leq |z| \sum_{k=2}^{\infty} \frac{(2+\beta-\alpha)(1+\gamma)b_2}{1-\alpha} a_k \leq |z|.$$

This completes the proof of Theorem 1.

By taking different choices of $g(z)$, α, β and γ in the above theorem, we can state the following integral means results for various subclasses.

Remarks. (i) Taking $g(z) = z + \sum\limits_{k=2}^{\infty} \Gamma_k z^k$, where Γ_k is given by (1.11) in Theorem 1, we obtain the result obtained by Ahuja et al. [1, Theorem 3.1];

(ii) Taking $g(z) = \frac{z}{1-z}$ and $\gamma = 0$ in Theorem 1, we obtain the result obtained by Ahuja et al. [1, Corollary 3.2];

(iii) Taking $g(z) = \frac{z}{1-z}$ and $\gamma = 1$ in Theorem 1, we obtain the result obtained by Ahuja et al. [1, Corollary 3.4];

(iv) Taking $g(z) = z + \sum\limits_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k$ $(\lambda > -1)$ and $\gamma = 0$ in Theorem 1, we obtain the result obtained by Ahuja et al. [1, Corollary 3.6];

(v) Taking $g(z) = z + \sum\limits_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) z^k$ $(c > -1)$ and $\gamma = 0$ in Theorem 1, we obtain the result obtained by Ahuja et al. [1, Corollary 3.7];

(vi) Taking $g(z) = z + \sum\limits_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k$ $(a > 0;\ c > 0)$ and $\gamma = 0$ in Theorem 1, we obtain the result obtained by Ahuja et al. [1, Corollary 3.8].

Corollary 2. *If* $f \in TS_0(f, z + \sum\limits_{k=2}^{\infty} k^n z^k; \alpha, \beta) = TS\,(n, \alpha, \beta)$ $(-1 \leq \alpha < 1,\ \beta \geq 0,$ $n \in N_0$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{2^n(2+\beta-\alpha)} z^2.$$

Corollary 3. *If* $f \in TS_0(f, z + \sum\limits_{k=2}^{\infty} [1+\lambda(k-1)]^n z^k; \alpha, \beta) = TS_\lambda\,(n, \alpha, \beta)$ $(-1 \leq$ $\alpha < 1,\ \beta \geq 0,\ \lambda \geq 0,\ n \in N_0$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{(2+\beta-\alpha)\,(1+\lambda)^n} z^2.$$

Corollary 4. *If* $f \in TS_\gamma(f, z + \sum\limits_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k; \alpha, \beta) = TS\,(\gamma, \alpha, \beta)$ $(0 \leq \gamma \leq 1,$ $-1 \leq \alpha < 1,\ \beta \geq 0,\ a > 0,\ c > 0$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{c\,(1-\alpha)}{a(2+\beta-\alpha)(1+\gamma)} z^2.$$

Corollary 5. *If* $f \in TS_0(f, g(z); \alpha, \beta) = H_T\,(g, \alpha, \beta)$ $(-1 \leq \alpha < 1,\ \beta \geq 0$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{(2+\beta-\alpha) b_2} z^2.$$

Corollary 6. *If* $f \in TS_\gamma(f, z+\sum\limits_{k=2}^{\infty} k^n z^k; \alpha, \beta) = TS_\gamma(n, \alpha, \beta)$ $(0 \leq \gamma \leq 1,$ $-1 \leq \alpha < 1,$ $\beta \geq 0,$ $n \in N_0$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{2^n(2+\beta-\alpha)(1+\gamma)} z^2.$$

Corollary 7. *If* $f \in TS_\gamma(f, z+\sum\limits_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right) z^k; \alpha, \beta) = TS_\gamma(c, \alpha, \beta)$ $(0 \leq \gamma \leq 1,$ $-1 \leq \alpha < 1,$ $\beta \geq 0,$ $c > -1$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(c+2)}{(2+\beta-\alpha)(1+\gamma)(c+1)} z^2.$$

Corollary 8. *If* $f \in TS_\gamma(f, z+\sum\limits_{k=2}^{\infty} \frac{(\mu)_{k-1}}{(\lambda+1)_{k-1}} z^k; \alpha, \beta) = TS_\gamma(\mu, \lambda; \alpha, \beta)$ $(0 \leq \gamma \leq 1,$ $-1 \leq \alpha < 1,$ $\beta \geq 0,$ $\lambda > -1,$ $\mu > 0$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(\lambda+1)}{(2+\beta-\alpha)(1+\gamma)\mu} z^2.$$

Corollary 9. *If* $f \in TS_\gamma(f, z+\sum\limits_{k=2}^{\infty} \frac{(c)_{k-1}}{(a)_{k-1}} \frac{(\lambda+1)_{k-1}}{(1)_{k-1}} z^k; \alpha, \beta) = TS_\gamma(a, c, \lambda; \alpha, \beta)$ $(0 \leq \gamma \leq 1,$ $-1 \leq \alpha < 1,$ $\beta \geq 0,$ $a, c \in R \backslash Z_0^-,$ $\lambda > -1$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{a(1-\alpha)}{c(2+\beta-\alpha)(1+\gamma)(\lambda+1)} z^2.$$

Corollary 10. *If* $f \in TS_\gamma(f, z+\sum\limits_{k=2}^{\infty} \frac{(2)_{k-1}}{(n+1)_{k-1}} z^k; \alpha, \beta) = TS_\gamma(n, \alpha, \beta)$ $(0 \leq \gamma \leq 1,$ $-1 \leq \alpha < 1,$ $\beta \geq 0,$ $n > -1$ *and* $\eta > 0)$, *then the assertion (3.1) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(n+1)}{2(2+\beta-\alpha)(1+\gamma)} z^2.$$

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