SUBORDINATION RESULTS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH THE CONVOLUTION STRUCTURE

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Abstract. The main object of the present paper is to derive subordination property of the class which was very recently $\mathcal{PT}_g(\lambda,\alpha,\beta,\gamma)$ introduced by Murugusundaramoorthy and Joshi (J. Math. Ineq., 3(1)(2009), pp. 293-303).

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \Delta := \{z \in C | \, |z| < 1\}) \tag{1.1}$$

which are analytic and univalent in the open unit disk. If $\mathcal{U} = \{z \in C : |z| < 1\}$ is given by (1.1) and $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{1.2}$$

the Hadamard product (or convolution) $(f * g)$ of f and g is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, z \in \mathcal{U} \tag{1.3}$$

Very recently, by using Hadamard product (or convolution) Murugusundaramoorthy and Joshi [5], introduced the subclass $\mathcal{P}_g(\lambda,\alpha,\beta,\gamma)$ of $\mathcal{A}$ consisting the functions of the form (1.1) and satisfying the following inequality,

$$\left|\frac{J_{g,\lambda}(z)-1}{2\gamma(J_{g,\lambda}(z)-\alpha)-(J_{g,\lambda}(z)-1)}\right| < \beta, \lambda \geq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1, z \in C)$$

where $J_{g,\lambda}(z) = \frac{(1-\lambda)(f*g)(z)+\lambda z(f*g)'}{z}$ $0 < \gamma \leq 1$, and $(f*g)(z)$ is given by (1.3) and g is fixed function for all $z \in \mathcal{U}$. We further assume that $\mathcal{PT}_g(\lambda,\alpha,\beta,\gamma) = P_g(\lambda,\alpha,\beta,\gamma) \cap T$, where

$$T := \left\{ f \in A : f(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k, z \in U \right\} \tag{1.4}$$

is a subclass of $\mathcal{A}$ introduced and studied by Silverman [6].

We observe that several known operators are deducible from the convolution. That is, for various choices of g in (1.3), we obtain some interesting operators studied by many authors. For examples, we illustrate the following two operators.

At first, for complex parameters $\alpha_1, ...\alpha_l$ and $\beta_1, ...\beta_q, (\beta_j \in C \backslash Z_0^-; Z_0^- = \{0,-1,-2,.......\}; j = 1,2,3.....)$ consider the function g defined by

$$g(z) = z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \cdots (\alpha_l)_{k-1}}{(\beta_1)_{k-1} \cdots (\beta_q)_{k-1}} \frac{z^k}{(k-1)!}, \quad (l \leq q+1; q, s \in N_0 := N \cup \{0\}; z \in \mathcal{U}) \tag{1.5}$$

where$(\nu)_k$ is the well known Pochhammer symbol.
Then for functions $f \in \mathcal{A}$, the convolution (1.3) with the function g defined by (1.5) gives the operator studied by Dziok and Srivastava ([2], see also [3,4]).
Next, if we define the function g by

$$g(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+l}{1+l}\right)^m z^k (m \geq 0, l \in Z). \tag{1.6}$$

Then for functions $f \in \mathcal{A}$, the convolution (1.3) with the function g defined by (1.6) reduces to the multiplier transformation studied by Cho and Srivastava [1]. In the present paper, we will obtain subordination property for the subclass $\mathcal{PT}_g(\lambda,\alpha,\beta,\gamma)$. To prove it, we need following definitions and lemma which are given below.

Definition 1.1 (Subordination Principle). *For the two functions f and g analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$ and denoted by $f(z) \prec g(z)$ $z \in \mathcal{U}$, if there exists a Schwarz function $w(z)$ analytic in $\mathcal{U}$ with*

$w(0) = 0$, and $|w(z)| < 1$ such that $f(z) = g(w(z))z \in \mathcal{U}$. In particular, if the function $g(z)$ is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

Definition 1.2 (Subordinating factor sequence, see [7]). *A sequence $\{b_k\}_{k=1}^{\infty}$ of complex number is called a subordinating factor sequence if, whenever $f(z) = \sum\limits_{k=1}^{\infty} a_K z^K$ ($a_1 = 1$) is analytic, univalent and convex in $\mathcal{U}$, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z) \ \ (z \in \mathcal{U}) \tag{1.7}$$

Lemma 1.1 (see [7]). *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \quad (z \in \mathcal{U}) \tag{1.8}$$

To prove our main result, we shall required the following lemma due to Murusundaramoorthy and Joshi.

Lemma 1.2 (see [5]). *Let the function f be defined by (1.4) then $\mathcal{PT}_g(\lambda, \alpha, \beta, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty} (1 + \lambda(k-1))[1 + \beta(2\gamma - 1)]a_k b_k \leq 2\beta\gamma(1-\alpha) \tag{1.9}$$

2. Subordination theorem

Theorem 2.1. *Let the function $f \in \mathcal{PT}_g(\lambda, \alpha, \beta, \gamma)$ satisfy the inequality (1.8), and K denote the familiar class of univalent and convex functions in $\mathcal{U}$. Then for every $g \in K$, we have*

$$\begin{gathered}\tfrac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]}(f * \phi)(z) \prec \phi(z),\\ (z \in U, b_k \geq b_2 > 0 (k \geq 2); \gamma \in C \backslash \{0\}; 0 \leq \alpha < 1)\end{gathered} \tag{2.1}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]}{2(1+\lambda)[1+\beta(2\gamma-1)]b_2} \quad (z \in \mathcal{U}) \tag{2.2}$$

The following constant factor $\frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{2[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]}$, in the subordination result (2.1) is the best dominant.

Proof. Let $f(z)$ satisfy the inequality and let $\phi(z) = z + \sum\limits_{k=2}^{\infty} c_k z^k \in K$, then

$$\frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{2[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]}(f * \phi)(z) \tag{2.3}$$

$$= \frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{2[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]}(z + \sum_{k=2}^{\infty} a_k c_k z^k).$$

By invoking definition (2.2), the subordination (2.1) our theorem will hold true if the sequence

$$\left\{ \frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{2[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]} a_k \right\}_{k=1}^{\infty} \tag{2.4}$$

is a subordination factor sequence. By virtue of Lemma (1.1), this is equivalent to the inequality

$$\operatorname{Re}\left\{ 1 + \sum_{k=1}^{\infty} \frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]} a_k z^k \right\} > 0 \tag{2.5}$$

Since $b_k \geq b_2 > 0$ for $k \geq 2$, we have

$$\begin{aligned}
&\operatorname{Re}\left\{ 1 + \sum_{k=1}^{\infty} \tfrac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} a_k z^k \right\} \\
&\geq 1 - \tfrac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} r \\
&\qquad - \tfrac{1}{(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} \sum_{k=2}^{\infty} (1+\lambda)[1+\beta(2\gamma-1)]b_2 a_k r^k
\end{aligned}$$

$$\begin{aligned}
&\geq 1 - \tfrac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} r \\
&\qquad - \tfrac{1}{(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} \sum_{k=2}^{\infty} (1+\lambda(k-1))[1+\beta(2\gamma-1)]b_k a_k r^k.
\end{aligned}$$

By using Lemma 1.2, we can easily seen that

$$\begin{aligned}
&\geq 1 - \tfrac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} r \\
&\qquad - \tfrac{2\beta\gamma(1-\alpha)}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]} r > 0, \quad (|z| = r < 1).
\end{aligned}$$

This establishes the inequality (2.5), and consequently the subordination relation (2.1) of Theorem 2.1 is proved. The inequality (2.2) follows from (2.1), upon setting

$$\phi(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \in K \quad (z \in \mathcal{U}), \tag{2.6}$$

and

$$f_0(z) = z - \frac{2\beta\gamma(1-\alpha)}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]} z^2 \quad (z \in \mathcal{U}), \tag{2.7}$$

which belongs to $\mathcal{PT}_g(\lambda, \alpha, \beta, \gamma)$. Using (2.1), we infer that

$$\frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2 + 2\beta\gamma(1-\alpha)]} f_0(z) \prec \frac{z}{1-z} \tag{2.8}$$

It can be easily verified for the function $f_0(z)$ defined by (2.7) that

$$\underset{|z|<1}{Min}\operatorname{Re}\left\{\frac{(1+\lambda)[1+\beta(2\gamma-1)]b_2}{[(1+\lambda)[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]}f_0(z)\right\}>-\frac{1}{2} \tag{2.9}$$

which completes the proof of theorem.

By substituting $\lambda = 0$, in Theorem (2.1), we easily get

Corollary 2.2. *Let the function $f(z) \in \mathcal{PT}_g(0,\alpha,\beta,\gamma)$ satisfy the inequality*

$$\sum_{k=2}^{\infty}[1+\beta(2\gamma-1)]a_k b_k \leq 2\beta\gamma(1-\alpha) \tag{2.10}$$

and K denote the familiar class of univalent and convex functions in $\mathcal{U}$. Then for every $\phi \in K$, we have

$$\begin{array}{c}\frac{[1+\beta(2\gamma-1)]b_2}{[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]}(f*\phi)(z) \prec \phi(z)\\ (z\in\mathcal{U}, b_k \geq b_2 > 0 (k\geq 2); \gamma \in C/\{0\}; 0\leq\alpha<1)\end{array} \tag{2.11}$$

and

$$\operatorname{Re}\{f(z)\} > -\frac{[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)]}{2[1+\beta(2\gamma-1)]b_2} \tag{2.12}$$

The following constant factor $\frac{[1+\beta(2\gamma-1)]b_2]}{2[1+\beta(2\gamma-1)]b_2+2\beta\gamma(1-\alpha)}$, in the subordination result (2.11) is the best dominant. By setting $\phi(z) = \frac{z}{1-z}$,$\beta = 1$ and $\lambda = 0$,

Corollary 2.3. *Let the function $f(z) \in \mathcal{PT}_g(0,\alpha,1,\gamma)$ satisfy the inequality*

$$\sum_{k=2}^{\infty}2\gamma a_k \leq 2(1-\alpha) \tag{2.13}$$

and K denote the familiar class of univalent and convex functions in $\mathcal{U}$. Then for every $\phi(z) \in K$, we have

$$\begin{array}{c}\frac{1}{2-\alpha}(f*\phi)(z) \prec \phi(z)\\ (z\in\mathcal{U}, b_k \geq b_2 > 0 (k\geq 2); \gamma \in C/\{0\}; 0\leq\alpha<1)\end{array} \tag{2.14}$$

and

$$\operatorname{Re}\{f(z)\} > \frac{\alpha-2}{2} \tag{2.15}$$

The following constant factor$\frac{2}{\alpha-2}$, in the subordination result (2.14) is the best dominant.

3. Further remarks and observation

Using Hadamard Product (or convolution) defined by (1.2) and using Wilf lemma, we obtained the subordination results for the subclass $\mathcal{PT}_g(\lambda, \alpha, \beta, \gamma)$ of $\mathcal{A}$. If we replace g (z) in Theorem 2.1 defined by the functions in (1.5) and (1.6) , we get the corresponding results of the Theorem 2.1.

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